# 3. Maxwell's Equations

## Differential form of Maxwell's equations

$$\boxed{\nabla \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t}} \tag{1}$$

$$\boxed{\nabla \times \vec{\mathbf{H}} = \vec{\mathbf{J}} + \frac{\partial \vec{\mathbf{D}}}{\partial t}} \tag{2}$$

$$\boxed{\nabla \cdot \vec{\mathbf{D}} = \rho} \tag{3}$$

$$\boxed{\nabla \cdot \vec{\mathbf{B}} = 0} \tag{4}$$

$$\nabla \cdot \vec{\mathbf{J}} = -\frac{\partial \rho}{\partial t} \qquad \text{(Equation of continuity; conservation of charge)} \tag{5}$$

$\vec{\mathbf{E}}$ : electric field intensity (V/m)

$\vec{\mathbf{B}}$ : magnetic flux density ($\mathrm{T} = \mathrm{Wb/m^2}$)

$\vec{\mathbf{H}}$ : magnetic field intensity (A/m)

$\vec{\mathbf{D}}$ : electric flux density ($\mathrm{C/m^2}$)

$\vec{\mathbf{J}}$ : current density ($\mathrm{A/m^2}$)

$\rho$ : volume charge density ($\mathrm{C/m^3}$)

$$\vec{\mathbf{F}} = q\,(\vec{\mathbf{E}} + \vec{\mathbf{u}} \times \vec{\mathbf{B}}) \qquad \text{(Lorentz's force equation)} \tag{6}$$

These four Maxwell's equations, together with the equation of continuity and Lorentz's force equation, form the foundation of electromagnetic theory. These equations can be used to explain and predict all *macroscopic* electromagnetic phenomena.

The term $\dfrac{\partial \vec{\mathbf{D}}}{\partial t}$ is called displacement current density (deplasman akımı; yer değiştirme akımı).

## Derivation of equation of continuity

$$\nabla \times \vec{\mathbf{H}} = \vec{\mathbf{J}} + \frac{\partial \vec{\mathbf{D}}}{\partial t} \tag{7}$$

$$\nabla \cdot (\nabla \times \vec{\mathbf{A}}) = 0 \qquad \text{(Identity II)} \tag{8}$$

$$\nabla \cdot (\nabla \times \vec{\mathbf{H}}) = \nabla \cdot \left( \vec{\mathbf{J}} + \frac{\partial \vec{\mathbf{D}}}{\partial t} \right) \tag{9}$$

$$\nabla \cdot (\nabla \times \vec{\mathbf{H}}) = 0 \tag{10}$$

$$\nabla \cdot \left( \vec{\mathbf{J}} + \frac{\partial \vec{\mathbf{D}}}{\partial t} \right) = 0 \tag{11}$$

$$\nabla \cdot \vec{\mathbf{J}} + \nabla \cdot \frac{\partial \vec{\mathbf{D}}}{\partial t} = 0 \tag{12}$$

$$\nabla \cdot \vec{\mathbf{J}} + \frac{\partial}{\partial t}(\nabla \cdot \vec{\mathbf{D}}) = 0 \tag{13}$$

$$\nabla \cdot \vec{\mathbf{J}} + \frac{\partial \rho}{\partial t} = 0 \tag{14}$$

$$\nabla \cdot \vec{\mathbf{J}} = -\frac{\partial \rho}{\partial t} \tag{15}$$

## Integral form of Maxwell's equations

$$\int_S (\nabla \times \vec{\mathbf{A}}) \cdot d\vec{\mathbf{s}} = \oint_C \vec{\mathbf{A}} \cdot d\vec{\ell} \qquad \text{(Stokes' theorem)} \tag{16}$$

1)

$$\nabla \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t} \tag{17}$$

$$\int_S (\nabla \times \vec{\mathbf{E}}) \cdot d\vec{\mathbf{s}} = -\int_S \frac{\partial \vec{\mathbf{B}}}{\partial t} \cdot d\vec{\mathbf{s}} \tag{18}$$

$$\boxed{\oint_C \vec{\mathbf{E}} \cdot d\vec{\ell} = -\int_S \frac{\partial \vec{\mathbf{B}}}{\partial t} \cdot d\vec{\mathbf{s}}} \qquad \text{(Faraday's law)} \tag{19}$$

$$\oint_C \vec{\mathbf{E}} \cdot d\vec{\ell} = -\frac{\partial}{\partial t} \int_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = -\frac{d\Phi}{dt} \tag{20}$$

2)

$$\nabla \times \vec{\mathbf{H}} = \vec{\mathbf{J}} + \frac{\partial \vec{\mathbf{D}}}{\partial t} \tag{21}$$

$$\int_S (\nabla \times \vec{\mathbf{H}}) \cdot d\vec{\mathbf{s}} = \int_S \left( \vec{\mathbf{J}} + \frac{\partial \vec{\mathbf{D}}}{\partial t} \right) \cdot d\vec{\mathbf{s}} \tag{22}$$

$$\boxed{\oint_C \vec{\mathbf{H}} \cdot d\vec{\ell} = \int_S \left( \vec{\mathbf{J}} + \frac{\partial \vec{\mathbf{D}}}{\partial t} \right) \cdot d\vec{\mathbf{s}}} \qquad \text{(Ampere's circuital law)} \tag{23}$$

$$\oint_C \vec{\mathbf{H}} \cdot d\vec{\ell} = \int_S \vec{\mathbf{J}} \cdot d\vec{\mathbf{s}} + \int_S \frac{\partial \vec{\mathbf{D}}}{\partial t} \cdot d\vec{\mathbf{s}} = I + \int_S \frac{\partial \vec{\mathbf{D}}}{\partial t} \cdot d\vec{\mathbf{s}} \tag{24}$$

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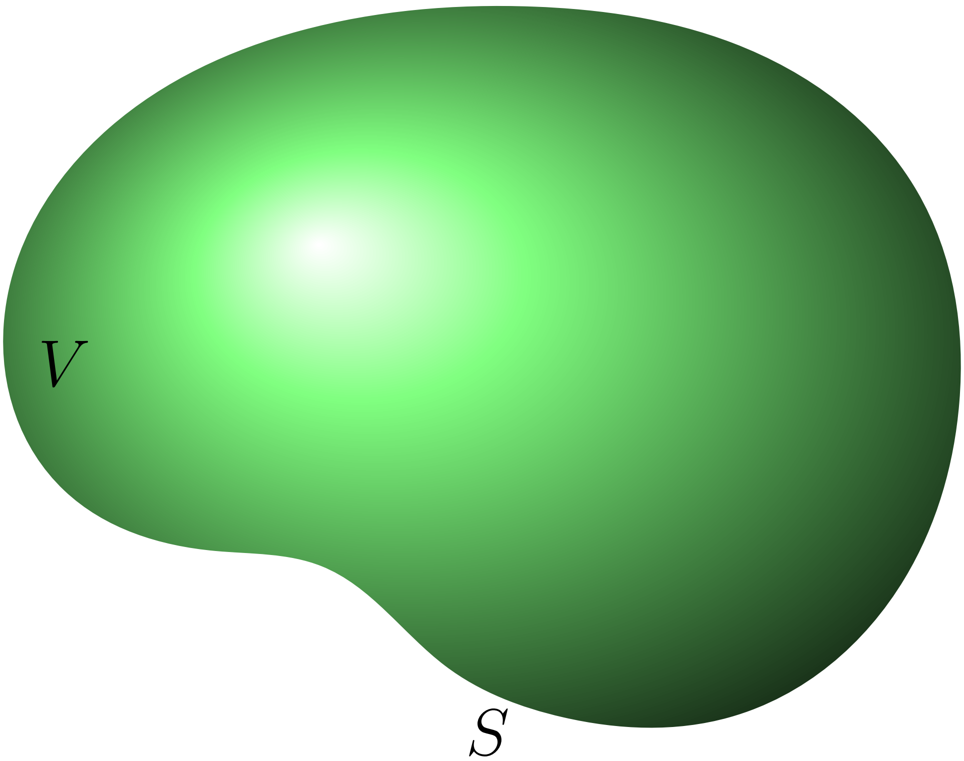
$$\int_V (\nabla \cdot \vec{\mathbf{A}})\, dv = \oint_S \vec{\mathbf{A}} \cdot d\vec{\mathbf{s}} \qquad \text{(Divergence theorem)} \tag{25}$$




3)

$$\nabla \cdot \vec{\mathbf{D}} = \rho \tag{26}$$

$$\int_V (\nabla \cdot \vec{\mathbf{D}})\, dv = \int_V \rho\, dv \tag{27}$$

$$\boxed{\oint_S \vec{\mathbf{D}} \cdot d\vec{\mathbf{s}} = \int_V \rho\, dv} \qquad \text{(Gauss's law)} \tag{28}$$

$$\oint_S \vec{\mathbf{D}} \cdot d\vec{\mathbf{s}} = \int_V \rho\, dv = Q \tag{29}$$

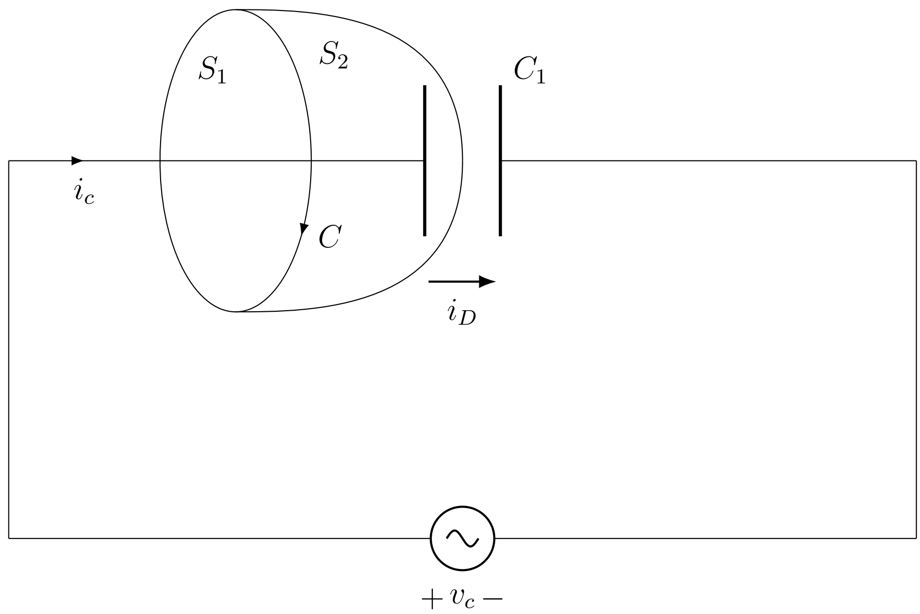
4)

$$\nabla \cdot \vec{\mathbf{B}} = 0 \tag{30}$$

$$\int_V (\nabla \cdot \vec{\mathbf{B}})\, dv = \int_V 0\, dv \tag{31}$$

$$\boxed{\oint_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = 0} \qquad \text{(No isolated magnetic charge)} \tag{32}$$

**Example 5:**
An ac voltage of amplitude $V_0$ and angular frequency $\omega$, $v_c = V_0\ \sin\omega t$, is connected across a parallel plate capacitor $C_1$, as shown in figure. a) Verify that the displacement current in the capacitor is the same as the conduction current in the wires. b) Determine the magnetic field intensity at a distance $r$ from the wire.

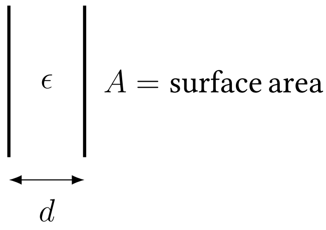



<u>Solution</u>
a)

$$i_c = C_1\, \frac{dv_c}{dt} \qquad \text{(conduction current in the wires)} \tag{33}$$

$$i_c = C_1\, \frac{d}{dt}\, (V_0\ \sin\omega t) = \omega\, V_0\, C_1\ \cos\omega t \tag{34}$$

$$i_D = \int_S \frac{\partial \vec{\mathbf{D}}}{\partial t} \cdot d\vec{s} \qquad \text{(displacement current in the capacitor)} \tag{35}$$

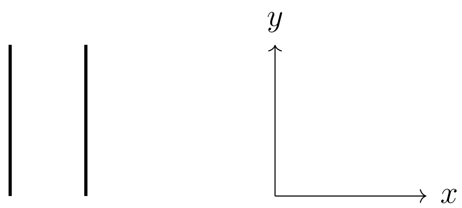



$$C_1 = \frac{\epsilon A}{d} \tag{36}$$

$$E = \frac{v_c}{d} \tag{37}$$

$$D = \epsilon\, E = \epsilon\, \frac{v_c}{d} = \frac{\epsilon}{d}\, V_0\, \sin \omega t \tag{38}$$

$$\frac{\partial D}{\partial t} = \frac{\epsilon\, V_0}{d}\, \omega \cos \omega t \tag{39}$$




$$\vec{\mathbf{D}} = D\, \hat{\mathbf{a}}_x \tag{40}$$

$$d\vec{\mathbf{s}} = \hat{\mathbf{a}}_x\, dy\, dz \tag{41}$$

$$\frac{\partial \vec{\mathbf{D}}}{\partial t} = \hat{\mathbf{a}}_x\, \frac{\partial D}{\partial t} = \hat{\mathbf{a}}_x\, \frac{\epsilon\, V_0}{d}\, \omega \cos \omega t \tag{42}$$

$$\frac{\partial \vec{\mathbf{D}}}{\partial t} \cdot d\vec{\mathbf{s}} = \frac{\epsilon\, V_0}{d}\, \omega \cos \omega t\, dy\, dz \tag{43}$$

$$i_D = \int \int \frac{\epsilon\, V_0}{d}\, \omega \cos \omega t\, dy\, dz = \frac{\epsilon\, V_0}{d}\, \omega \cos \omega t \int \int dy\, dz \tag{44}$$

$$\int \int dy\, dz = A \tag{45}$$

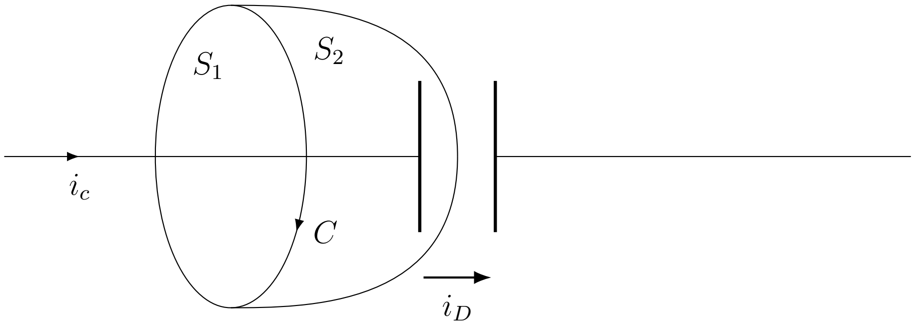
$$i_D = \frac{\epsilon\, A}{d}\, V_0\, \omega \cos \omega t = C_1\, V_0\, \omega \cos \omega t \tag{46}$$

$$i_c = i_D = \omega\, V_0\, C_1 \cos \omega t \tag{47}$$

The displacement current is equal to the conduction current.

**b)** The magnetic field intensity at a distance $r$ from the conducting wire can be found by applying the generalized Ampere's circuital law:

$$\oint_C \vec{\mathbf{H}} \cdot d\vec{\ell} = \int_S \vec{\mathbf{J}} \cdot d\vec{\mathbf{s}} + \int_S \frac{\partial \vec{\mathbf{D}}}{\partial t} \cdot d\vec{\mathbf{s}} \tag{48}$$




First let's choose planar disk surface $S_1$ for $S$. For this case $\vec{\mathbf{D}} = 0$, because no charges are deposited along the wire. So we have

$$\oint_C \vec{\mathbf{H}} \cdot d\vec{\ell} = \int_{S_1} \vec{\mathbf{J}} \cdot d\vec{\mathbf{s}} \tag{49}$$

$$\oint_C \vec{\mathbf{H}} \cdot d\vec{\ell} = \int_0^{2\pi} (H_\phi\, \hat{\mathbf{a}}_\phi) \cdot (r\, d\phi\, \hat{\mathbf{a}}_\phi) = \int_0^{2\pi} H_\phi\, r\, d\phi \;=\; H_\phi\, 2\pi r \tag{50}$$

$$\int_{S_1} \vec{\mathbf{J}} \cdot d\vec{\mathbf{s}} = i_c \;=\; \omega\, V_0\, C_1\, \cos \omega t \tag{51}$$

$$\Rightarrow H_\phi = \frac{C_1\, V_0}{2\pi r}\, \omega\, \cos \omega t \tag{52}$$

Now let's choose curved surface $S_2$ passing through dielectric medium.

$$\oint_C \vec{\mathbf{H}} \cdot d\vec{\ell} = \int_{S_2} \frac{\partial \vec{\mathbf{D}}}{\partial t} \cdot d\vec{\mathbf{s}} = i_D \tag{53}$$

$$\Rightarrow H_\phi\, 2\pi r = \omega\, V_0\, C_1\, \cos \omega t \tag{54}$$

$$\Rightarrow H_\phi = \frac{C_1\, V_0}{2\pi r}\, \omega\, \cos \omega t \tag{55}$$