EEE351 Electromagnetic Waves
2023-2024 Fall Semester

Textbook:

1) Field and Wave Electromagnetics (2nd Edition)
David K. Cheng
Addison-Wesley, 1989.
(Chapter 7,8)

Supplementary:

2) Fundamentals of Engineering Electromagnetism
David K. Cheng
Prentice–Hall, 1993.

3) Mühendislik Elektromanyetiğinin Temelleri
David K. Cheng
Çeviri: Adnan Köksal, Birsen Saka
Palme Yayıncılık, 2006.

4) Elektromanyetik
J. A. Edminister
Çeviri: M. Timur Aydemir, Erkan Afacan, K. Cem Nakiboğlu
Nobel Yayın Dağıtım, 2000.

Date of Current Version: 9th October 2023

TIME–VARYING FIELDS AND MAXWELL'S EQUATIONS

1. Introduction

Electrostatic Model:

$$\nabla \times \vec{\mathbf{E}} = 0 \tag{1}$$

$$\nabla \cdot \vec{\mathbf{D}} = \rho \tag{2}$$

$\vec{\mathbf{E}}$: electric field intensity (V/m)

$\vec{\mathbf{D}}$: electric flux density (C/m^2)

ρ : charge density (C/m^3)

For linear (*doğrusal*) and isotropic (*yön bağımsız*) media we have

$$\vec{\mathbf{D}} = \epsilon \vec{\mathbf{E}} \tag{3}$$

The electric field intensity $\vec{\mathbf{E}}$ for an electrostatic model is conservative ($\nabla \times \vec{\mathbf{E}} = 0$) and can be expressed as the gradient of a scalar potential.

$$\vec{\mathbf{E}} = -\nabla V \qquad \text{(Electrostatic model)} \tag{4}$$

Magnetostatic Model:

$$\nabla \cdot \vec{\mathbf{B}} = 0 \tag{5}$$

$$\nabla \times \vec{\mathbf{H}} = \vec{\mathbf{J}} \tag{6}$$

$\vec{\mathbf{B}}$: magnetic flux density (T = Wb/m^2)

$\vec{\mathbf{H}}$: magnetic field intensity (A/m)

$\vec{\mathbf{J}}$: current density (A/m^2)

For linear and isotropic media we have

$$\vec{\mathbf{B}} = \mu \vec{\mathbf{H}} \tag{7}$$

A changing magnetic field gives rise to an electric field, and a changing electric field gives rise to a magnetic field.

2. Faraday's Law of Electromagnetic Induction

Fundamental Postulate for Electromagnetic Induction

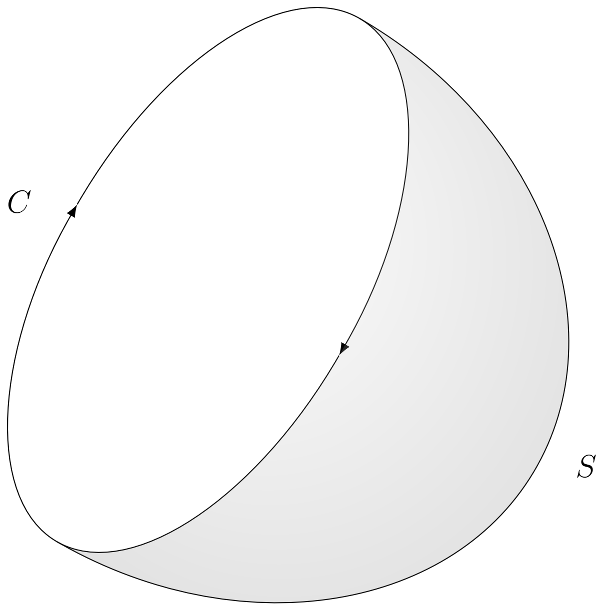
$$\boxed{\nabla \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t}} \tag{8}$$

The electric field intensity $\vec{\mathbf{E}}$ in a region of time–varying magnetic flux density $\vec{\mathbf{B}}$ is non-conservative ($\nabla \times \vec{\mathbf{E}} \neq 0$) and cannot be expressed as the gradient of a scalar potential. We will see that

$$\vec{\mathbf{E}} = -\nabla V + ? \qquad \text{(Time–varying fields)} \tag{9}$$

Now, we will obtain the integral form of Faraday's law (Eq. 8). We will use Stokes' theorem.

$$\int_S (\nabla \times \vec{\mathbf{A}}) \cdot d\vec{\mathbf{s}} = \oint_C \vec{\mathbf{A}} \cdot d\vec{\ell} \tag{10}$$



Taking the surface integral of both sides of Eq. 8 over an open surface we have

$$\int_S (\nabla \times \vec{\mathbf{E}}) \cdot d\vec{\mathbf{s}} = -\int_S \frac{\partial \vec{\mathbf{B}}}{\partial t} \cdot d\vec{\mathbf{s}} \tag{11}$$

Applying Stokes' theorem

$$\int_S (\nabla \times \vec{\mathbf{E}}) \cdot d\vec{\mathbf{s}} = \oint_C \vec{\mathbf{E}} \cdot d\vec{\ell} \tag{12}$$

$$\boxed{\oint_C \vec{\mathbf{E}} \cdot d\vec{\ell} = -\int_S \frac{\partial \vec{\mathbf{B}}}{\partial t} \cdot d\vec{\mathbf{s}}} \qquad \text{(Faraday's law)} \tag{13}$$

2.1 A Stationary Circuit in a Time–Varying Magnetic Field

For a stationary circuit with a contour C and surface S we have

$$\oint_C \vec{\mathbf{E}} \cdot d\vec{\ell} = -\frac{\partial}{\partial t} \int_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} \tag{14}$$

Let's define

$$V = \oint_C \vec{\mathbf{E}} \cdot d\vec{\ell} \tag{15}$$

where V is the electromotive force (emf) induced in a circuit with contour C (V), and

$$\Phi = \int_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} \tag{16}$$

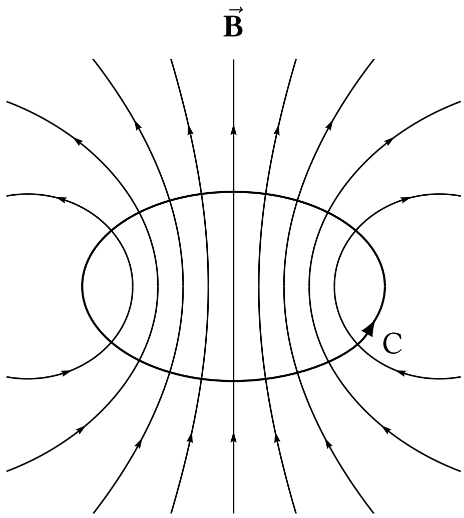
where Φ is the magnetic flux crossing surface S (Wb). So we have

$$\boxed{V = -\frac{d\Phi}{dt}} \qquad \text{(Faraday's law of electromagnetic induction)} \tag{17}$$

The electromotive force induced in a stationary closed circuit is equal to the negative rate of increase of the magnetic flux linking the circuit.

The induced emf will cause a current to flow in the closed loop in a such a direction as to oppose the change in the linking magnetic flux.

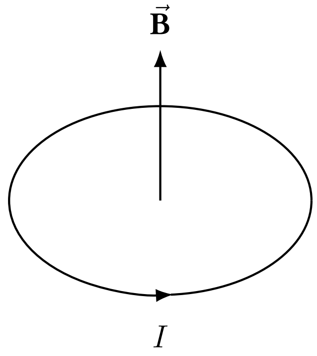
The current induced by the changing flux is called the induced current. The negative sign is known as **Lenz's law**.



Let's assume that the flux through a circuit C is decreasing.

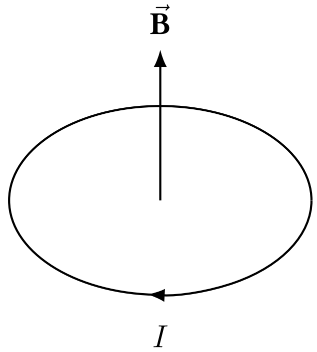
$$\frac{d\Phi}{dt} < 0, \qquad V > 0 \tag{18}$$

Then by Lenz's law, the induced emf produces a current I in the circuit which gives rise to a magnetic field so directed as to increase the flux through C.



The reverse situation is shown in the following figure when the flux through C is increasing.

$$\frac{d\Phi}{dt} > 0, \qquad V < 0 \tag{19}$$



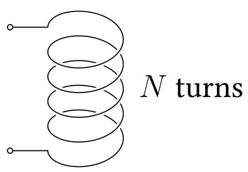
The change in flux can occur for a variety of reasons. For instance
i) the magnetic induction field $\vec{\mathbf{B}}$ may be changing in time,
ii) the size or shape of the circuit may be altering,
iii) the circuit may be moving in a manner which continually alters the flux passing through it.

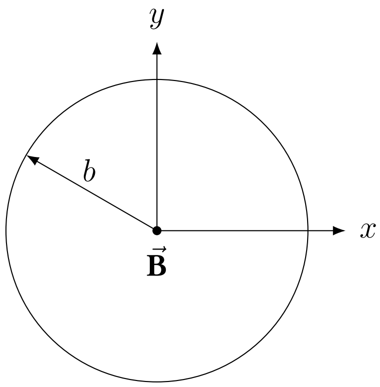
Example 1: A circular loop (*çembersel halka*) of N turns (*sarım*) of conducting wire lies in the $x - y$ plane with its center at the origin of a magnetic field specified by

$$\vec{\mathbf{B}} = \hat{\mathbf{a}}_z\, B_0 \,\cos\left(\frac{\pi r}{2b}\right) \sin\omega t \tag{20}$$

where b is the radius of the loop and ω is the angular frequency. Find the emf induced in the loop.

Solution:





$$\vec{\mathbf{B}} = \hat{\mathbf{a}}_z\, B_0\, \cos\left(\frac{\pi r}{2b}\right) \sin\omega t$$

$$V = -\frac{d\Phi}{dt} \tag{21}$$

$$\Phi = \int_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} \tag{22}$$

$$\vec{\mathbf{B}} = \hat{\mathbf{a}}_z\, B_0\, \cos\left(\frac{\pi r}{2b}\right) \sin\omega t \tag{23}$$

$$d\vec{\mathbf{s}} = \hat{\mathbf{a}}_z\, r\, d\phi\, dr \tag{24}$$

$$\hat{\mathbf{a}}_z \cdot \hat{\mathbf{a}}_z = 1 \tag{25}$$

$$\vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \left[B_0\, \cos\left(\frac{\pi r}{2b}\right) \sin\omega t\right]\, [r\, d\phi\, dr] \tag{26}$$

$$\Phi = \int_0^{2\pi} \int_0^b B_0\, \cos\left(\frac{\pi r}{2b}\right) \sin\omega t\, r\, dr\, d\phi \tag{27}$$

$$\Phi = B_0\, \sin\omega t\, \int_0^{2\pi} d\phi \int_0^b \cos\left(\frac{\pi r}{2b}\right)\, r\, dr \tag{28}$$

$$\int_0^{2\pi} d\phi = 2\pi \tag{29}$$

$$\Phi = 2\pi\, B_0\, \sin\omega t\, \int_0^b \cos\left(\frac{\pi r}{2b}\right)\, r\, dr \tag{30}$$

$$I = \int_0^b r\, \cos\left(\frac{\pi r}{2b}\right)\, dr \tag{31}$$

$$u = \frac{\pi r}{2b} \Rightarrow r = \frac{2b}{\pi}\, u \tag{32}$$

$$du = \frac{\pi}{2b}\, dr \Rightarrow dr = \frac{2b}{\pi}\, du \tag{33}$$

$$I = \int \left[\frac{2b}{\pi}\, u\right]\, \cos(u)\, \left[\frac{2b}{\pi}\, du\right] \tag{34}$$

$$I = \left(\frac{2b}{\pi}\right)^2 \int u\, \cos u\, du \tag{35}$$

$$J = \int x \cos x \, dx \tag{36}$$

$$\int u \, dv = uv - \int v \, du \qquad \text{(integration by parts)} \tag{37}$$

$$u = x \qquad dv = \cos x \, dx \tag{38}$$

$$du = dx \qquad v = \sin x \tag{39}$$

$$J = x \sin x - \int \sin x \, dx = x \sin x + \cos x \tag{40}$$

$$I = \left(\frac{2b}{\pi}\right)^2 J = \left(\frac{2b}{\pi}\right)^2 (u \sin u + \cos u) \tag{41}$$

$$I = \left(\frac{2b}{\pi}\right)^2 \left[\left(\frac{\pi r}{2b}\right) \sin\left(\frac{\pi r}{2b}\right) + \cos\left(\frac{\pi r}{2b}\right)\right]_0^b \tag{42}$$

$$\begin{aligned} I = \left(\frac{2b}{\pi}\right)^2 & \left[\left(\frac{\pi b}{2b}\right) \sin\left(\frac{\pi b}{2b}\right) + \cos\left(\frac{\pi b}{2b}\right)\right. \\ & \left. - \left(\frac{\pi 0}{2b}\right) \sin\left(\frac{\pi 0}{2b}\right) - \cos\left(\frac{\pi 0}{2b}\right)\right] \end{aligned} \tag{43}$$

$$I = \left(\frac{2b}{\pi}\right)^2 \left[\left(\frac{\pi}{2}\right) \sin\left(\frac{\pi}{2}\right) + \cos\left(\frac{\pi}{2}\right) - 0 - \cos(0)\right] \tag{44}$$

$$I = \left(\frac{2b}{\pi}\right)^2 \left[\frac{\pi}{2} - 1\right] \tag{45}$$

$$\Phi = 2\pi B_0 \sin \omega t \int_0^b \cos\left(\frac{\pi r}{2b}\right) r \, dr \tag{46}$$

$$\Phi = 2\pi B_0 \sin \omega t \left(\frac{2b}{\pi}\right)^2 \left(\frac{\pi}{2} - 1\right) \tag{47}$$

$$\Phi = \frac{8b^2}{\pi} \left(\frac{\pi}{2} - 1\right) B_0 \sin \omega t \tag{48}$$

$$V = -\frac{d\Phi}{dt} \tag{49}$$

$$V = -\omega \frac{8b^2}{\pi} \left(\frac{\pi}{2} - 1\right) B_0 \cos \omega t \qquad \text{(for one turn)} \tag{50}$$

$$V = -N \omega \frac{8b^2}{\pi} \left(\frac{\pi}{2} - 1\right) B_0 \cos \omega t \quad \text{(V)} \qquad \text{(for } N \text{ turns)} \tag{51}$$