6. Wave Equations and Their Solutions

For given charge and current distributions, ρ and $\vec{\mathbf{J}}$, we first solve the following nonhomogeneous wave equations for potentials V and $\vec{\mathbf{A}}$.

$$\boxed{\nabla^2 V - \mu\epsilon\frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon}}$$

$$\boxed{\nabla^2 \vec{\mathbf{A}} - \mu\epsilon\frac{\partial^2 \vec{\mathbf{A}}}{\partial t^2} = -\mu\vec{\mathbf{J}}}$$

With V and $\vec{\mathbf{A}}$ determined, $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ can be found from the following equations by differentiation.

$$\vec{\mathbf{E}} = -\nabla V - \frac{\partial \vec{\mathbf{A}}}{\partial t}$$

$$\vec{\mathbf{B}} = \nabla \times \vec{\mathbf{A}}$$

6.1 Solution of Wave Equations for Potentials

We now consider the solution of the nonhomogeneous wave equation for scalar electric potential V.

$$\nabla^2 V - \mu\epsilon\frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon} \tag{1}$$

First, let's find the solution for a point charge at time t, located at the origin of the coordinates. Then by summing the effects of all charge elements in a given region we can find the total solution. For a point charge at the origin it is convenient to use spherical coordinates. Because of spherical symmetry, V depends only on R and t (not on θ and ϕ). $V(R,t)$ satisfies the following homogenous equation:

$$\nabla^2 V = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial R}\right) + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{R^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} \tag{2}$$

$$\nabla^2 V = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial R}\right) \tag{3}$$

$$\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial R}\right) - \mu\epsilon\frac{\partial^2 V}{\partial t^2} = 0 \qquad \text{(Except at the origin)} \tag{4}$$

Let's introduce a new variable

$$V(R,t) = \frac{1}{R} U(R,t) \tag{5}$$

$$\frac{\partial V}{\partial R} = \frac{\partial}{\partial R}\left(\frac{U}{R}\right) = \frac{\left(\frac{\partial U}{\partial R}\right) R - U}{R^2} \tag{6}$$

$$R^2 \frac{\partial V}{\partial R} = R \frac{\partial U}{\partial R} - U \tag{7}$$

$$\frac{\partial V}{\partial t} = \frac{1}{R} \frac{\partial U}{\partial t} \tag{8}$$

$$\frac{\partial^2 V}{\partial t^2} = \frac{1}{R} \frac{\partial^2 U}{\partial t^2} \tag{9}$$

$$\frac{1}{R^2} \frac{\partial}{\partial R}\left(R \frac{\partial U}{\partial R} - U\right) - \mu\epsilon \frac{1}{R} \frac{\partial^2 U}{\partial t^2} = 0 \tag{10}$$

$$\frac{\partial}{\partial R}\left(R \frac{\partial U}{\partial R} - U\right) = \frac{\partial U}{\partial R} + R \frac{\partial^2 U}{\partial R^2} - \frac{\partial U}{\partial R} = R \frac{\partial^2 U}{\partial R^2} \tag{11}$$

$$\frac{1}{R^2} \frac{\partial}{\partial R}\left(R \frac{\partial U}{\partial R} - U\right) = \frac{1}{R} \frac{\partial^2 U}{\partial R^2} \tag{12}$$

$$\frac{1}{R} \frac{\partial^2 U}{\partial R^2} - \mu\epsilon \frac{1}{R} \frac{\partial^2 U}{\partial t^2} = 0 \tag{13}$$

$$\boxed{\frac{\partial^2 U}{\partial R^2} - \mu\epsilon \frac{\partial^2 U}{\partial t^2} = 0} \tag{14}$$

One-dimensional homogeneous wave equation.

$$U = f\left(t - \frac{R}{c}\right) \tag{15}$$

$U = f\left(t + \frac{R}{c}\right)$ does not corresspond to a physically useful solution. So we have

$$U(R,t) = f\left(t - \frac{R}{c}\right), \qquad c = \frac{1}{\sqrt{\mu\epsilon}} \tag{16}$$

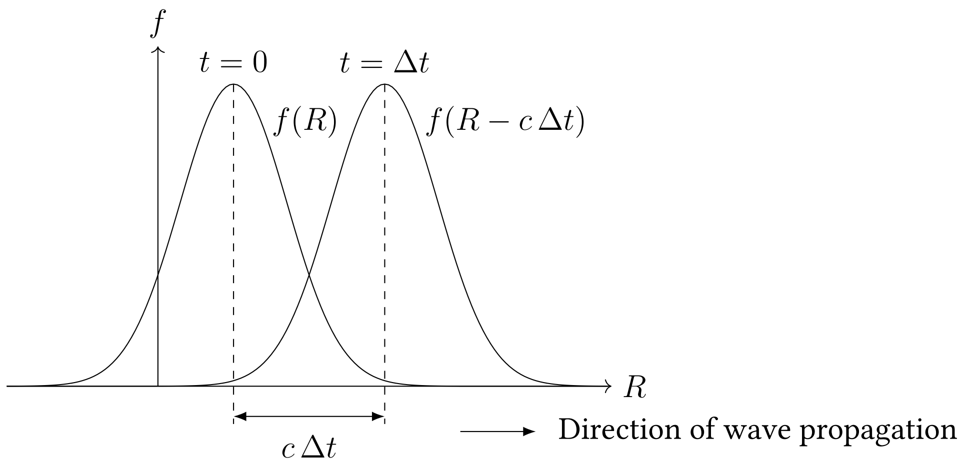
This represents a wave traveling in the positive R direction with a velocity $c = \frac{1}{\sqrt{\mu\epsilon}}$.

$$V(R,t) = \frac{1}{R} U(R,t) \tag{17}$$

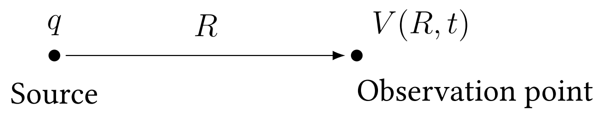
$$V(R,t) = \frac{1}{R} f\left(t - \frac{R}{c}\right) \tag{18}$$

It can be also shown that

$$V(R,t) = \frac{1}{R} f(R - ct) \tag{19}$$



$$V(R,t) = \frac{1}{R}\, f\left(t - \frac{R}{c}\right)$$



At an instant t, the potential at a distance R is a function of the charge that existed at the instant $\left(t - \frac{R}{c}\right)$. A time interval $\Delta t = \frac{R}{c}$ elapses before an observer at a distance R from the charge is able to notice any change occuring in the charge. This potential is therefore referred to as the retarded (gecikmeli) scalar potential.

To determine the function $f\left(t - \frac{R}{c}\right)$ more precisely, let us consider a point very close to the charge. In this case, the retardation may be ignored. If the charge varies according to the law $q(t)$, the potential is

$$V(R,t) = \frac{q(t)}{4\pi\epsilon R} \qquad \text{(Close to the charge)} \tag{20}$$

We have found the solution of wave equation as

$$V(R,t) = \frac{1}{R}\, f\left(t - \frac{R}{c}\right) \tag{21}$$

Comparing the last two equations we see that,

$$f\left(t - \frac{R}{c}\right) = \frac{q\left(t - \frac{R}{c}\right)}{4\pi\epsilon} \tag{22}$$

The resulting potential created by a varying point charge is

$$V(R,t) = \frac{q\left(t - \frac{R}{c}\right)}{4\pi\epsilon R}, \qquad c = \frac{1}{\sqrt{\mu\epsilon}} \tag{23}$$

The retarded potential at a point due to a cloud of charges of density $\rho(t)$ is given by

$$\boxed{V(R,t) = \frac{1}{4\pi\epsilon}\int_{V'} \frac{\rho\left(t-\frac{R}{c}\right)}{R}\,dv' \qquad (\mathrm{V})} \tag{24}$$

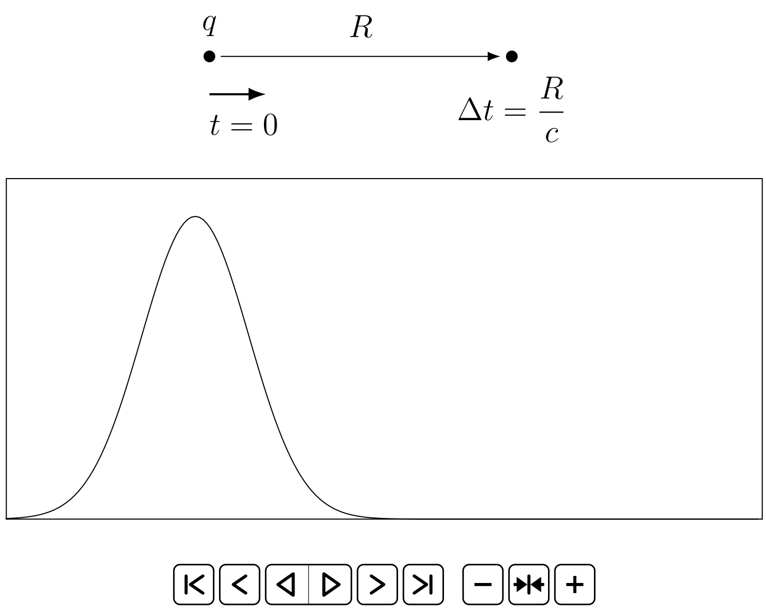
Retarded scalar potential

In a similar way

$$\boxed{\vec{\mathbf{A}}(R,t) = \frac{\mu}{4\pi}\int_{V'} \frac{\vec{\mathbf{J}}\left(t-\frac{R}{c}\right)}{R}\,dv' \qquad (\mathrm{Wb/m})} \tag{25}$$

Retarded vector potential

The electric and magnetic fields in the case of varying charges and currents need some time to change at points distant from the sources. In the quasi-static approximation we ignore this time-retardation effect and assume instant response. This assumption is implicit in dealing with circuit problems.



6.2 Source-Free Wave Equation

In problems of wave propagation we are interested in how an electromagnetic wave propagates in a source-free region where ρ and $\vec{\mathbf{J}}$ are both zero. In a simple (linear, isotropic,homogeneous) nonconducting medium ($\sigma = 0$) characterized by ϵ and μ, Maxwell's equations reduce to

$$\nabla \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t} \tag{26}$$

$$\vec{\mathbf{B}} = \mu\,\vec{\mathbf{H}} \tag{27}$$

$$\boxed{\nabla \times \vec{\mathbf{E}} = -\mu \frac{\partial \vec{\mathbf{H}}}{\partial t}} \tag{28}$$

$$\nabla \times \vec{\mathbf{H}} = \vec{\mathbf{J}} + \frac{\partial \vec{\mathbf{D}}}{\partial t} \tag{29}$$

$$\vec{\mathbf{D}} = \epsilon \vec{\mathbf{E}} \tag{30}$$

$$\vec{\mathbf{J}} = 0 \tag{31}$$

$$\boxed{\nabla \times \vec{\mathbf{H}} = \epsilon \frac{\partial \vec{\mathbf{E}}}{\partial t}} \tag{32}$$

$$\nabla \cdot \vec{\mathbf{D}} = \rho \tag{33}$$

$$\rho = 0 \tag{34}$$

$$\vec{\mathbf{D}} = \epsilon \vec{\mathbf{E}} \tag{35}$$

$$\boxed{\nabla \cdot \vec{\mathbf{E}} = 0} \tag{36}$$

$$\nabla \cdot \vec{\mathbf{B}} = 0 \tag{37}$$

$$\vec{\mathbf{B}} = \mu \vec{\mathbf{H}} \tag{38}$$

$$\boxed{\nabla \cdot \vec{\mathbf{H}} = 0} \tag{39}$$

$$\nabla \times \vec{\mathbf{E}} = -\mu \frac{\partial \vec{\mathbf{H}}}{\partial t} \tag{40}$$

$$\nabla \times (\nabla \times \vec{\mathbf{E}}) = -\mu \nabla \times \left(\frac{\partial \vec{\mathbf{H}}}{\partial t} \right) \tag{41}$$

$$\nabla \times (\nabla \times \vec{\mathbf{E}}) = -\mu \frac{\partial}{\partial t} (\nabla \times \vec{\mathbf{H}}) = -\mu \frac{\partial}{\partial t} \left(\epsilon \frac{\partial \vec{\mathbf{E}}}{\partial t} \right) = -\mu\epsilon \frac{\partial^2 \vec{\mathbf{E}}}{\partial t^2} \tag{42}$$

$$\nabla \times \nabla \times \vec{\mathbf{E}} = \nabla(\nabla \cdot \vec{\mathbf{E}}) - \nabla^2 \vec{\mathbf{E}} \tag{43}$$

$$\nabla \cdot \vec{\mathbf{E}} = 0 \tag{44}$$

$$\nabla \times \nabla \times \vec{\mathbf{E}} = -\nabla^2 \vec{\mathbf{E}} \tag{45}$$

$$-\nabla^2 \vec{\mathbf{E}} = -\mu\epsilon \frac{\partial^2 \vec{\mathbf{E}}}{\partial t^2} \tag{46}$$

$$\nabla^2 \vec{\mathbf{E}} - \mu\epsilon \frac{\partial^2 \vec{\mathbf{E}}}{\partial t^2} = 0 \tag{47}$$

$$c = \frac{1}{\sqrt{\mu\epsilon}} \tag{48}$$

$$\boxed{\nabla^2 \vec{\mathbf{E}} - \frac{1}{c^2}\frac{\partial^2 \vec{\mathbf{E}}}{\partial t^2} = 0} \tag{49}$$

Homogeneous vector wave equation

In a similar way

$$\boxed{\nabla^2 \vec{\mathbf{H}} - \frac{1}{c^2}\frac{\partial^2 \vec{\mathbf{H}}}{\partial t^2} = 0} \tag{50}$$

Homogeneous vector wave equation