5. Potential Functions

Wave Equation for Vector Potential $\vec{A}$

$$\nabla \cdot \vec{\mathbf{B}} = 0 \qquad \text{(4th Maxwell's equation)} \tag{1}$$

$$\nabla \cdot (\nabla \times \vec{\mathbf{A}}) = 0 \qquad \text{(Identity II)} \tag{2}$$

$\vec{\mathbf{B}}$ is solenoidal ($\nabla \cdot \vec{\mathbf{B}} = 0$). So $\vec{\mathbf{B}}$ can be expressed as the curl of another vector field.

$$\boxed{\vec{\mathbf{B}} = \nabla \times \vec{\mathbf{A}}} \tag{3}$$

The vector field $\vec{\mathbf{A}}$ is called the vector magnetic potential (Wb/m).

$$\nabla \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t} \qquad \text{(1st Maxwell's equation)} \tag{4}$$

$$\nabla \times \vec{\mathbf{E}} = -\frac{\partial}{\partial t}(\nabla \times \vec{\mathbf{A}}) \tag{5}$$

$$\nabla \times \left(\vec{\mathbf{E}} + \frac{\partial \vec{\mathbf{A}}}{\partial t}\right) = 0 \tag{6}$$

$$\nabla \times (\nabla V) = 0 \qquad \text{(Identity I)} \tag{7}$$

$$\vec{\mathbf{E}} + \frac{\partial \vec{\mathbf{A}}}{\partial t} = -\nabla V \tag{8}$$

$$\boxed{\vec{\mathbf{E}} = -\nabla V - \frac{\partial \vec{\mathbf{A}}}{\partial t} \qquad (\mathrm{V/m})} \tag{9}$$

$$\nabla \times \vec{\mathbf{H}} = \vec{\mathbf{J}} + \frac{\partial \vec{\mathbf{D}}}{\partial t} \qquad \text{(2nd Maxwell's equation)} \tag{10}$$

$$\vec{\mathbf{B}} = \nabla \times \vec{\mathbf{A}} \tag{11}$$

$$\vec{\mathbf{E}} = -\nabla V - \frac{\partial \vec{\mathbf{A}}}{\partial t} \tag{12}$$

Let's substitute Eqs. 3 and 9 into Eq. 10 and make use of the constitutive relations.

$$\vec{\mathbf{H}} = \frac{\vec{\mathbf{B}}}{\mu} \tag{13}$$

$$\vec{\mathbf{D}} = \epsilon \vec{\mathbf{E}} \tag{14}$$

For a homogenous medium we have

$$\nabla \times \left(\frac{\vec{\mathbf{B}}}{\mu}\right) = \vec{\mathbf{J}} + \frac{\partial}{\partial t}(\epsilon \vec{\mathbf{E}}) \tag{15}$$

$$\nabla \times \left(\frac{1}{\mu}\nabla \times \vec{\mathbf{A}}\right) = \vec{\mathbf{J}} + \frac{\partial}{\partial t}\left[\epsilon\left(-\nabla V - \frac{\partial \vec{\mathbf{A}}}{\partial t}\right)\right] \tag{16}$$

$$\frac{1}{\mu}\nabla \times \nabla \times \vec{\mathbf{A}} = \vec{\mathbf{J}} + \epsilon\frac{\partial}{\partial t}\left(-\nabla V - \frac{\partial \vec{\mathbf{A}}}{\partial t}\right) \tag{17}$$

$$\nabla \times \nabla \times \vec{\mathbf{A}} = \mu\vec{\mathbf{J}} + \mu\epsilon\frac{\partial}{\partial t}\left(-\nabla V - \frac{\partial \vec{\mathbf{A}}}{\partial t}\right) \tag{18}$$

$$\nabla \times \nabla \times \vec{\mathbf{A}} = \nabla(\nabla \cdot \vec{\mathbf{A}}) - \nabla^2\vec{\mathbf{A}} \tag{19}$$

$$\nabla(\nabla \cdot \vec{\mathbf{A}}) - \nabla^2\vec{\mathbf{A}} = \mu\vec{\mathbf{J}} + \mu\epsilon\frac{\partial}{\partial t}\left(-\nabla V - \frac{\partial \vec{\mathbf{A}}}{\partial t}\right) \tag{20}$$

$$\nabla(\nabla \cdot \vec{\mathbf{A}}) - \nabla^2\vec{\mathbf{A}} = \mu\vec{\mathbf{J}} + \mu\epsilon\frac{\partial}{\partial t}(-\nabla V) - \mu\epsilon\frac{\partial}{\partial t}\left(\frac{\partial \vec{\mathbf{A}}}{\partial t}\right) \tag{21}$$

$$\nabla(\nabla \cdot \vec{\mathbf{A}}) - \nabla^2\vec{\mathbf{A}} = \mu\vec{\mathbf{J}} - \nabla\left(\mu\epsilon\frac{\partial V}{\partial t}\right) - \mu\epsilon\frac{\partial^2 \vec{\mathbf{A}}}{\partial t^2} \tag{22}$$

$$\nabla(\nabla \cdot \vec{\mathbf{A}}) + \nabla\left(\mu\epsilon\frac{\partial V}{\partial t}\right) - \mu\vec{\mathbf{J}} = \nabla^2\vec{\mathbf{A}} - \mu\epsilon\frac{\partial^2 \vec{\mathbf{A}}}{\partial t^2} \tag{23}$$

$$\nabla^2\vec{\mathbf{A}} - \mu\epsilon\frac{\partial^2 \vec{\mathbf{A}}}{\partial t^2} = \nabla(\nabla \cdot \vec{\mathbf{A}}) + \nabla\left(\mu\epsilon\frac{\partial V}{\partial t}\right) - \mu\vec{\mathbf{J}} \tag{24}$$

$$\nabla^2\vec{\mathbf{A}} - \mu\epsilon\frac{\partial^2 \vec{\mathbf{A}}}{\partial t^2} = \nabla\left(\nabla \cdot \vec{\mathbf{A}} + \mu\epsilon\frac{\partial V}{\partial t}\right) - \mu\vec{\mathbf{J}} \tag{25}$$

The definiton of a vector requires the specification of both its curl and its divergence. The curl of $\vec{\mathbf{A}}$ is equal to $\vec{\mathbf{B}}$, i.e. ($\nabla \times \vec{\mathbf{A}} = \vec{\mathbf{B}}$). We can choose the divergence of $\vec{\mathbf{A}}$ as follows:

$$\boxed{\nabla \cdot \vec{\mathbf{A}} + \mu\epsilon\frac{\partial V}{\partial t} = 0} \qquad \text{(Lorentz condition for potentials)} \tag{26}$$

So we obtain

$$\boxed{\nabla^2\vec{\mathbf{A}} - \mu\epsilon\frac{\partial^2 \vec{\mathbf{A}}}{\partial t^2} = -\mu\vec{\mathbf{J}}} \tag{27}$$

This is the nonhomogenous wave equation for vector potential $\vec{\mathbf{A}}$

Wave Equation for Scalar Potential V

Now let's derive the wave equation for scalar potential V.

$$\vec{\mathbf{E}} = -\nabla V - \frac{\partial \vec{\mathbf{A}}}{\partial t} \tag{28}$$

$$\nabla \cdot \vec{\mathbf{D}} = \rho \tag{29}$$

$$\vec{\mathbf{D}} = \epsilon\,\vec{\mathbf{E}} \tag{30}$$

$$\nabla \cdot (\epsilon\,\vec{\mathbf{E}}) = \rho \tag{31}$$

$$\nabla \cdot \left(\epsilon \left[-\nabla V - \frac{\partial \vec{\mathbf{A}}}{\partial t} \right] \right) = \rho \tag{32}$$

$$-\nabla \cdot \epsilon \left(\nabla V + \frac{\partial \vec{\mathbf{A}}}{\partial t} \right) = \rho \tag{33}$$

For a constant ϵ, we obtain

$$-\epsilon\,\nabla \cdot \left(\nabla V + \frac{\partial \vec{\mathbf{A}}}{\partial t} \right) = \rho \tag{34}$$

$$\nabla \cdot \left(\nabla V + \frac{\partial \vec{\mathbf{A}}}{\partial t} \right) = -\frac{\rho}{\epsilon} \tag{35}$$

$$\nabla \cdot (\nabla V) = \nabla^2 V \tag{36}$$

$$\nabla^2 V + \frac{\partial}{\partial t} (\nabla \cdot \vec{\mathbf{A}}) = -\frac{\rho}{\epsilon} \tag{37}$$

Using Lorentz condition

$$\nabla \cdot \vec{\mathbf{A}} + \mu\epsilon\,\frac{\partial V}{\partial t} = 0 \tag{38}$$

$$\nabla \cdot \vec{\mathbf{A}} = -\mu\epsilon\,\frac{\partial V}{\partial t} \tag{39}$$

$$\nabla^2 V + \frac{\partial}{\partial t} \left(-\mu\epsilon\,\frac{\partial V}{\partial t} \right) = -\frac{\rho}{\epsilon} \tag{40}$$

$$\boxed{\nabla^2 V - \mu\epsilon\,\frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon}} \tag{41}$$

This is the nonhomogenous wave equation for scalar potential V.