PLANE ELECTROMAGNETIC WAVES

A uniform plane wave (*düzlemsel dalga*) is a particular solution of Maxwell's equations with $\vec{\mathbf{E}}$ ($\vec{\mathbf{H}}$) assuming the same direction, same magnitude, and same phase in infinite planes perpendicular to the direction of propagation.

1. Plane Waves in Lossless Media

We focus our attention on wave behavior in the sinusoidal steady state. We will investigate the solutions of the homogeneous vector Helmholtz's equation in free space.

$$\nabla^2 \vec{\mathbf{E}} + k_0^2 \vec{\mathbf{E}} = 0 \tag{1}$$

k_0 : free-space wavenumber

$$k_0 = \omega\sqrt{\mu_0\epsilon_0} = \frac{\omega}{c} \qquad \left(\frac{\text{rad}}{\text{m}}\right) \tag{2}$$

$$\vec{\mathbf{E}} = \hat{\mathbf{a}}_x\, E_x + \hat{\mathbf{a}}_y\, E_y + \hat{\mathbf{a}}_z\, E_z \tag{3}$$

$$\nabla^2 E_x + k_0^2 E_x = 0 \tag{4}$$

$$\nabla^2 E_y + k_0^2 E_y = 0 \tag{5}$$

$$\nabla^2 E_z + k_0^2 E_z = 0 \tag{6}$$

For E_x component we have

$$\nabla^2 E_x + k_0^2 E_x = 0 \tag{7}$$

$$\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} + k_0^2 E_x = 0 \tag{8}$$

Consider a uniform plane wave characterized by a uniform E_x (uniform magnitude and constant phase) over plane surfaces perpendicular to z; that is

$$\frac{\partial^2 E_x}{\partial x^2} = 0 \tag{9}$$

$$\frac{\partial^2 E_x}{\partial y^2} = 0 \tag{10}$$

In this case we have

$$\frac{d^2 E_x}{dz^2} + k_0^2 E_x = 0 \tag{11}$$

$$E_x(z) = E_x^+(z) + E_x^-(z) \tag{12}$$

$$E_x(z) = E_0^+ e^{-jk_0 z} + E_0^- e^{jk_0 z} \tag{13}$$

E_0^+, E_0^- : arbitrary complex constants

E_0^+ and E_0^- are determined by boundary conditions.

$$E_x^+(z,t) = \text{Re}\left[E_x^+(z)\, e^{j\omega t}\right] \tag{14}$$

$$E_x^+(z,t) = \text{Re}\left[E_0^+\, e^{-jk_0 z}\, e^{j\omega t}\right] \tag{15}$$

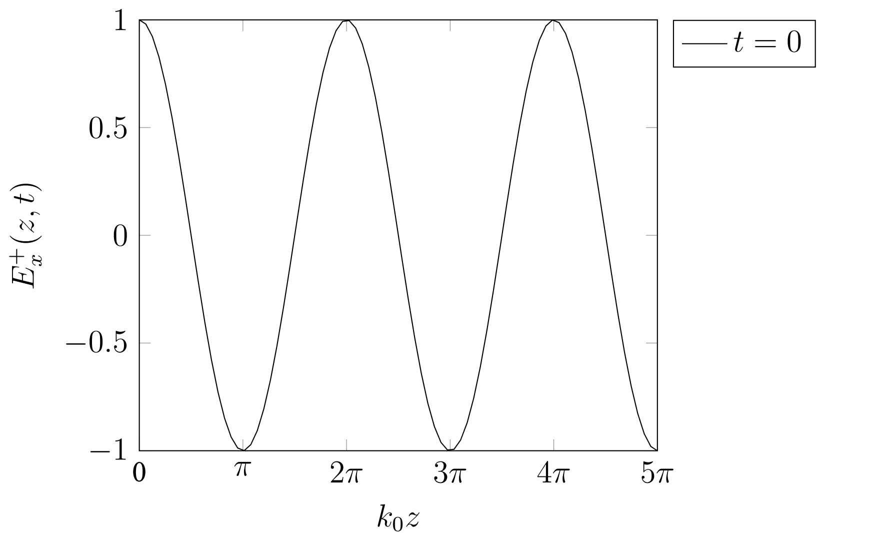
$$E_x^+(z,t) = E_0^+\, \cos(\omega t - k_0 z) \qquad \text{(V/m)} \tag{16}$$

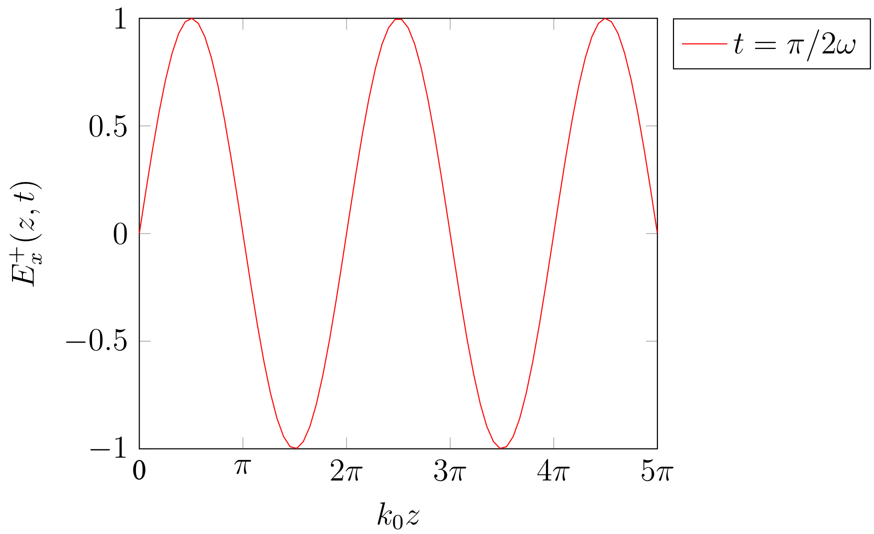
$$t = 0 \Rightarrow E_x^+(z,0) = E_0^+\, \cos(-k_0 z) = E_0^+\, \cos(k_0 z) \tag{17}$$

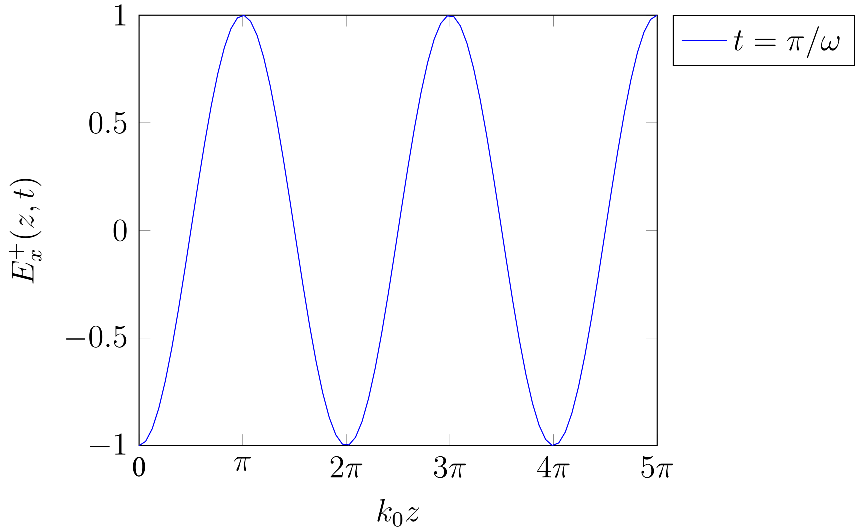
$$t = \frac{\pi}{2\,\omega} \Rightarrow E_x^+\left(z, \frac{\pi}{2\,\omega}\right) = E_0^+\, \cos\left(\frac{\pi}{2} - k_0 z\right) = E_0^+\, \sin(k_0 z) \tag{18}$$

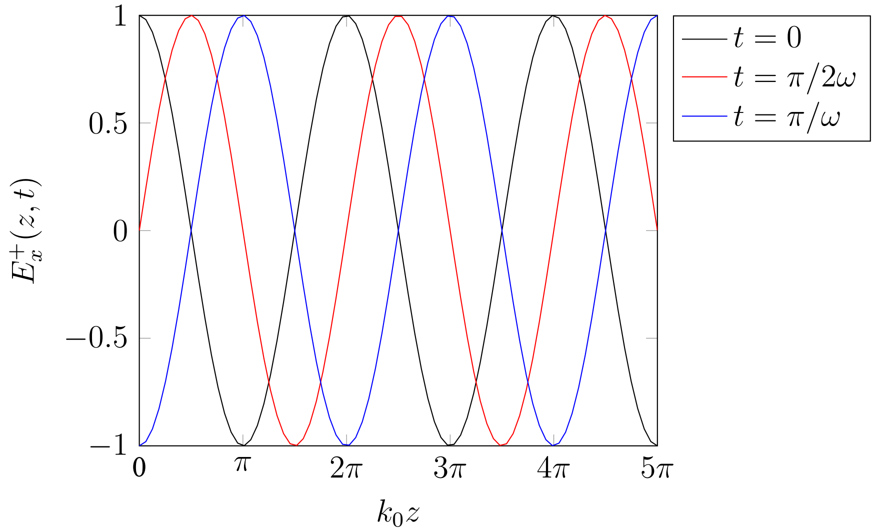
$$t = \frac{\pi}{\omega} \Rightarrow E_x^+\left(z, \frac{\pi}{\omega}\right) = E_0^+\, \cos(\pi - k_0 z) = -E_0^+\, \cos(k_0 z) \tag{19}$$

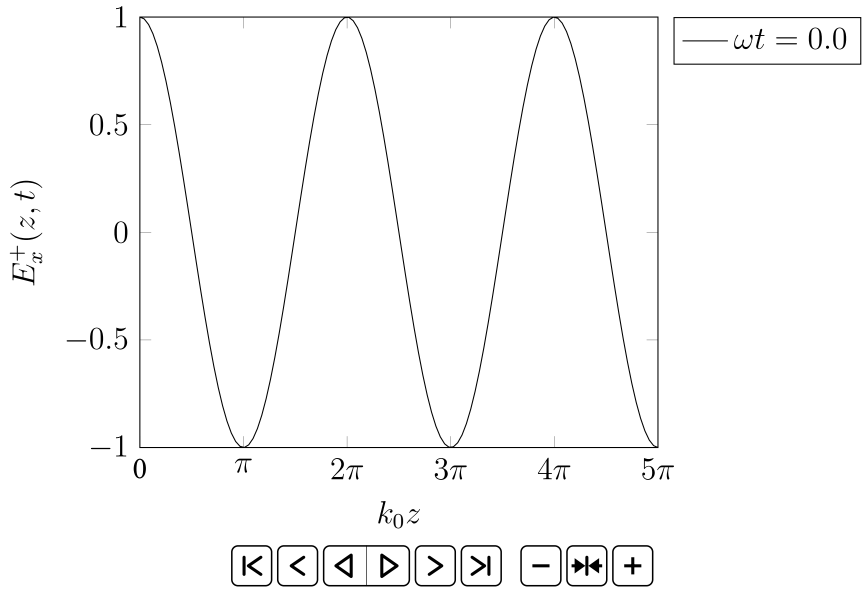
The curve travels in the $+z$ direction and we have a traveling wave.






$E_x^+(z,t)$
$k_0 z$
$t = \pi/\omega$


$E_x^+(z,t)$
$k_0 z$
$t = 0$
$t = \pi/2\omega$
$t = \pi/\omega$


$E_x^+(z,t)$
$k_0 z$
$\omega t = 0.0$

$$E_x^+(z,t) = E_0^+ \cos(\omega t - k_0 z) \tag{20}$$

Let's fix our attention on a particular point on the wave:

$$\cos(\omega t - k_0 z) = \text{constant} \tag{21}$$

$$\Rightarrow \omega t - k_0 z = \text{constant phase} \tag{22}$$

$$u_p = \frac{dz}{dt} \quad \text{(phase velocity)} \tag{23}$$

$$\frac{d}{dt}(\omega t - k_0 z) = 0 \tag{24}$$

$$\omega - k_0 \frac{dz}{dt} = 0 \tag{25}$$

$$\frac{dz}{dt} = \frac{\omega}{k_0} = \frac{\omega}{\frac{w}{c}} = c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \tag{26}$$

$$u_p = c \tag{27}$$

The velocity of propagation of equiphase front (the phase velocity) in free space is equal to the velocity of light. The term $E_0^- \, e^{jk_0 z}$ represents a cosinusoidal wave traveling in $-z$ direction with velocity c.

$$E_x^+(z) = E_0^+ \, e^{-jk_0 z} \tag{28}$$

Now let's find $\vec{\mathbf{H}}$.

$$\nabla \times \vec{\mathbf{E}} = -j\omega\mu_0 \, \vec{\mathbf{H}} \tag{29}$$

$$\nabla \times \vec{\mathbf{E}} = \begin{vmatrix} \hat{\mathbf{a}}_x & \hat{\mathbf{a}}_y & \hat{\mathbf{a}}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} \tag{30}$$

$$\nabla \times \vec{\mathbf{E}} = \hat{\mathbf{a}}_x \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right) - \hat{\mathbf{a}}_y \left(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} \right) + \hat{\mathbf{a}}_z \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) \tag{31}$$

$$E_x = E_x^+(z) = E_0^+ \, e^{-jk_0 z} \tag{32}$$

$$E_y = 0 \tag{33}$$

$$E_z = 0 \tag{34}$$

$$\frac{\partial E_x}{\partial y} = 0 \tag{35}$$

$$\nabla \times \vec{\mathbf{E}} = \hat{\mathbf{a}}_y \, \frac{\partial E_x}{\partial z} = \hat{\mathbf{a}}_y \, \frac{\partial E_x^+(z)}{\partial z} = \hat{\mathbf{a}}_y \, \frac{\partial}{\partial z} \left[E_0^+ \, e^{-jk_0 z} \right] \tag{36}$$

$$\nabla \times \vec{\mathbf{E}} = \hat{\mathbf{a}}_y \, (-jk_0) \, E_0^+ \, e^{-jk_0 z} = \hat{\mathbf{a}}_y \, (-jk_0) \, E_x^+(z) \tag{37}$$

$$\vec{\mathbf{H}} = \frac{1}{-j\omega\mu_0} \nabla \times \vec{\mathbf{E}} \tag{38}$$

$$\vec{\mathbf{H}} = \frac{1}{-j\omega\mu_0} \hat{\mathbf{a}}_y \left(-jk_0\right) E_x^+(z) \tag{39}$$

$$\vec{\mathbf{H}} = \hat{\mathbf{a}}_y H_y^+ \tag{40}$$

$$H_y^+(z) = \frac{k_0}{\omega\mu_0} E_x^+(z) \tag{41}$$

$$\frac{k_0}{\omega\mu_0} = \frac{\frac{\omega}{c}}{\omega\mu_0} = \frac{1}{c\mu_0} = \frac{\sqrt{\mu_0\epsilon_0}}{\mu_0} = \sqrt{\frac{\epsilon_0}{\mu_0}} = \frac{1}{\eta_0} \tag{42}$$

$$\eta_0 \triangleq \sqrt{\frac{\mu_0}{\epsilon_0}} = 120\pi \simeq 377\,(\Omega) \tag{43}$$

η_0 : the intrinsic impedance of the free space

$$H_y^+(z) = \frac{1}{\eta_0} E_x^+(z) \qquad \text{(A/m)} \tag{44}$$

$$H_y^+(z,t) = \text{Re}\left[H_y^+(z)\, e^{j\omega t}\right] \tag{45}$$

$$H_y^+(z,t) = \text{Re}\left[\frac{1}{\eta_0} E_0^+ e^{-jk_0 z} e^{j\omega t}\right] \tag{46}$$

$$H_y^+(z,t) = \frac{E_0^+}{\eta_0} \cos(\omega t - k_0 z) \qquad \text{(A/m)} \tag{47}$$

$$\vec{\mathbf{E}}(z,t) = \hat{\mathbf{a}}_x E_x^+ = \hat{\mathbf{a}}_x E_0^+ \cos(\omega t - k_0 z) \qquad \text{(V/m)} \tag{48}$$

$$\vec{\mathbf{H}}(z,t) = \hat{\mathbf{a}}_y H_y^+ = \hat{\mathbf{a}}_y \frac{E_0^+}{\eta_0} \cos(\omega t - k_0 z) \qquad \text{(A/m)} \tag{49}$$

For a uniform plane wave

$$\frac{|\vec{\mathbf{E}}|}{|\vec{\mathbf{H}}|} = \frac{|E_0^+ \cos(\omega t - k_0 z)|}{\frac{1}{\eta_0}|E_0^+ \cos(\omega t - k_0 z)|} = \eta_0 \tag{50}$$