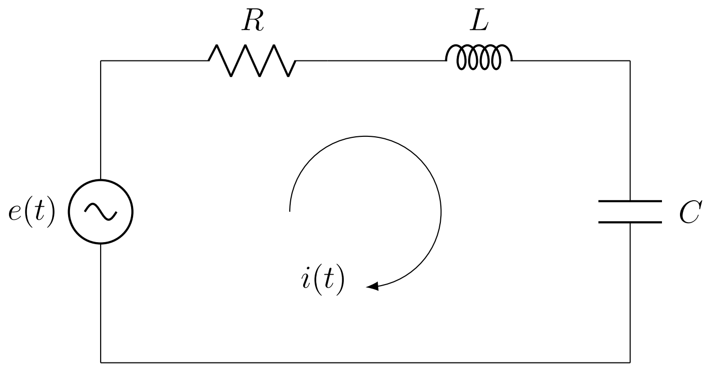
7. Time-Harmonic Fields

7.1 The Use of Phasors

For time harmonic (steady-state sinusoidal) fields it is convenient to use a phasor notation.



$$R\,i + L\,\frac{di}{dt} + \frac{1}{C}\int i\,dt = e(t) \tag{1}$$

$$e(t) = E\cos\omega t = \text{Re}\left[E\,e^{j\omega t}\right] \qquad \text{(peak value)} \tag{2}$$

$$i(t) = I\cos(\omega t + \phi) = \text{Re}\left[I\,e^{j\phi}\,e^{j\omega t}\right] \qquad \text{(peak value)} \tag{3}$$

Re : the real part of

$$\frac{di(t)}{dt} = \text{Re}\left[I\,e^{j\phi}\,\frac{d}{dt}\,e^{j\omega t}\right] = \text{Re}\left[j\omega I\,e^{j\phi}\,e^{j\omega t}\right] \tag{4}$$

$$\int i(t)\,dt = \text{Re}\left[\int I\,e^{j\phi}\,e^{j\omega t}\,dt\right] = \text{Re}\left[\frac{1}{j\omega}\,I\,e^{j\phi}\,e^{j\omega t}\right] \tag{5}$$

$$R\,\text{Re}\left[I\,e^{j\phi}\,e^{j\omega t}\right] + L\,\text{Re}\left[j\omega I\,e^{j\phi}\,e^{j\omega t}\right] + \frac{1}{C}\,\text{Re}\left[\frac{1}{j\omega}\,I\,e^{j\phi}\,e^{j\omega t}\right] = \text{Re}\left[E\,e^{j\omega t}\right] \tag{6}$$

$$\left(R + j\omega L + \frac{1}{j\omega C}\right)\,I\,e^{j\phi} = E \tag{7}$$

Example 6:
Express $3\cos\omega t - 4\sin\omega t$ as first a) $A_1\ \cos(\omega t+\theta_1)$; b) $A_2\ \sin(\omega t+\theta_2)$. Determine A_1, θ_1, A_2 and θ_2.

Solution
a)

$$3\cos\omega t = \text{Re}\left[3\,e^{j0^\circ}\ e^{j\omega t}\right] \tag{8}$$

$$\cos(x+y) = \cos x\ \cos y - \sin x\ \sin y \tag{9}$$

$$\cos(x-y) = \cos x\ \cos y + \sin x\ \sin y \tag{10}$$

$$\cos(x-90^\circ) = \cos x\ \cos(90^\circ) + \sin x\ \sin(90^\circ) \tag{11}$$

$$\cos(x-90^\circ) = (\cos x)\,(0) + (\sin x)\,(1) = \sin x \tag{12}$$

$$\sin\omega t = \cos(\omega t - 90^\circ) = \text{Re}\left[e^{-j90^\circ}\ e^{j\omega t}\right] \tag{13}$$

$$-4\ \sin\omega t = -4\ \cos(\omega t - 90^\circ) = \text{Re}\left[4\,e^{j180^\circ}\ e^{-j90^\circ}\ e^{j\omega t}\right] = \text{Re}\left[4\,e^{j90^\circ}\ e^{j\omega t}\right] \tag{14}$$

$$3\cos\omega t - 4\sin\omega t = \text{Re}\left[3\,e^{j0^\circ}\ e^{j\omega t}\right] + \text{Re}\left[4\,e^{j90^\circ}\ e^{j\omega t}\right] \tag{15}$$

$$3\cos\omega t - 4\sin\omega t = \text{Re}\left[\left(3\,e^{j0^\circ} + 4\,e^{j90^\circ}\right)\ e^{j\omega t}\right] \tag{16}$$

$$3\cos\omega t - 4\sin\omega t = \text{Re}\left[(3+j4)\ e^{j\omega t}\right] \tag{17}$$

$$3\cos\omega t - 4\sin\omega t = \text{Re}\left[\left(5\,e^{j\tan^{-1}(4/3)}\right)\ e^{j\omega t}\right] \tag{18}$$

$$3\cos\omega t - 4\sin\omega t = \text{Re}\left[\left(5\,e^{j53.13^\circ}\right)\ e^{j\omega t}\right] \tag{19}$$

$$3\cos\omega t - 4\sin\omega t = 5\cos(\omega t + 53.13^\circ) \tag{20}$$

$A_1 = 5, \theta_1 = 53.13^\circ$

b)

$$3\cos\omega t - 4\sin\omega t = A_2\ \sin(\omega t + \theta_2) \tag{21}$$

$$\sin\omega t \to 1\angle 0^\circ \tag{22}$$

$$-4\sin\omega t \to 4\angle 180^\circ \tag{23}$$

$$\sin(x+y) = \sin x\ \cos y + \cos x\ \sin y \tag{24}$$

$$\sin(x+90^\circ) = \sin x\ \cos(90^\circ) + \cos x\ \sin(90^\circ) = \cos x \tag{25}$$

$$\cos\omega t = \sin(\omega t + 90^\circ) \to 1\angle 90^\circ \tag{26}$$

$$3\cos\omega t \to 3\angle 90^\circ \tag{27}$$

$$3\angle 90^\circ + 4\angle 180^\circ = j3 - 4 = -4 + j3 = 5\angle(-\tan^{-1} 3/4) = 5\angle 143.13^\circ \tag{28}$$

$$3\cos\omega t - 4\sin\omega t = 5\ \sin(\omega t + 143.13^\circ) \tag{29}$$

$A_2 = 5, \theta_2 = 143.13^\circ$

7.2 Time-Harmonic Electromagnetics

We can write a time-harmonic $\vec{\mathbf{E}}$-field as

$$\vec{\mathbf{E}}(x,y,z,t) = \text{Re}\left[\vec{\mathbf{E}}(x,y,z)\, e^{j\omega t}\right] \qquad \text{(peak value)} \tag{30}$$

where $\vec{\mathbf{E}}(x,y,z)$ is a vector phasor that contains information on direction, magnitude and phase.

$$\frac{d\vec{\mathbf{E}}}{dt} = \text{Re}\left[\vec{\mathbf{E}}(x,y,z)\,\frac{d}{dt}\,e^{j\omega t}\right] = \text{Re}\left[j\omega\,\vec{\mathbf{E}}(x,y,z)\,e^{j\omega t}\right] \tag{31}$$

$$\frac{d\vec{\mathbf{E}}}{dt} \rightarrow j\omega\vec{\mathbf{E}} \tag{32}$$

$$\frac{d^2\vec{\mathbf{E}}}{dt^2} = \text{Re}\left[\vec{\mathbf{E}}(x,y,z)\,\frac{d^2}{dt^2}\,e^{j\omega t}\right] = \text{Re}\left[-\omega^2\,\vec{\mathbf{E}}(x,y,z)\,e^{j\omega t}\right] \tag{33}$$

$$\frac{d^2\vec{\mathbf{E}}}{dt^2} \rightarrow -\omega^2\vec{\mathbf{E}} \tag{34}$$

$$\int \vec{\mathbf{E}}\,dt = \int \text{Re}\left[\vec{\mathbf{E}}(x,y,z)\,e^{j\omega t}dt\right] = \text{Re}\left[\vec{\mathbf{E}}(x,y,z)\int e^{j\omega t}\,dt\right] = \text{Re}\left[\frac{\vec{\mathbf{E}}(x,y,z)}{j\omega}\,e^{j\omega t}\right] \tag{35}$$

$$\int \vec{\mathbf{E}}\,dt \rightarrow \frac{\vec{\mathbf{E}}}{j\omega} \tag{36}$$

We can write time-harmonic Maxwell's equations in terms of vector field phasors ($\vec{\mathbf{E}}, \vec{\mathbf{H}}$) and source phasors ($\rho, \vec{\mathbf{J}}$) in a simple (linear, isotropic, and homogeneous) medium as follows:

$$\nabla\times\vec{\mathbf{E}} = -\frac{\partial\vec{\mathbf{B}}}{\partial t} \qquad \rightarrow \nabla\times\vec{\mathbf{E}} = -j\omega\,\vec{\mathbf{B}} \qquad \Rightarrow \nabla\times\vec{\mathbf{E}} = -j\omega\mu\,\vec{\mathbf{H}} \tag{37}$$

$$\nabla\times\vec{\mathbf{H}} = \vec{\mathbf{J}} + \frac{\partial\vec{\mathbf{D}}}{\partial t} \qquad \rightarrow \nabla\times\vec{\mathbf{H}} = \vec{\mathbf{J}} + j\omega\,\vec{\mathbf{D}} \qquad \Rightarrow \nabla\times\vec{\mathbf{H}} = \vec{\mathbf{J}} + j\omega\epsilon\,\vec{\mathbf{E}} \tag{38}$$

$$\nabla\cdot\vec{\mathbf{D}} = \rho \qquad \rightarrow \nabla\cdot\vec{\mathbf{D}} = \rho \qquad \Rightarrow \nabla\cdot\vec{\mathbf{E}} = \frac{\rho}{\epsilon} \tag{39}$$

$$\nabla\cdot\vec{\mathbf{B}} = 0 \qquad \rightarrow \nabla\cdot\vec{\mathbf{B}} = 0 \qquad \Rightarrow \nabla\cdot\vec{\mathbf{H}} = 0 \tag{40}$$

$$\boxed{\nabla\times\vec{\mathbf{E}} = -j\omega\mu\,\vec{\mathbf{H}}} \tag{41}$$

$$\boxed{\nabla\times\vec{\mathbf{H}} = \vec{\mathbf{J}} + j\omega\epsilon\,\vec{\mathbf{E}}} \tag{42}$$

$$\boxed{\nabla\cdot\vec{\mathbf{E}} = \frac{\rho}{\epsilon}} \tag{43}$$

$$\boxed{\nabla\cdot\vec{\mathbf{H}} = 0} \tag{44}$$

$$\nabla^2 \vec{\mathbf{A}} - \mu\epsilon \frac{\partial^2 \vec{\mathbf{A}}}{\partial t^2} = -\mu \vec{\mathbf{J}} \tag{45}$$

Nonhomogeneous wave equation for vector potential $\vec{\mathbf{A}}$.

$$\Rightarrow \nabla^2 \vec{\mathbf{A}} - \mu\epsilon\,(-\omega^2 \vec{\mathbf{A}}) = -\mu \vec{\mathbf{J}} \tag{46}$$

$$\Rightarrow \nabla^2 \vec{\mathbf{A}} + \mu\epsilon\,\omega^2 \vec{\mathbf{A}} = -\mu \vec{\mathbf{J}} \tag{47}$$

$$k^2 = \omega^2 \mu\epsilon \tag{48}$$

$$k = \omega\sqrt{\mu\epsilon} = \frac{\omega}{c} \tag{49}$$

k : wavenumber

$$\boxed{\nabla^2 \vec{\mathbf{A}} + k^2 \vec{\mathbf{A}} = -\mu \vec{\mathbf{J}}} \tag{50}$$

Nonhomogeneous Helmholtz equation.

$$\nabla^2 V - \mu\epsilon \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon} \tag{51}$$

Nonhomogeneous wave equation for scalar potential V.

$$\Rightarrow \nabla^2 V - \mu\epsilon\,(-\omega^2 V) = -\frac{\rho}{\epsilon} \tag{52}$$

$$\Rightarrow \nabla^2 V + \mu\epsilon\,\omega^2 V = -\frac{\rho}{\epsilon} \tag{53}$$

$$\boxed{\nabla^2 V + k^2 V = -\frac{\rho}{\epsilon}} \tag{54}$$

Nonhomogeneous Helmholtz equation.

Lorentz condition

$$\nabla \cdot \vec{\mathbf{A}} + \mu\epsilon \frac{\partial V}{\partial t} = 0 \qquad \rightarrow \nabla \cdot \vec{\mathbf{A}} + j\omega\mu\epsilon\, V = 0 \tag{55}$$

The phasor solution of nonhomogeneous Helmholtz equation

$$\vec{\mathbf{A}}(R,t) = \frac{\mu}{4\pi} \int_{V'} \frac{\vec{\mathbf{J}}\left(t - \frac{R}{c}\right)}{R}\, dv' \tag{56}$$

$$\vec{\mathbf{J}}(x,y,z,t) = \mathrm{Re}\left[\vec{\mathbf{J}}(x,y,z)\, e^{j\omega t}\right] \tag{57}$$

$$\vec{\mathbf{J}}(x,y,z,t - R/c) = \mathrm{Re}\left[\vec{\mathbf{J}}(x,y,z)\, e^{j\omega(t-R/c)}\right] \tag{58}$$

$$e^{j\omega(t-R/c)} = e^{j\omega t}\, e^{-j\omega R/c} = e^{j\omega t}\, e^{-jkR} \tag{59}$$

$$k = \omega/c \tag{60}$$

$$\vec{\mathbf{J}}(x,y,z,t-R/c) = \mathrm{Re}\left[\vec{\mathbf{J}}(x,y,z)\, e^{-jkR}\, e^{j\omega t}\right] \tag{61}$$

$$\vec{\mathbf{J}}(t-R/c) \to \vec{\mathbf{J}}\, e^{-jkR} \tag{62}$$

$$\boxed{\vec{\mathbf{A}}(R) = \frac{\mu}{4\pi}\int_{V'} \frac{\vec{\mathbf{J}}\, e^{-jkR}}{R}\, dv' \qquad (\mathrm{Wb/m})} \tag{63}$$

In a similar way

$$\boxed{V(R) = \frac{1}{4\pi\epsilon}\int_{V'} \frac{\rho\, e^{-jkR}}{R}\, dv' \qquad (\mathrm{V})} \tag{64}$$

The procedure for determining the electric and magnetic fields due to time harmonic charge and current distributions is as follows:
1) Find phasors $V(R)$ and $\vec{\mathbf{A}}(R)$.

$$V(R) = \frac{1}{4\pi\epsilon}\int_{V'} \frac{\rho\, e^{-jkR}}{R}\, dv' \tag{65}$$

$$\vec{\mathbf{A}}(R) = \frac{\mu}{4\pi}\int_{V'} \frac{\vec{\mathbf{J}}\, e^{-jkR}}{R}\, dv' \tag{66}$$

2) Find $\vec{\mathbf{E}}(R)$ and $\vec{\mathbf{B}}(R)$.

$$\vec{\mathbf{E}} = -\nabla V - \frac{\partial \vec{\mathbf{A}}}{\partial t} \to \vec{\mathbf{E}}(R) = -\nabla V - j\omega\, \vec{\mathbf{A}} \tag{67}$$

$$\vec{\mathbf{B}} = \nabla \times \vec{\mathbf{A}} \to \vec{\mathbf{B}}(R) = \nabla \times \vec{\mathbf{A}} \tag{68}$$

3) Find instantaneous $\vec{\mathbf{E}}(R,t)$ and $\vec{\mathbf{B}}(R,t)$.

$$\vec{\mathbf{E}}(R,t) = \mathrm{Re}\left[\vec{\mathbf{E}}(R)\, e^{j\omega t}\right] \tag{69}$$

$$\vec{\mathbf{B}}(R,t) = \mathrm{Re}\left[\vec{\mathbf{B}}(R)\, e^{j\omega t}\right] \tag{70}$$

7.3 Source-Free Fields in Simple Media

General Maxwell's equations are given as follows:

$$\boxed{\nabla \times \vec{\mathbf{E}} = -j\omega\mu\,\vec{\mathbf{H}}} \tag{71}$$

$$\boxed{\nabla \times \vec{\mathbf{H}} = \vec{\mathbf{J}} + j\omega\epsilon\,\vec{\mathbf{E}}} \tag{72}$$

$$\boxed{\nabla \cdot \vec{\mathbf{E}} = \frac{\rho}{\epsilon}} \tag{73}$$

$$\boxed{\nabla \cdot \vec{\mathbf{H}} = 0} \tag{74}$$

In a simple, nonconducting source-free medium ($\rho = 0, \vec{\mathbf{J}} = 0, \sigma = 0$) time-harmonic Maxwell's equations are

$$\nabla \times \vec{\mathbf{E}} = -j\omega\mu\,\vec{\mathbf{H}} \tag{75}$$

$$\nabla \times \vec{\mathbf{H}} = j\omega\epsilon\,\vec{\mathbf{E}} \tag{76}$$

$$\nabla \cdot \vec{\mathbf{E}} = 0 \tag{77}$$

$$\nabla \cdot \vec{\mathbf{H}} = 0 \tag{78}$$

$$\nabla \times \vec{\mathbf{E}} = -j\omega\mu\,\vec{\mathbf{H}} \tag{79}$$

$$\nabla \times \nabla \times \vec{\mathbf{E}} = -j\omega\mu\,\nabla \times \vec{\mathbf{H}} = -j\omega\mu\,(j\omega\epsilon\,\vec{\mathbf{E}}) = \omega^2\mu\epsilon\vec{\mathbf{E}} \tag{80}$$

$$\nabla \times \nabla \times \vec{\mathbf{E}} - \omega^2\mu\epsilon\vec{\mathbf{E}} = 0 \tag{81}$$

$$\nabla \times \nabla \times \vec{\mathbf{E}} = \nabla(\nabla \cdot \vec{\mathbf{E}}) - \nabla^2\,\vec{\mathbf{E}} \tag{82}$$

$$\nabla \cdot \vec{\mathbf{E}} = 0 \tag{83}$$

$$-\nabla^2\,\vec{\mathbf{E}} - \omega^2\mu\epsilon\vec{\mathbf{E}} = 0 \tag{84}$$

$$\nabla^2\,\vec{\mathbf{E}} + \omega^2\mu\epsilon\vec{\mathbf{E}} = 0 \tag{85}$$

$$\boxed{\nabla^2\,\vec{\mathbf{E}} + k^2\vec{\mathbf{E}} = 0} \tag{86}$$

Homogeneous vector Helmholtz's equation

$$k = \omega\sqrt{\mu\epsilon} = \frac{\omega}{c} \tag{87}$$

In a similar way

$$\boxed{\nabla^2\,\vec{\mathbf{H}} + k^2\vec{\mathbf{H}} = 0} \tag{88}$$

Homogeneous vector Helmholtz's equation