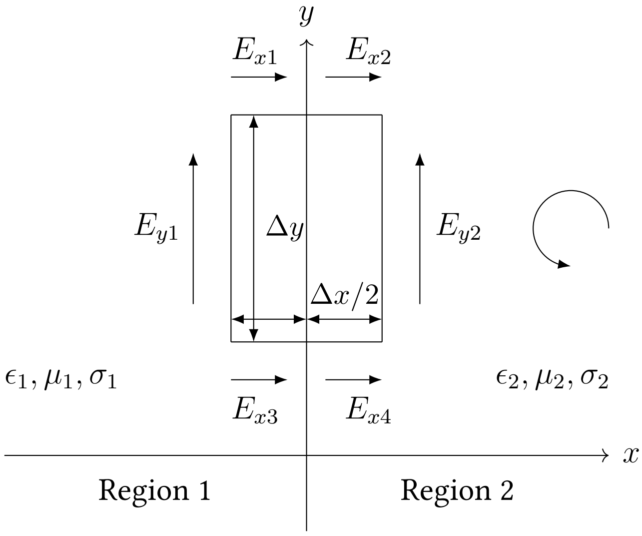
4. Electromagnetic Boundary Conditions

Conditions on the Tangential Components of $\vec{E}$ and $\vec{H}$

$$\nabla \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t} \tag{1}$$

$$\oint_C \vec{\mathbf{E}} \cdot d\vec{\ell} = -\int_S \frac{\partial \vec{\mathbf{B}}}{\partial t} \cdot d\vec{\mathbf{s}} \qquad \text{(Faraday's law)} \tag{2}$$



Suppose the surface of discontinuity to be the plane $x = 0$ as shown in figure. Consider the small rectangle of width Δx and length Δy enclosing a small portion of each media (1) and (2).

$$\oint_C \vec{\mathbf{E}} \cdot d\vec{\ell} = -\int_S \frac{\partial \vec{\mathbf{B}}}{\partial t} \cdot d\vec{\mathbf{s}} \tag{3}$$

$$d\vec{\mathbf{s}} = \hat{\mathbf{a}}_z \, dx \, dy = \hat{\mathbf{a}}_z \, \Delta x \, \Delta y \tag{4}$$

$$E_{y2} \, \Delta y - E_{x2} \frac{\Delta x}{2} - E_{x1} \frac{\Delta x}{2} - E_{y1} \, \Delta y + E_{x3} \frac{\Delta x}{2} + E_{x4} \frac{\Delta x}{2} = -\frac{\partial B_z}{\partial t} \Delta x \, \Delta y \tag{5}$$

B_z is the average magnetic flux density through the rectangle $\Delta x \, \Delta y$. Now let $\Delta x \to 0$. So we obtain

$$E_{y2} \, \Delta_y - E_{y1} \, \Delta_y = 0 \tag{6}$$

$$\boxed{E_{y1} = E_{y2}} \tag{7}$$

The tangential components of an $\vec{\mathbf{E}}$ field is continuous across an interface.

$$E_{y1} - E_{y2} = 0 \tag{8}$$

$$\hat{\mathbf{a}}_n \times (\vec{\mathbf{E}}_1 - \vec{\mathbf{E}}_2) = 0 \tag{9}$$

$\hat{\mathbf{a}}_n = -\hat{\mathbf{a}}_x$: outward unit normal from medium 2 at the interface.

$$\vec{\mathbf{E}}_1 = \hat{\mathbf{a}}_x \, E_{x1} + \hat{\mathbf{a}}_y \, E_{y1} + \hat{\mathbf{a}}_z \, E_{z1} \tag{10}$$

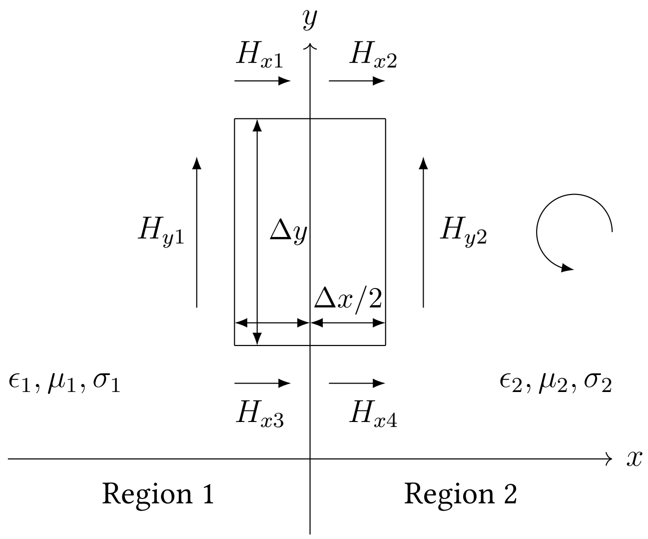
$$\vec{\mathbf{E}}_2 = \hat{\mathbf{a}}_x \, E_{x2} + \hat{\mathbf{a}}_y \, E_{y2} + \hat{\mathbf{a}}_z \, E_{z2} \tag{11}$$

$$\begin{aligned}
&\hat{\mathbf{a}}_n \times (\vec{\mathbf{E}}_1 - \vec{\mathbf{E}}_2) = \\
&\qquad = -\hat{\mathbf{a}}_x \times (\hat{\mathbf{a}}_x \, E_{x1} + \hat{\mathbf{a}}_y \, E_{y1} + \hat{\mathbf{a}}_z \, E_{z1} - \hat{\mathbf{a}}_x \, E_{x2} - \hat{\mathbf{a}}_y \, E_{y2} - \hat{\mathbf{a}}_z \, E_{z2}) \\
&\qquad = -\hat{\mathbf{a}}_z \, E_{y1} + \hat{\mathbf{a}}_y \, E_{z1} + \hat{\mathbf{a}}_z \, E_{y2} - \hat{\mathbf{a}}_y \, E_{z2} \\
&\qquad = \hat{\mathbf{a}}_y (E_{z1} - E_{z2}) + \hat{\mathbf{a}}_z \, (E_{y2} - E_{y1}) = 0
\end{aligned} \tag{12}$$

$$\Rightarrow \; E_{y1} = E_{y2} \tag{13}$$

$$\Rightarrow \; E_{z1} = E_{z2} \tag{14}$$

$$\oint_C \vec{\mathbf{H}} \cdot d\vec{\ell} = \int_S \left(\vec{\mathbf{J}} + \frac{\partial \vec{\mathbf{D}}}{\partial t} \right) \cdot d\vec{\mathbf{s}} \qquad \text{(Ampere's circuital law)} \tag{15}$$



$$\oint_C \vec{\mathbf{H}} \cdot d\vec{\ell} = H_{y2} \, \Delta y - H_{x2} \, \frac{\Delta x}{2} - H_{x1} \, \frac{\Delta x}{2} - H_{y1} \, \Delta_y + H_{x3} \, \frac{\Delta x}{2} + H_{x4} \, \frac{\Delta x}{2} \tag{16}$$

$$\lim_{\Delta x \to 0} \oint_C \vec{\mathbf{H}} \cdot d\vec{\ell} = H_{y2} \, \Delta y - H_{y1} \, \Delta y \tag{17}$$

$$\int_S \vec{\mathbf{J}} \cdot d\vec{\mathbf{s}} = \vec{\mathbf{J}} \cdot \hat{\mathbf{a}}_z \, \Delta x \, \Delta y \qquad d\vec{\mathbf{s}} = \hat{\mathbf{a}}_z \, \Delta x \, \Delta y \tag{18}$$

$$\lim_{\Delta x \to 0} \int_S \vec{\mathbf{J}} \cdot d\vec{\mathbf{s}} = \lim_{\Delta x \to 0} [\vec{\mathbf{J}} \cdot \hat{\mathbf{a}}_z \, \Delta x \, \Delta y] = \lim_{\Delta x \to 0} [(\vec{\mathbf{J}} \, \Delta x) \cdot \hat{\mathbf{a}}_z \, \Delta y] \tag{19}$$

$$\lim_{\Delta x \to 0} \vec{\mathbf{J}} \, \Delta x = \vec{\mathbf{J}}_s \tag{20}$$

$\vec{\mathbf{J}}$: volume current density (A/m^2)

$\vec{\mathbf{J}}_s$: surface current density (A/m) (current sheet)

$$\lim_{\Delta x \to 0} \int_S \vec{\mathbf{J}} \cdot d\vec{\mathbf{s}} = \vec{\mathbf{J}}_s \cdot \hat{\mathbf{a}}_z \, \Delta y = J_{sz} \, \Delta y \tag{21}$$

$\vec{\mathbf{J}}_s \cdot \hat{\mathbf{a}}_z = J_{sz}$ (z component of $\vec{\mathbf{J}}_s$)

$$\lim_{\Delta x \to 0} \int_S \frac{\partial \vec{\mathbf{D}}}{\partial t} \cdot d\vec{\mathbf{s}} = \lim_{\Delta x \to 0} \frac{\partial D_z}{\partial t} \Delta x \, \Delta y = 0 \tag{22}$$

$$\Rightarrow H_{y2} \, \Delta y - H_{y1} \, \Delta y = J_{sz} \, \Delta y \tag{23}$$

$$\boxed{H_{y2} - H_{y1} = J_{sz}} \tag{24}$$

The tangential components of an $\vec{\mathbf{H}}$ field is discontinuous across an interface where a surface current exists.

$$H_{y2} - H_{y1} = J_{sz} \tag{25}$$

$$\hat{\mathbf{a}}_n \times (\vec{\mathbf{H}}_1 - \vec{\mathbf{H}}_2) = \vec{\mathbf{J}}_s \tag{26}$$

$\hat{\mathbf{a}}_n = -\hat{\mathbf{a}}_x$: outward unit normal from medium 2 at the interface.

$$\vec{\mathbf{H}}_1 = \hat{\mathbf{a}}_x \, H_{x1} + \hat{\mathbf{a}}_y \, H_{y1} + \hat{\mathbf{a}}_z \, H_{z1} \tag{27}$$

$$\vec{\mathbf{H}}_2 = \hat{\mathbf{a}}_x \, H_{x2} + \hat{\mathbf{a}}_y \, H_{y2} + \hat{\mathbf{a}}_z \, H_{z2} \tag{28}$$

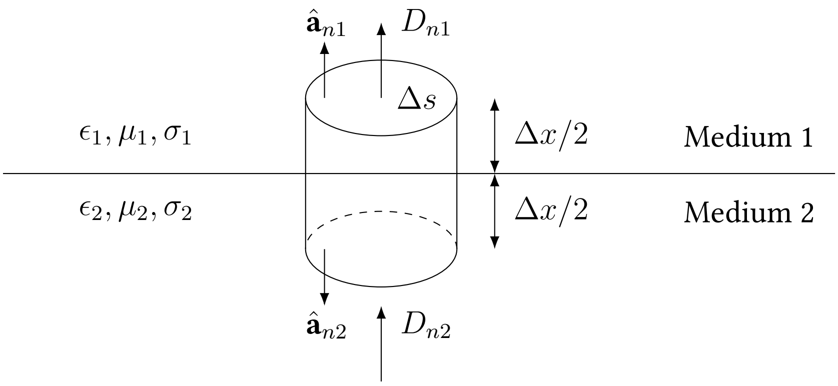
$$\vec{\mathbf{J}}_s = \hat{\mathbf{a}}_x \, J_{sx} + \hat{\mathbf{a}}_y \, J_{sy} + \hat{\mathbf{a}}_z \, J_{sz} \tag{29}$$

$$\begin{aligned} \hat{\mathbf{a}}_n \times (\vec{\mathbf{H}}_1 - \vec{\mathbf{H}}_2) &= \\ &= -\hat{\mathbf{a}}_x \times (\hat{\mathbf{a}}_x \, H_{x1} + \hat{\mathbf{a}}_y \, H_{y1} + \hat{\mathbf{a}}_z \, H_{z1} - \hat{\mathbf{a}}_x \, H_{x2} - \hat{\mathbf{a}}_y \, H_{y2} - \hat{\mathbf{a}}_z \, H_{z2}) \\ &= -\hat{\mathbf{a}}_z \, H_{y1} + \hat{\mathbf{a}}_y \, H_{z1} + \hat{\mathbf{a}}_z \, H_{y2} - \hat{\mathbf{a}}_y \, H_{z2} \\ &= \hat{\mathbf{a}}_y \, (H_{z1} - H_{z2}) + \hat{\mathbf{a}}_z \, (H_{y2} - H_{y1}) = \vec{\mathbf{J}}_s \end{aligned} \tag{30}$$

$$\Rightarrow H_{y2} - H_{y1} = J_{sz} \tag{31}$$

$$\Rightarrow H_{z1} - H_{z2} = J_{sy} \tag{32}$$

Conditions on the Normal Components of $\vec{\mathbf{B}}$ and $\vec{\mathbf{D}}$



$$\oint_S \vec{\mathbf{D}} \cdot d\vec{\mathbf{s}} = \int_V \rho \, dv \tag{33}$$

$$\vec{\mathbf{D}} \cdot \hat{\mathbf{a}}_{n1} \, \Delta s + \vec{\mathbf{D}} \cdot \hat{\mathbf{a}}_{n2} \, \Delta s + \Psi_{\text{edge}} = \rho \, \Delta x \, \Delta s \tag{34}$$

$$D_{n1} \, \Delta s - D_{n2} \, \Delta s + \Psi_{\text{edge}} = \rho \, \Delta x \, \Delta s \tag{35}$$

Ψ_{edge} : the outward electric flux through the curved edge surface of the pillbox.

$$\lim_{\Delta x \to 0} \Psi_{\text{edge}} = 0 \tag{36}$$

$$\lim_{\Delta x \to 0} \rho \, \Delta x = \rho_s \tag{37}$$

ρ : volume charge density (C/m^3)

ρ_s : surface charge density (C/m^2)

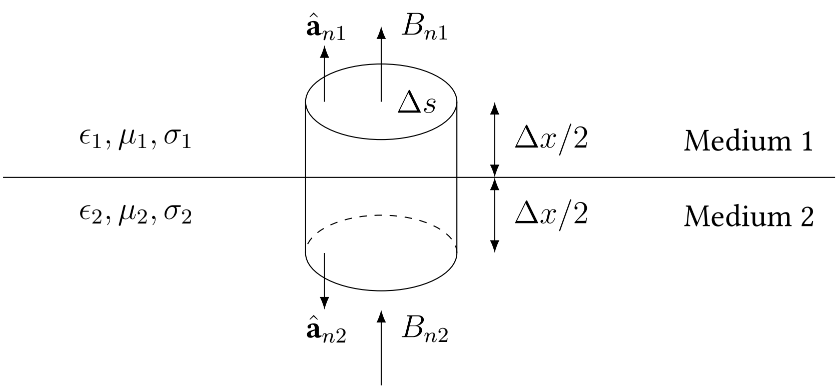
$$D_{n1} \, \Delta s - D_{n2} \, \Delta s = \rho_s \, \Delta s \tag{38}$$

$$\boxed{D_{n1} - D_{n2} = \rho_s} \tag{39}$$

The normal component of a $\vec{\mathbf{D}}$ field is discontinuous across an interface where a surface charge exists.

$$\hat{\mathbf{a}}_n \cdot (\vec{\mathbf{D}}_1 - \vec{\mathbf{D}}_2) = \rho_s \tag{40}$$

$\hat{\mathbf{a}}_n$: outward unit normal from medium 2 at the interface.



$$\oint_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = 0 \tag{41}$$

In a similar way

$$\vec{\mathbf{B}} \cdot \hat{\mathbf{a}}_{n1}\, \Delta s + \vec{\mathbf{B}} \cdot \hat{\mathbf{a}}_{n2}\, \Delta s + \Phi_{\text{edge}} = 0 \tag{42}$$

Φ_{edge} : the outward magnetic flux through the curved edge surface of the pillbox.

$$\lim_{\Delta x \to 0} \Phi_{\text{edge}} = 0 \tag{43}$$

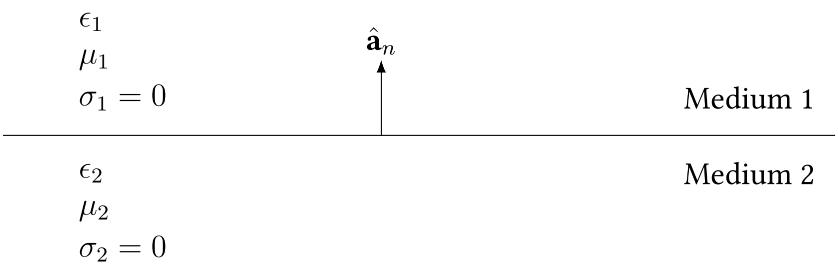
$$B_{n1}\, \Delta s - B_{n2}\, \Delta s = 0 \tag{44}$$

$$\boxed{B_{n1} = B_{n2}} \tag{45}$$

The normal component of a $\vec{\mathbf{B}}$ field is continuous across an interface.

4.1 Interface Between Two Lossless Linear Media

A lossless linear medium can be specified by a permittivity ϵ and permeability μ, with $\sigma = 0$. We set $\rho_s = 0$, $\vec{\mathbf{J}}_s = 0$.



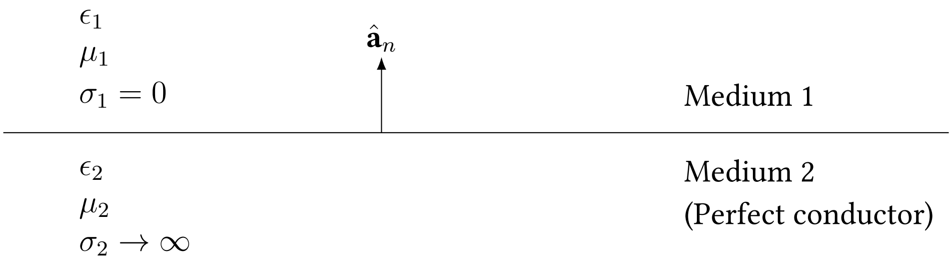
$$E_{t1} = E_{t2} \Rightarrow \frac{D_{t1}}{\epsilon_1} = \frac{D_{t2}}{\epsilon_2} \Rightarrow \frac{D_{t1}}{D_{t2}} = \frac{\epsilon_1}{\epsilon_2} \tag{46}$$

$$H_{t1} = H_{t2} \Rightarrow \frac{B_{t1}}{\mu_1} = \frac{B_{t2}}{\mu_2} \Rightarrow \frac{B_{t1}}{B_{t2}} = \frac{\mu_1}{\mu_2} \tag{47}$$

$$D_{n1} = D_{n2} \Rightarrow \epsilon_1\, E_{n1} = \epsilon_2\, E_{n2} \tag{48}$$

$$B_{n1} = B_{n2} \Rightarrow \mu_1\, H_{n1} = \mu_2\, H_{n2} \tag{49}$$

4.2 Interface Between a Dielectric and a Perfect Conductor



In the interior of a perfect conductor the electric field is zero, and charges reside on the surface only.

$$\begin{aligned} &\vec{\mathbf{E}}_2 = 0 \\ &\vec{\mathbf{H}}_2 = 0 \\ &\vec{\mathbf{D}}_2 = 0 \\ &\vec{\mathbf{B}}_2 = 0 \end{aligned} \tag{50}$$

$$\begin{aligned} &E_{t1} = E_{t2} = 0 \\ &\hat{\mathbf{a}}_n \times (\vec{\mathbf{H}}_1 - \vec{\mathbf{H}}_2) = \vec{\mathbf{J}}_s \Rightarrow \hat{\mathbf{a}}_n \times \vec{\mathbf{H}}_1 = \vec{\mathbf{J}}_s \\ &\hat{\mathbf{a}}_n \cdot (\vec{\mathbf{D}}_1 - \vec{\mathbf{D}}_2) = \rho_s \Rightarrow \hat{\mathbf{a}}_n \cdot \vec{\mathbf{D}}_1 = \rho_s \\ &B_{n1} = B_{n2} = 0 \end{aligned} \tag{51}$$

$\hat{\mathbf{a}}_n$: outward unit normal from medium 2 at the interface.