2. Transverse Electromagnetic Waves (Enine Elektromanyetik Dalgalar)

$$\vec{\mathbf{E}} = \hat{\mathbf{a}}_x \, E_x \tag{1}$$

$$\vec{\mathbf{H}} = \hat{\mathbf{a}}_y \, H_y \tag{2}$$

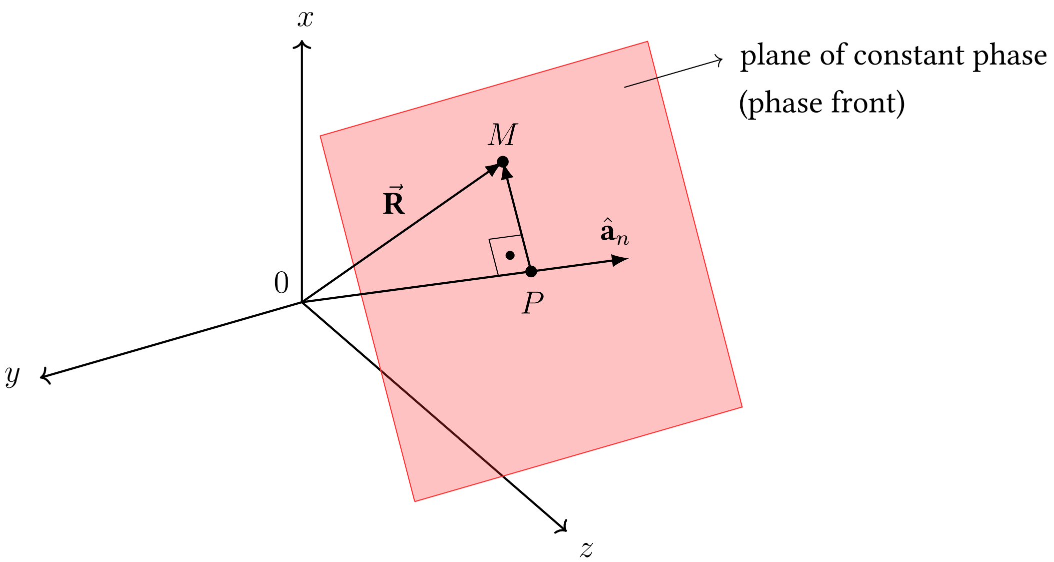
$\vec{\mathbf{E}}$ and $\vec{\mathbf{H}}$ are perpendicular to each other, and both are transverse to the direction of propagation. It is a particular case of a transverse electromagnetic (TEM) wave. The phasor form E_x and H_y are functions of only the distance z.

Now we consider the propagation of a uniform plane wave along an arbitrary direction that does not necessarily coincide with a coordinate axis.

The phasor electric field intensity for a uniform plane wave propagating in the $+z$ direction is

$$\vec{\mathbf{E}}(z) = \vec{\mathbf{E}}_0 \, e^{-jkz} \tag{3}$$

$\vec{\mathbf{E}}_0$: constant vector



The phasor electric field intensity for a uniform plane wave propagating in the $\hat{\mathbf{a}}_n$ direction is

$$\vec{\mathbf{E}} = \vec{\mathbf{E}}_0 \, e^{-jk|\overrightarrow{\mathbf{OP}}|} \tag{4}$$

$\hat{\mathbf{a}}_n$ is the unit vector in the direction of propagation. The position vector $\vec{\mathbf{R}}$ shows any point M on the phase front and it is given by

$$\vec{\mathbf{R}} = \hat{\mathbf{a}}_x \, x + \hat{\mathbf{a}}_y \, y + \hat{\mathbf{a}}_z \, z \tag{5}$$

The equation of the plane

$$\hat{\mathbf{a}}_n \cdot \overrightarrow{\mathbf{PM}} = 0 \tag{6}$$

$$\overrightarrow{\mathbf{PM}} - \vec{\mathbf{R}} + \overrightarrow{\mathbf{OP}} = 0 \tag{7}$$

$$\Rightarrow \overrightarrow{\mathbf{PM}} = \vec{\mathbf{R}} - \overrightarrow{\mathbf{OP}} = \vec{\mathbf{R}} - \hat{\mathbf{a}}_n\,|\overrightarrow{\mathbf{OP}}| \tag{8}$$

$$\hat{\mathbf{a}}_n \cdot \overrightarrow{\mathbf{PM}} = \hat{\mathbf{a}}_n \cdot (\vec{\mathbf{R}} - \hat{\mathbf{a}}_n\,|\overrightarrow{\mathbf{OP}}|) = 0 \tag{9}$$

$$\hat{\mathbf{a}}_n \cdot \vec{\mathbf{R}} - |\overrightarrow{\mathbf{OP}}| = 0 \tag{10}$$

$$\hat{\mathbf{a}}_n \cdot \vec{\mathbf{R}} = |\overrightarrow{\mathbf{OP}}| \tag{11}$$

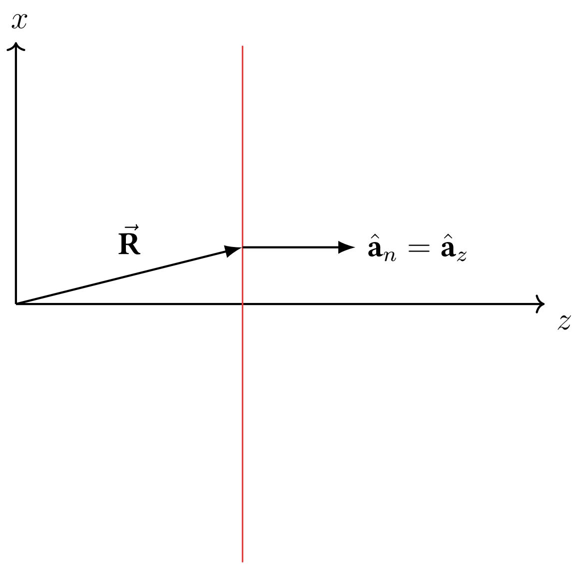
$$\vec{\mathbf{E}} = \vec{\mathbf{E}}_0\, e^{-jk|\overrightarrow{\mathbf{OP}}|} \tag{12}$$

$$\boxed{\vec{\mathbf{E}} = \vec{\mathbf{E}}_0\, e^{-jk\,\hat{\mathbf{a}}_n\cdot\vec{\mathbf{R}}} \quad (\mathrm{V/m})} \tag{13}$$

$\hat{\mathbf{a}}_n \cdot \vec{\mathbf{R}} =$ constant is a plane of constant phase and uniform amplitude for the wave

$$\vec{\mathbf{E}} = \vec{\mathbf{E}}_0\, e^{-jk\,\hat{\mathbf{a}}_n\cdot\vec{\mathbf{R}}} \tag{14}$$

As a special case let $\hat{\mathbf{a}}_n = \hat{\mathbf{a}}_z$.



$$\hat{\mathbf{a}}_n = \hat{\mathbf{a}}_z \tag{15}$$

$$\vec{\mathbf{R}} = \hat{\mathbf{a}}_x\,x + \hat{\mathbf{a}}_y\,y + \hat{\mathbf{a}}_z\,z \tag{16}$$

$$\hat{\mathbf{a}}_n \cdot \vec{\mathbf{R}} = z \tag{17}$$

$$\vec{\mathbf{E}} = \vec{\mathbf{E}}_0\, e^{-jk\,\hat{\mathbf{a}}_n\cdot\vec{\mathbf{R}}} = \vec{\mathbf{E}}_0\, e^{-jkz} \tag{18}$$

$$\vec{\mathbf{E}} = \vec{\mathbf{E}}_0 \, e^{-jk\,\hat{\mathbf{a}}_n \cdot \vec{\mathbf{R}}} \tag{19}$$

$$\vec{\mathbf{k}} = k\,\hat{\mathbf{a}}_n = \hat{\mathbf{a}}_x k_x + \hat{\mathbf{a}}_y k_y + \hat{\mathbf{a}}_z k_z \tag{20}$$

$\vec{\mathbf{k}}$: wavenumber vector

$$\vec{\mathbf{k}} \cdot \vec{\mathbf{R}} = (\hat{\mathbf{a}}_x k_x + \hat{\mathbf{a}}_y k_y + \hat{\mathbf{a}}_z k_z) \cdot (\hat{\mathbf{a}}_x\, x + \hat{\mathbf{a}}_y\, y + \hat{\mathbf{a}}_z\, z) \tag{21}$$

$$\vec{\mathbf{k}} \cdot \vec{\mathbf{R}} = k_x x + k_y y + k_z z \tag{22}$$

$$\vec{\mathbf{E}} = \vec{\mathbf{E}}_0 \, e^{-j\,\vec{\mathbf{k}} \cdot \vec{\mathbf{R}}} = \vec{\mathbf{E}}_0 \, e^{-j(k_x x + k_y y + k_z z)} = \vec{\mathbf{E}}_0 \, e^{-jk_x x} \, e^{-jk_y y} \, e^{-jk_z z} \tag{23}$$

Let’s substitute this expression into the homogeneous Helmholtz’s equation:

$$\nabla^2 \vec{\mathbf{E}} + k^2 \vec{\mathbf{E}} = 0 \tag{24}$$

$$\vec{\mathbf{E}} = \vec{\mathbf{E}}_0 \, e^{-jk_x x} \, e^{-jk_y y} \, e^{-jk_z z} \tag{25}$$

$$\psi = e^{-jk_x x} \, e^{-jk_y y} \, e^{-jk_z z} \tag{26}$$

$$\vec{\mathbf{E}} = \vec{\mathbf{E}}_0 \, e^{-jk_x x} \, e^{-jk_y y} \, e^{-jk_z z} = \vec{\mathbf{E}}_0 \, \psi \tag{27}$$

$$\nabla^2 \vec{\mathbf{E}} = \nabla^2 \, (\vec{\mathbf{E}}_0 \, \psi) = \vec{\mathbf{E}}_0 \, \nabla^2 \psi \tag{28}$$

$$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} \tag{29}$$

$$\frac{\partial \psi}{\partial x} = \frac{\partial (e^{-jk_x x} \, e^{-jk_y y} \, e^{-jk_z z})}{\partial x} = -jk_x (e^{-jk_x x} \, e^{-jk_y y} \, e^{-jk_z z}) \tag{30}$$

$$\frac{\partial \psi}{\partial x} = -jk_x \psi \tag{31}$$

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial \psi}{\partial x} \right) \tag{32}$$

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{\partial}{\partial x} \, (-jk_x \psi) \tag{33}$$

$$\frac{\partial^2 \psi}{\partial x^2} = -jk_x \frac{\partial \psi}{\partial x} = -jk_x (-jk_x \psi) = -k_x^2 \psi \tag{34}$$

In a similar way

$$\frac{\partial^2 \psi}{\partial y^2} = -k_y^2 \psi \tag{35}$$

$$\frac{\partial^2 \psi}{\partial z^2} = -k_z^2 \psi \tag{36}$$

$$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = -k_x^2 \psi - k_y^2 \psi - k_z^2 \psi \tag{37}$$

$$\nabla^2 \psi = (-k_x^2 - k_y^2 - k_z^2) \, \psi \tag{38}$$

$$\nabla^2 \vec{\mathbf{E}} = \vec{\mathbf{E}}_0 \nabla^2 \psi = \vec{\mathbf{E}}_0 (-k_x^2 - k_y^2 - k_z^2)\, \psi \tag{39}$$

$$\nabla^2 \vec{\mathbf{E}} + k^2 \vec{\mathbf{E}} = \vec{\mathbf{E}}_0 (-k_x^2 - k_y^2 - k_z^2)\, \psi + k^2 \vec{\mathbf{E}}_0\, \psi = 0 \tag{40}$$

$$\Rightarrow \boxed{k_x^2 + k_y^2 + k_z^2 = k^2 = \omega^2 \mu \epsilon} \tag{41}$$

In a charge free region

$$\nabla \cdot \vec{\mathbf{E}} = 0 \tag{42}$$

$$\vec{\mathbf{E}} = \vec{\mathbf{E}}_0\, e^{-jk\, \hat{\mathbf{a}}_n \cdot \vec{\mathbf{R}}} \tag{43}$$

$$\nabla \cdot \vec{\mathbf{E}} = \nabla \cdot \left(\vec{\mathbf{E}}_0\, e^{-jk\, \hat{\mathbf{a}}_n \cdot \vec{\mathbf{R}}} \right) \tag{44}$$

$$\nabla \cdot \left(\psi \vec{\mathbf{A}} \right) = \psi\, \nabla \cdot \vec{\mathbf{A}} + \vec{\mathbf{A}} \cdot \nabla \psi \tag{45}$$

$$\vec{\mathbf{A}} \rightarrow \vec{\mathbf{E}}_0 \tag{46}$$

$$\psi \rightarrow e^{-jk\, \hat{\mathbf{a}}_n \cdot \vec{\mathbf{R}}} \tag{47}$$

$$\nabla \cdot \vec{\mathbf{E}} = \nabla \cdot \left(\vec{\mathbf{E}}_0\, e^{-jk\, \hat{\mathbf{a}}_n \cdot \vec{\mathbf{R}}} \right) \tag{48}$$

$$\nabla \cdot \vec{\mathbf{E}} = e^{-jk\, \hat{\mathbf{a}}_n \cdot \vec{\mathbf{R}}}\, \nabla \cdot \vec{\mathbf{E}}_0 + \vec{\mathbf{E}}_0 \cdot \nabla e^{-jk\, \hat{\mathbf{a}}_n \cdot \vec{\mathbf{R}}} \tag{49}$$

$$\nabla \cdot \vec{\mathbf{E}}_0 = 0 \quad (\vec{\mathbf{E}}_0 = \text{constant vector}) \tag{50}$$

$$\nabla \cdot \vec{\mathbf{E}} = \vec{\mathbf{E}}_0 \cdot \nabla e^{-jk\, \hat{\mathbf{a}}_n \cdot \vec{\mathbf{R}}} = \vec{\mathbf{E}}_0 \cdot \nabla \psi \tag{51}$$

$$\psi = e^{-jk\, \hat{\mathbf{a}}_n \cdot \vec{\mathbf{R}}} = e^{-jk_x x}\, e^{-jk_y y}\, e^{-jk_z z} \tag{52}$$

$$\nabla \psi = \hat{\mathbf{a}}_x \frac{\partial \psi}{\partial x} + \hat{\mathbf{a}}_y \frac{\partial \psi}{\partial y} + \hat{\mathbf{a}}_z \frac{\partial \psi}{\partial z} \tag{53}$$

$$\frac{\partial \psi}{\partial x} = -jk_x \psi \tag{54}$$

$$\frac{\partial \psi}{\partial y} = -jk_y \psi \tag{55}$$

$$\frac{\partial \psi}{\partial z} = -jk_z \psi \tag{56}$$

$$\nabla \psi = \hat{\mathbf{a}}_x (-jk_x) \psi + \hat{\mathbf{a}}_y (-jk_y) \psi + \hat{\mathbf{a}}_z (-jk_z) \psi \tag{57}$$

$$\nabla \psi = -j (\hat{\mathbf{a}}_x k_x + \hat{\mathbf{a}}_y k_y + \hat{\mathbf{a}}_z k_z) \psi \tag{58}$$

$$\hat{\mathbf{a}}_x k_x + \hat{\mathbf{a}}_y k_y + \hat{\mathbf{a}}_z k_z = \vec{\mathbf{k}} = k\, \hat{\mathbf{a}}_n \tag{59}$$

$$k = \sqrt{k_x^2 + k_y^2 + k_z^2} \tag{60}$$

$$\nabla \psi = -jk\, \hat{\mathbf{a}}_n \psi \tag{61}$$

$$\nabla \cdot \vec{\mathbf{E}} = \vec{\mathbf{E}}_0 \cdot \nabla \psi = \vec{\mathbf{E}}_0 \cdot (-jk\, \hat{\mathbf{a}}_n)\, \psi \tag{62}$$

$$\nabla \cdot \vec{\mathbf{E}} = \vec{\mathbf{E}}_0 \cdot (-jk\, \hat{\mathbf{a}}_n)\, e^{-jk\, \hat{\mathbf{a}}_n \cdot \vec{\mathbf{R}}} = 0 \tag{63}$$

$$\Rightarrow \boxed{\hat{\mathbf{a}}_n \cdot \vec{\mathbf{E}}_0 = 0} \tag{64}$$

$\vec{\mathbf{E}}_0$ is transverse to the direction of propagation. Now, let's find $\vec{\mathbf{H}}$.

$$\nabla \times \vec{\mathbf{E}} = -j\omega\mu\,\vec{\mathbf{H}} \tag{65}$$

$$\vec{\mathbf{H}} = -\frac{1}{j\omega\mu}\,\nabla \times \vec{\mathbf{E}} \tag{66}$$

$$\nabla \times \vec{\mathbf{E}} = \nabla \times \left(\vec{\mathbf{E}}_0\, e^{-jk\,\hat{\mathbf{a}}_n \cdot \vec{\mathbf{R}}}\right) \tag{67}$$

$$\nabla \times \left(\psi\vec{\mathbf{A}}\right) = \psi\,\nabla \times \vec{\mathbf{A}} + \nabla\psi \times \vec{\mathbf{A}} \tag{68}$$

$$\vec{\mathbf{A}} \to \vec{\mathbf{E}}_0 \tag{69}$$

$$\psi \to e^{-jk\,\hat{\mathbf{a}}_n \cdot \vec{\mathbf{R}}} \tag{70}$$

$$\nabla \times \left(\vec{\mathbf{E}}_0\, e^{-jk\,\hat{\mathbf{a}}_n \cdot \vec{\mathbf{R}}}\right) = e^{-jk\,\hat{\mathbf{a}}_n \cdot \vec{\mathbf{R}}}\,\nabla \times \vec{\mathbf{E}}_0 + \left(\nabla e^{-jk\,\hat{\mathbf{a}}_n \cdot \vec{\mathbf{R}}}\right) \times \vec{\mathbf{E}}_0 \tag{71}$$

$$\nabla \times \vec{\mathbf{E}}_0 = 0 \quad (\vec{\mathbf{E}}_0 = \text{constant vector}) \tag{72}$$

$$\nabla \times \vec{\mathbf{E}} = \left(\nabla e^{-jk\,\hat{\mathbf{a}}_n \cdot \vec{\mathbf{R}}}\right) \times \vec{\mathbf{E}}_0 \tag{73}$$

$$\nabla e^{-jk\,\hat{\mathbf{a}}_n \cdot \vec{\mathbf{R}}} = \nabla\psi = -jk\, e^{-jk\,\hat{\mathbf{a}}_n \cdot \vec{\mathbf{R}}}\,\hat{\mathbf{a}}_n \tag{74}$$

$$\nabla \times \vec{\mathbf{E}} = -jk\, e^{-jk\,\hat{\mathbf{a}}_n \cdot \vec{\mathbf{R}}}\,\hat{\mathbf{a}}_n \times \vec{\mathbf{E}}_0 \tag{75}$$

$$\nabla \times \vec{\mathbf{E}} = -jk\,\hat{\mathbf{a}}_n \times \vec{\mathbf{E}}_0\, e^{-jk\,\hat{\mathbf{a}}_n \cdot \vec{\mathbf{R}}} \tag{76}$$

$$\nabla \times \vec{\mathbf{E}} = -jk\,\hat{\mathbf{a}}_n \times \vec{\mathbf{E}} \tag{77}$$

$$\vec{\mathbf{H}} = -\frac{1}{j\omega\mu}\,\nabla \times \vec{\mathbf{E}} = -\frac{1}{j\omega\mu}\,(-jk)\,\hat{\mathbf{a}}_n \times \vec{\mathbf{E}} \tag{78}$$

$$\vec{\mathbf{H}} = \frac{k}{\omega\mu}\,\hat{\mathbf{a}}_n \times \vec{\mathbf{E}} \tag{79}$$

$$\frac{k}{\omega\mu} = \frac{\frac{\omega}{c}}{\omega\mu} = \frac{1}{c\mu} = \frac{\sqrt{\mu\epsilon}}{\mu} = \sqrt{\frac{\epsilon}{\mu}} = \frac{1}{\eta} \tag{80}$$

$$\boxed{\vec{\mathbf{H}} = \frac{1}{\eta}\,\hat{\mathbf{a}}_n \times \vec{\mathbf{E}} \qquad (\mathrm{A/m})} \tag{81}$$

$$\eta = \frac{\omega\mu}{k} = \sqrt{\frac{\mu}{\epsilon}} \quad (\Omega) \tag{82}$$

η : intrinsic impedance of the medium (wave impedance; karakteristik empedans)

A uniform plane wave propagating in an arbitrary direction $\hat{\mathbf{a}}_n$, is a transverse electromagnetic (TEM) wave with $\vec{\mathbf{E}} \perp \vec{\mathbf{H}}$ and that both $\vec{\mathbf{E}}$ and $\vec{\mathbf{H}}$ are normal to $\hat{\mathbf{a}}_n$.