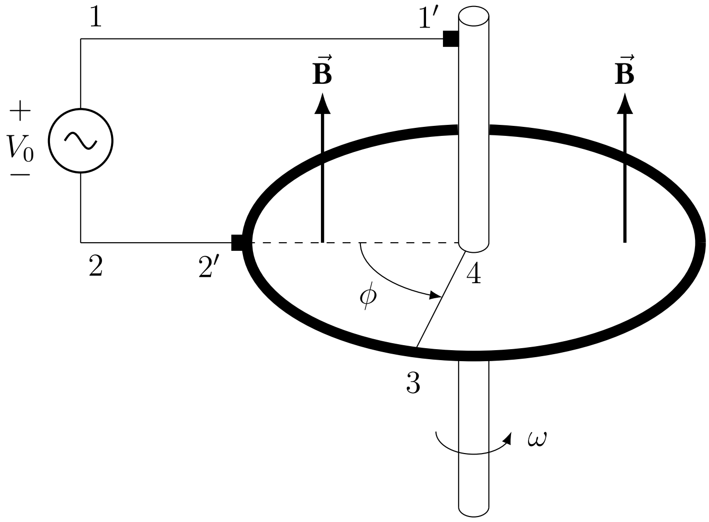
Example 3:
The Faraday disk generator consists of a circular metal disk rotating with a constant angular velocity ω in a uniform and constant magnetic flux density $\vec{\mathbf{B}} = \hat{\mathbf{a}}_z B_0$ that is parallel to the axis of rotation. Brush contacts are provided at the axis and on the rim of the disk. Determine the open-circuit voltage of the generator if the radius of the disk is b.



Faraday disk generator

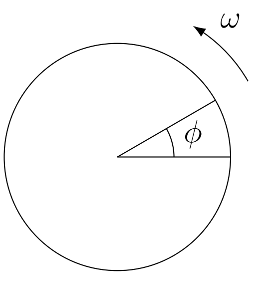
Solution I

$$\Phi = \int_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} \tag{1}$$

$$\vec{\mathbf{B}} = \hat{\mathbf{a}}_z B_0 \tag{2}$$

$$d\vec{\mathbf{s}} = \hat{\mathbf{a}}_z \, r \, d\phi \, dr \tag{3}$$

$$\vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = B_0 \, d\phi \, r \, dr \tag{4}$$



$$\phi = \omega t, \quad \omega = \frac{d\phi}{dt}$$

$$\Phi = \int_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \int_0^b \int_0^{\omega t} B_0 \, d\phi \, r \, dr \tag{5}$$

$$\Phi = B_0 \int_0^{\omega t} d\phi \int_0^b r \, dr \tag{6}$$

$$\int_0^{\omega t} d\phi = \omega t \tag{7}$$

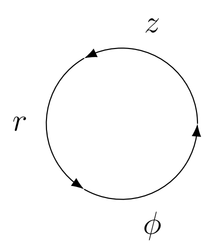
$$\int_0^{b} r\, dr = \frac{b^2}{2} \tag{8}$$

$$\Phi = \frac{1}{2}\, b^2\, B_0\, \omega\, t \tag{9}$$

$$V_0 = -\frac{d\Phi}{dt} = -\frac{1}{2}\, b^2\, B_0\, \omega \tag{10}$$

<u>Solution II</u>

$$V_0 = \oint_C (\vec{\mathbf{u}} \times \vec{\mathbf{B}}) \cdot d\vec{\ell} \tag{11}$$



$$\vec{\mathbf{u}} = \omega\, r\, \hat{\mathbf{a}}_\phi \tag{12}$$

$$\vec{\mathbf{B}} = \hat{\mathbf{a}}_z\, B_0 \tag{13}$$

$$\vec{\mathbf{u}} \times \vec{\mathbf{B}} = \hat{\mathbf{a}}_r\, \omega\, r\, B_0 \tag{14}$$

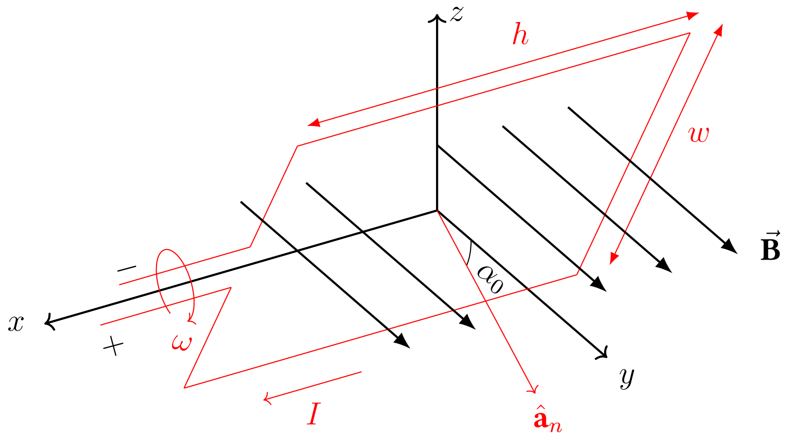
$$d\vec{\ell} = d\vec{\mathbf{r}} = \hat{\mathbf{a}}_r\, dr \tag{15}$$

$$\vec{\mathbf{u}} \times \vec{\mathbf{B}} \cdot d\vec{\ell} = B_0\, \omega\, r\, dr \tag{16}$$

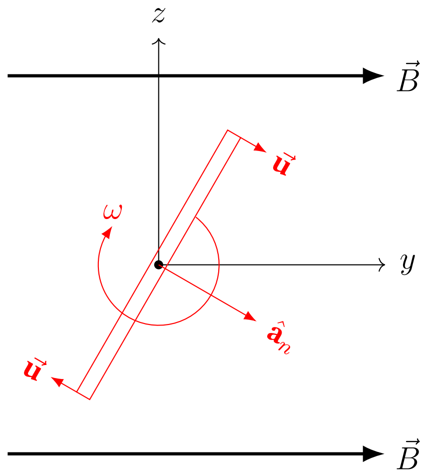
$$V_0 = \int_3^4 B_0\, \omega\, r\, dr = \omega\, B_0 \int_b^0 r\, dr = \omega\, B_0 \left[\frac{1}{2} r^2\right]_b^0 \tag{17}$$

$$V_0 = -\frac{1}{2} b^2\, B_0\, \omega \tag{18}$$

Example 4:
An h by w rectangular conducting loop is situated in a changing magnetic field $\vec{\mathbf{B}} = \hat{\mathbf{a}}_y\, B_0\ \sin\omega t$. The normal of the loop initially makes an angle α_0 with $\hat{\mathbf{a}}_y$. Find the induced emf in the loop a) when the loop is at rest, b) when the loop rotates with an angular velocity ω about the axis.



a) Perspective view



b) View from $+x$ direction

a)

$$V = -\frac{d\Phi}{dt} \tag{19}$$

$$\Phi = \int_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} \tag{20}$$

$$\vec{\mathbf{B}} = \hat{\mathbf{a}}_y\, B_0\ \sin\omega t \tag{21}$$

$$d\vec{\mathbf{s}} = \hat{\mathbf{a}}_n\, ds = \hat{\mathbf{a}}_n\, dx\, dz \tag{22}$$

$$\hat{\mathbf{a}}_y \cdot \hat{\mathbf{a}}_n = \cos\alpha_0 \tag{23}$$

$$\vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = B_0\ \cos\alpha_0 \sin\omega t\, dx\, dz \tag{24}$$

$$\Phi = \int_{-w/2}^{w/2} \int_{-h/2}^{h/2} B_0 \, \cos \alpha_0 \sin \omega t \, dx \, dz \tag{25}$$

$$\Phi = B_0 \, \cos \alpha_0 \sin \omega t \int_{-w/2}^{w/2} \int_{-h/2}^{h/2} dx \, dz \tag{26}$$

$$\int_{-w/2}^{w/2} \int_{-h/2}^{h/2} dx \, dz = w \, h = S \qquad \text{(area of the loop)} \tag{27}$$

$$\Phi = B_0 \, S \, \cos \alpha_0 \sin \omega t \tag{28}$$

$$V = -\frac{d\Phi}{dt} = -\omega \, B_0 \, S \, \cos \alpha_0 \cos \omega t \tag{29}$$

If the circuit is completed through an external load, V will produce a current that will oppose the change in Φ.

b)

$$V = -\frac{d\Phi}{dt} \tag{30}$$

$$\Phi = \int_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} \tag{31}$$

$$\vec{\mathbf{B}} = \hat{\mathbf{a}}_y \, B_0 \, \sin \omega t \tag{32}$$

$$d\vec{\mathbf{s}} = \hat{\mathbf{a}}_n \, ds = \hat{\mathbf{a}}_n \, dx \, dz \tag{33}$$

$\hat{\mathbf{a}}_n$ rotates with angular velocity ω, so the angle between $\hat{\mathbf{a}}_n$ and $\hat{\mathbf{a}}_y$ changes with time.

$$\alpha = \alpha_0 + \omega t \tag{34}$$

$$\hat{\mathbf{a}}_y \cdot \hat{\mathbf{a}}_n = \cos \alpha = \cos(\alpha_0 + \omega t) \tag{35}$$

$$\Phi = B_0 \, S \, \cos(\alpha_0 + \omega t) \sin \omega t \tag{36}$$

$$V = -\frac{d\Phi}{dt} = -B_0 \, S \, \frac{d}{dt} [\cos(\alpha_0 + \omega t) \sin \omega t] \tag{37}$$

$$\frac{d}{dt} [\cos(\alpha_0 + \omega t) \sin \omega t] = -\omega \, \sin(\alpha_0 + \omega t) \, \sin \omega t + \omega \, \cos(\alpha_0 + \omega t) \cos \omega t \tag{38}$$

$$V = -B_0 \, S \, [-\omega \, \sin(\alpha_0 + \omega t) \, \sin \omega t + \omega \, \cos(\alpha_0 + \omega t) \cos \omega t] \tag{39}$$

$$V = -\omega \, B_0 \, S \, [\cos(\alpha_0 + \omega t) \cos \omega t - \sin(\alpha_0 + \omega t) \, \sin \omega t] \tag{40}$$

$$\cos(x + y) = \cos x \, \cos y - \sin x \, \sin y \tag{41}$$

$$V = -\omega \, B_0 \, S \, \cos[(\alpha_0 + \omega t) + \omega t] \tag{42}$$

$$V = -\omega \, B_0 \, S \, \cos(\alpha_0 + 2\omega t) = -\omega \, B_0 \, S \, \cos(2\omega t + \alpha_0) \tag{43}$$

$$V = -\omega \, B_0 \, S \, \cos(2\omega t + \alpha_0) \tag{44}$$