HEAT AND MASS TRANSFER

Introduction and Basic Concepts

OUTLINE

- Thermodynamics and Heat Transfer
- A Review of Basic Thermodynamic Concepts
- Conduction
- Convection
- Radiation
- Simultaneous Heat Transfer Mechanisms
- Conclusions

Thermodynamics and Heat Transfer

- The science of thermodynamics deals with the amount of heat transfer as a system undergoes a process from one equilibrium state to another, and makes no reference to how long the process will take.

- The science of heat transfer deals with the determination of the rates of energy that can be transferred from one system to another as a result of temperature difference.

Application Areas of Heat Transfer

The human body

Air conditioning systems

Airplanes

Car radiators

Power plants

Refrigeration systems

Heat and Other Forms of Energy

- Energy can exist in numerous forms such as:
  - thermal,
  - mechanical,
  - kinetic,
  - potential,
  - electrical,
  - magnetic,
  - chemical, and
  - nuclear.

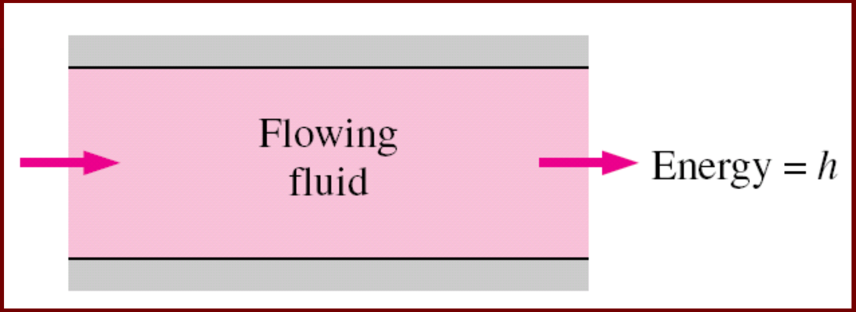
- Their sum constitutes the **total energy** *E* (or *e* on a unit mass basis) of a system.

- The sum of all microscopic forms of energy is called the **internal energy** of a system.

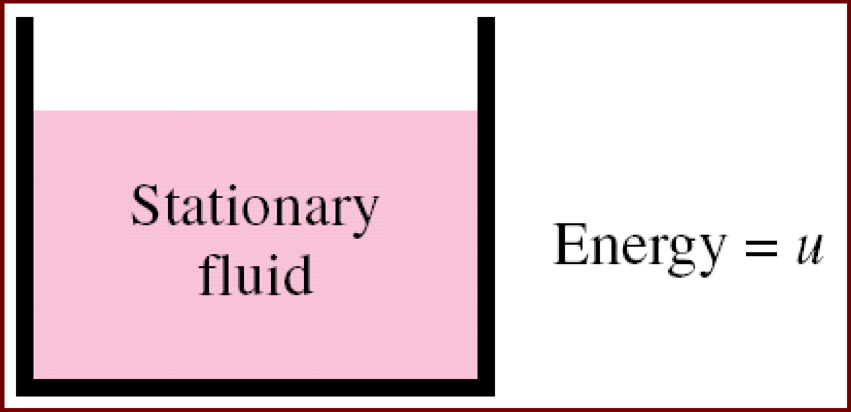
- **Internal energy** may be viewed as the sum of the kinetic and potential energies of the molecules.
- The kinetic energy of the molecules is called **sensible heat**.
- The internal energy associated with the phase of a system is called **latent heat**.
- The internal energy associated with the atomic bonds in a molecule is called **chemical** (or **bond**) **energy**.
- The internal energy associated with the bonds within the nucleus of the atom itself is called **nuclear energy**.

Internal Energy and Enthalpy

- In the analysis of systems that involve fluid flow, we frequently encounter the combination of properties u and Pv.



- The combination is defined as **enthalpy** ($h=u+Pv$).



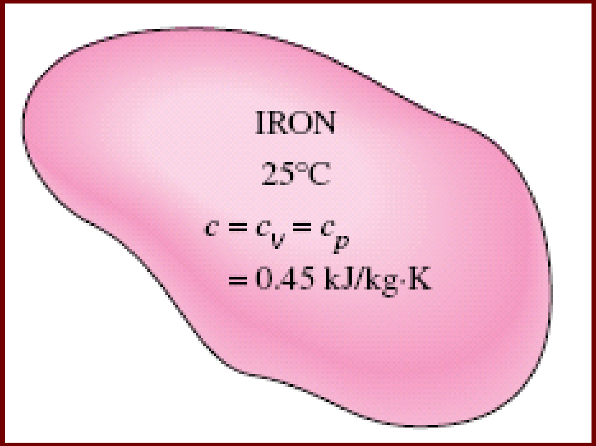
- The term Pv represents the flow energy of the fluid (also called the flow work).

Specific Heats of Gases, Liquids, and Solids

- **Specific heat** is defined as *the energy required to raise the temperature of a unit mass of a substance by one degree.*
- Two kinds of specific heats:
  - specific heat at constant volume c_v, and
  - specific heat at constant pressure c_p.
- The specific heats of a substance, in general, depend on two independent properties such as temperature and pressure.
- For an *ideal gas*, however, they depend on *temperature* only.

Specific Heats

- At low pressures all real gases approach ideal gas behavior, and therefore their specific heats depend on temperature only.
- A substance whose specific volume (or density) does not change with temperature or pressure is called an **incompressible substance**.
- The constant-volume and constant-pressure specific heats are identical for incompressible substances.



- The specific heats of incompressible substances depend on temperature only.

Energy Transfer

- Energy can be transferred to or from a given mass by two mechanisms:
  - heat transfer, and
  - work.
- The amount of heat transferred during a process is denoted by Q.
- The amount of heat transferred per unit time is called **heat transfer rate**, and is denoted by $\dot{Q}$
- The total amount of heat transfer Q during a time interval Δt can be determined from

$$Q = \int_0^{\Delta t} \dot{Q}\, dt \qquad \text{(J)}$$

- The rate of heat transfer per unit area normal to the direction of heat transfer is called **heat flux**, and the average heat flux is expressed as

$$\dot{q} = \frac{\dot{Q}}{A} \qquad (\mathrm{W/m^2})$$

The First Law of Thermodynamics

- The **first law of thermodynamics** states that *energy can neither be created nor destroyed during a process; it can only change forms.*

$$\begin{pmatrix}\text{Total energy}\\ \text{entering the}\\ \text{system}\end{pmatrix} - \begin{pmatrix}\text{Total energy}\\ \text{leaving the}\\ \text{system}\end{pmatrix} = \begin{pmatrix}\text{Change in the}\\ \text{total energy of}\\ \text{the system}\end{pmatrix}$$

- The **energy balance** for any system undergoing any process can be expressed as (in the rate form)

$$\underbrace{\dot{E}_{\text{in}} - \dot{E}_{\text{out}}}_{\substack{\text{Rate of net energy transfer}\\ \text{by heat, work, and mass}}} = \underbrace{dE_{\text{system}}/dt}_{\substack{\text{Rate of change in internal}\\ \text{kinetic, potential, etc., energies}}} \qquad \text{(W)}$$

- In heat transfer problems it is convenient to write a **heat balance** and to treat the conversion of nuclear, chemical, mechanical, and electrical energies into thermal energy as *heat generation.*
- The *energy balance* in that case can be expressed as

$$\underbrace{Q_{in} - Q_{out}}_{\substack{\text{Net heat}\\ \text{transfer}}} + \underbrace{E_{gen}}_{\substack{\text{Heat}\\ \text{generation}}} = \underbrace{\Delta E_{\text{thermal, system}}}_{\substack{\text{Change in thermal}\\ \text{energy of the system}}} \qquad \text{(J)}$$

Energy Balance

Closed systems

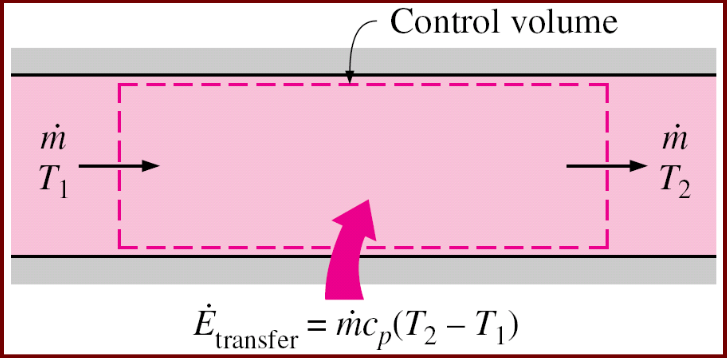
- *Stationary closed system, no work:*

$$Q = mc_v\Delta T$$

Steady-Flow Systems

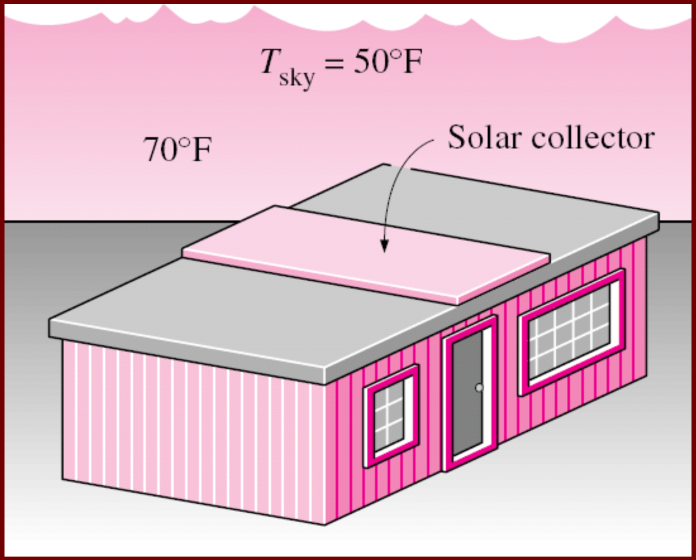
When kinetic and potential energies are negligible, and there is no work interaction

$$\dot{Q} = \dot{m}\Delta h = \dot{m}c_p\Delta T$$



Heat Transfer Mechanisms

- Heat can be transferred in three basic modes:
  - conduction,
  - convection,
  - radiation.



- All modes of heat transfer require the existence of a temperature difference.
- All modes are from the high-temperature medium to a lower-temperature one.

CONDUCTION

- **Conduction:** the transfer of heat from one part of a material to another part of the same material, or from one material to another in physical contact with it, without any appreciable displacement of the molecules.

The *rate* of heat conduction through a medium depends on:

- the *geometry* of the medium
- the *thickness*
- the *material* of the medium
- the *temperature difference* across the medium.

The rate of heat conduction through a plane layer is proportional to the temperature difference across the layer and the heat transfer area, but is inversely proportional to the thickness of the layer.

$$\text{Rate of heat conduction} \propto \frac{(\text{Area})(\text{Temperature difference})}{\text{Thickness}}$$

or

$$\dot{Q}_{\text{cond}} = kA\frac{T_1 - T_2}{\Delta x} = -kA\frac{\Delta T}{\Delta x} \qquad \text{(W)}$$

where the constant of proportionality k is the **thermal conductivity** of the material, which is a measure of the *ability of a material to conduct heat*

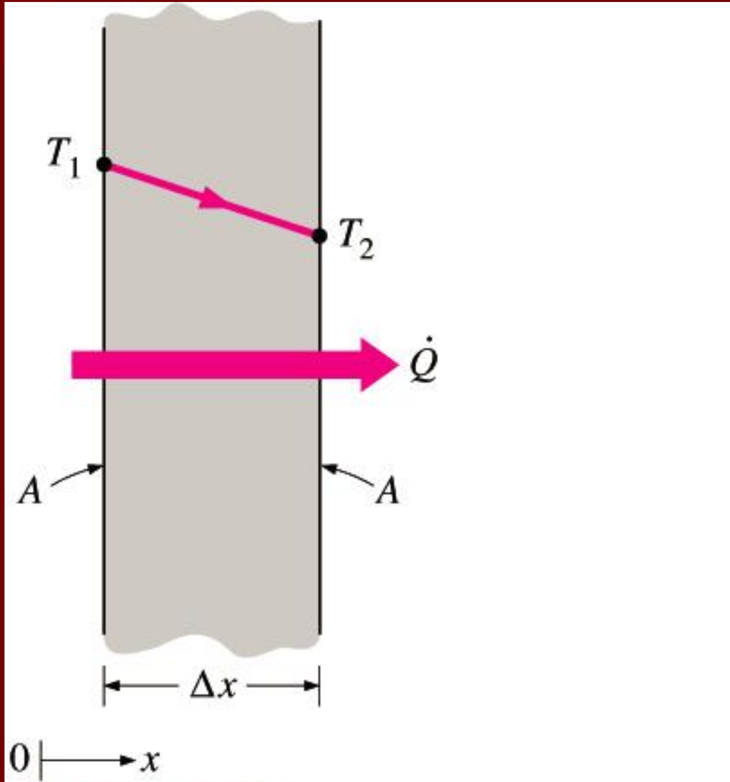


FIGURE 1–22
Heat conduction through a large plane wall of thickness Δx and area A.

In the limiting case of $\Delta x \to 0,$ reduces to

$$\dot{Q}_{\text{cond}} = -kA \frac{dT}{dx} \qquad \text{(W)}$$

which is called **Fourier's law of heat conduction** after J. Fourier, who expressed it first in his heat transfer text in 1822.

Here, *dT/dx* is the **temperature gradient**, which is the slope of the temperature curve on a *T-x* diagram at location *x*.

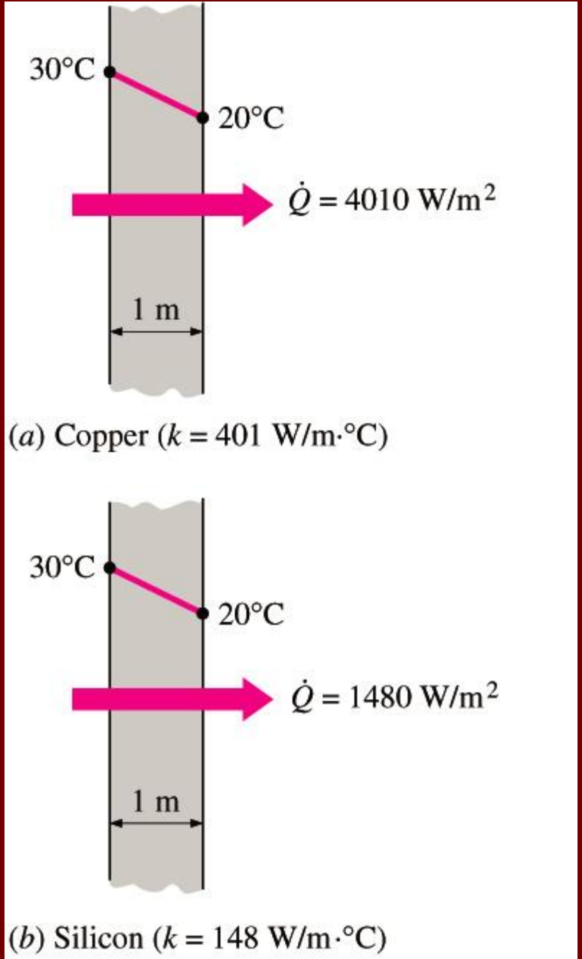


FIGURE 1–23
The rate of heat conduction through a solid is directly proportional to its thermal conductivity.

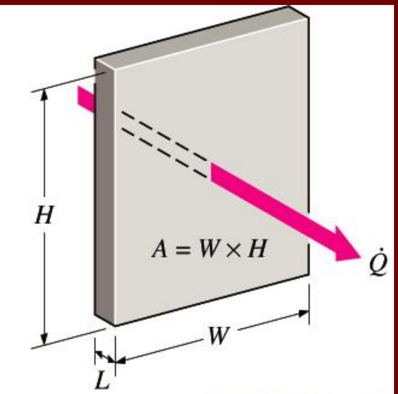


FIGURE 1–24
In heat conduction analysis, *A* represents the area *normal* to the direction of heat transfer.

The *negative sign* in the above equation ensures that heat transfer in the positive *x* direction is a positive quantity.

EXAMPLE 16–1 The Cost of Heat Loss through a Roof

The roof of an electrically heated home is 6 m long, 8 m wide, and 0.25 m thick, and is made of a flat layer of concrete whose thermal conductivity is $k = 0.8$ W/m · °C (Fig. 16–4). The temperatures of the inner and the outer surfaces of the roof one night are measured to be 15°C and 4°C, respectively, for a period of 10 hours. Determine (*a*) the rate of heat loss through the roof that night and (*b*) the cost of that heat loss to the home owner if the cost of electricity is $0.08/kWh.

SOLUTION The inner and outer surfaces of the flat concrete roof of an electrically heated home are maintained at specified temperatures during a night. The heat loss through the roof and its cost that night are to be determined.

Assumptions **1** Steady operating conditions exist during the entire night since the surface temperatures of the roof remain constant at the specified values. **2** Constant properties can be used for the roof.

Properties The thermal conductivity of the roof is given to be $k = 0.8$ W/m · °C.

Analysis (*a*) Noting that heat transfer through the roof is by conduction and the area of the roof is $A = 6\text{ m} \times 8\text{ m} = 48\text{ m}^2$, the steady rate of heat transfer through the roof is determined to be

$$\dot{Q} = kA\frac{T_1 - T_2}{L} = (0.8\text{ W/m}\cdot{}^\circ\text{C})(48\text{ m}^2)\frac{(15-4)^\circ\text{C}}{0.25\text{ m}} = 1690\text{ W} = 1.69\text{ kW}$$

(*b*) The amount of heat lost through the roof during a 10-h period and its cost are determined from

$$Q = \dot{Q}\,\Delta t = (1.69\text{ kW})(10\text{ h}) = 16.9\text{ kWh}$$

$$\text{Cost} = (\text{Amount of energy})(\text{Unit cost of energy})$$

$$= (16.9\text{ kWh})(\$0.08/\text{kWh}) = \$1.35$$

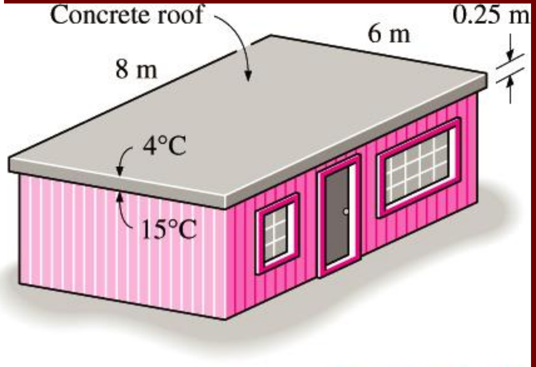


FIGURE 1–25
Schematic for Example 1–5.

Discussion The cost to the home owner of the heat loss through the roof that night was $1.35. The total heating bill of the house will be much larger since the heat losses through the walls are not considered in these calculations.

Thermal Conductivity

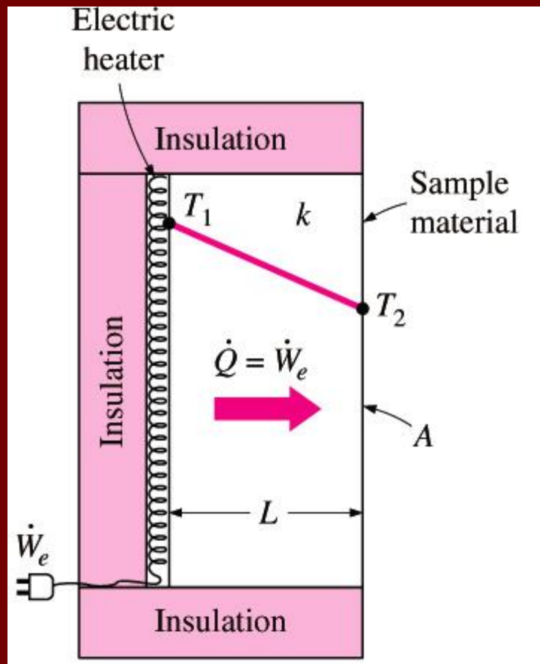
Thermal Conductivity: a measure of a solid material to conduct heat *(the rate of heat transfer through a unit thickness of the material per unit area per unit temperature difference).*

TABLE 1–1

The thermal conductivities of some materials at room temperature

Material	k, W/m · °C*
Diamond	2300
Silver	429
Copper	401
Gold	317
Aluminum	237
Iron	80.2
Mercury (l)	8.54
Glass	0.78
Brick	0.72
Water (l)	0.607
Human skin	0.37
Wood (oak)	0.17
Helium (g)	0.152
Soft rubber	0.13
Glass fiber	0.043
Air (g)	0.026
Urethane, rigid foam	0.026

*Multiply by 0.5778 to convert to Btu/h · ft · °F.



$$k = \frac{L}{A(T_1 - T_2)}\dot{Q}$$

FIGURE 1–26

A simple experimental setup to determine the thermal conductivity of a material.

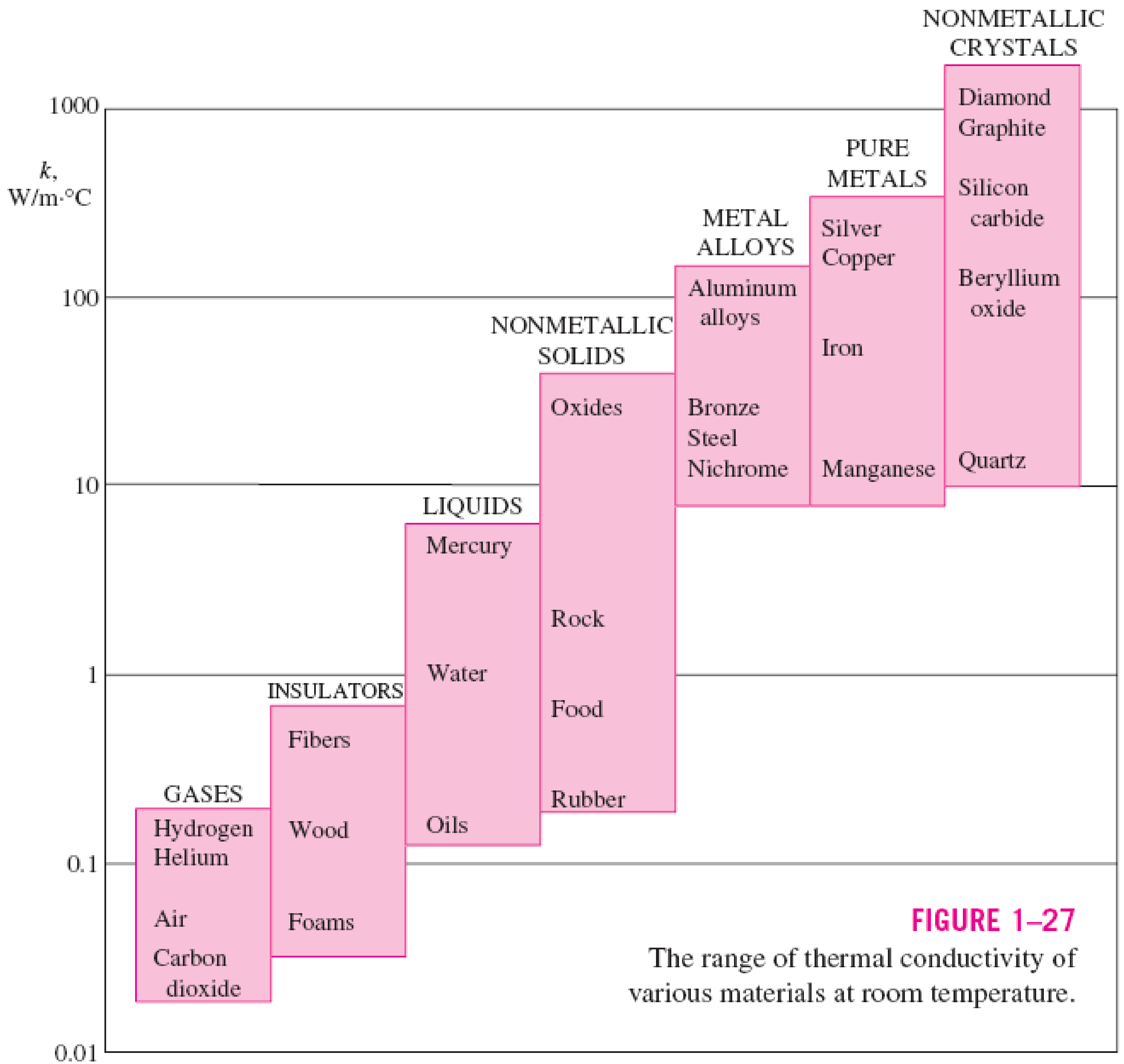


FIGURE 1–27
The range of thermal conductivity of various materials at room temperature.

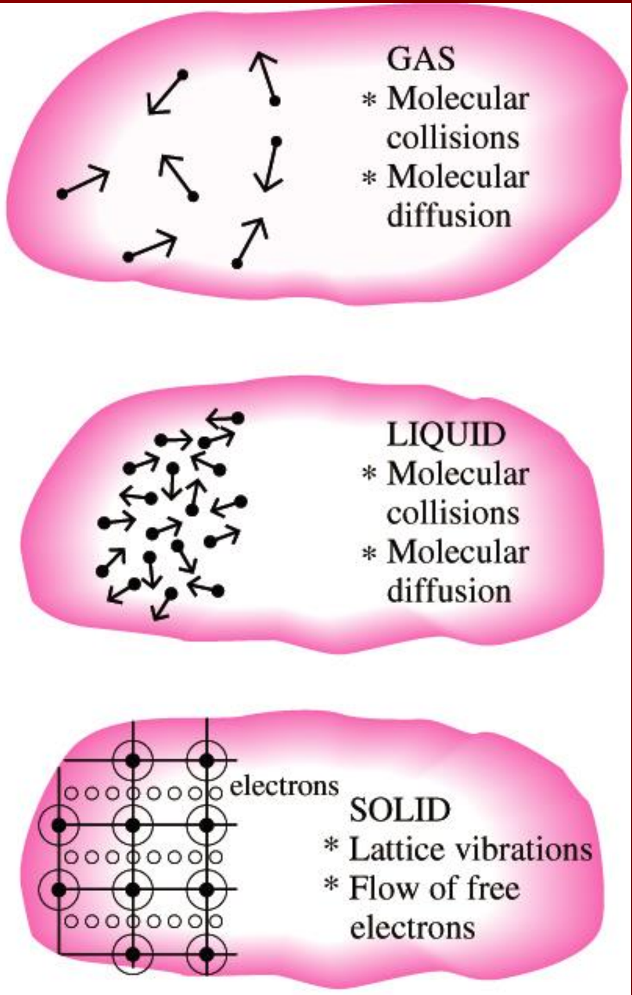


FIGURE 1–28

The mechanisms of heat conduction in different phases of a substance.

TABLE 1–2

The thermal conductivity of an alloy is usually much lower than the thermal conductivity of either metal of which it is composed

Pure metal or alloy	k, W/m · °C, at 300 K
Copper	401
Nickel	91
Constantan (55% Cu, 45% Ni)	23
Copper	401
Aluminum	237
Commercial bronze (90% Cu, 10% Al)	52

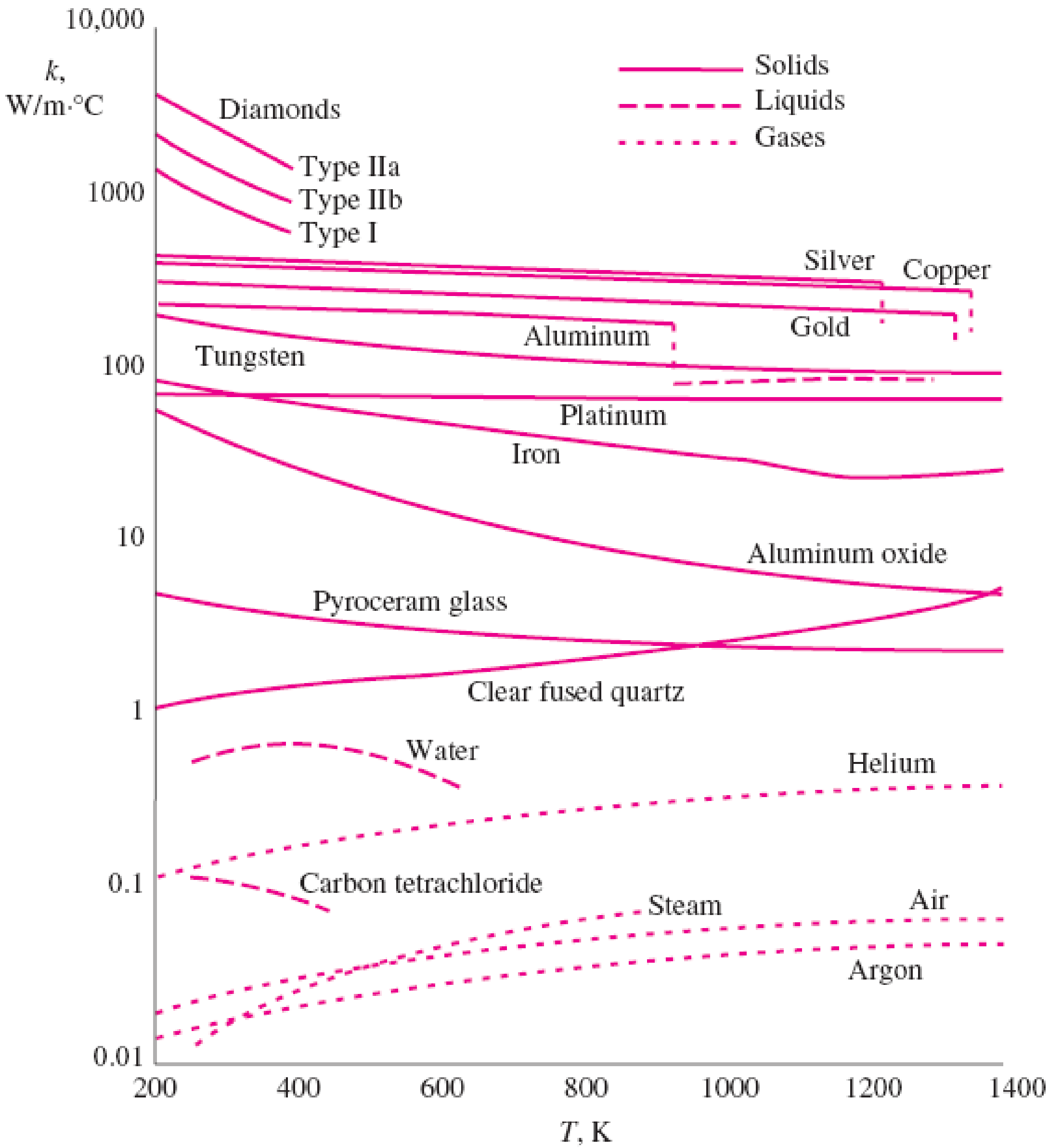


FIGURE 1–29
The variation of the thermal conductivity of various solids, liquids, and gases with temperature.

TABLE 1–3
Thermal conductivities of materials vary with temperature

	k, W/m · °C	
T, K	Copper	Aluminum
100	482	302
200	413	237
300	401	237
400	393	240
600	379	231
800	366	218

Thermal Diffusivity

Heat Capacity of a Material: the product ρC_p which is frequently encountered in heat transfer analysis.

Thermal Diffusivity: another material property that appears in the transient heat conduction analysis, representing how fast heat diffuses through a material.

$$\alpha = \frac{\text{Heat conducted}}{\text{Heat stored}} = \frac{k}{\rho C_p} \qquad (\text{m}^2/\text{s})$$

where the thermal conductivity k represents how well a material conducts heat, and the heat capacity ρC_p represents how much energy a material stores per unit volume.

The thermal diffusivity of a material can be viewed as the ratio of the *heat conducted* through the material to the *heat stored* per unit volume.

TABLE 1–4

The thermal diffusivities of some materials at room temperature

Material	α, m^2/s*
Silver	149×10^{-6}
Gold	127×10^{-6}
Copper	113×10^{-6}
Aluminum	97.5×10^{-6}
Iron	22.8×10^{-6}
Mercury (l)	4.7×10^{-6}
Marble	1.2×10^{-6}
Ice	1.2×10^{-6}
Concrete	0.75×10^{-6}
Brick	0.52×10^{-6}
Heavy soil (dry)	0.52×10^{-6}
Glass	0.34×10^{-6}
Glass wool	0.23×10^{-6}
Water (l)	0.14×10^{-6}
Beef	0.14×10^{-6}
Wood (oak)	0.13×10^{-6}

*Multiply by 10.76 to convert to ft^2/s.

CONVECTION

Convection: the mode of energy transfer between a solid surface and the adjacent, moving liquid or gas (involving the combined effects of *conduction* and *fluid motion*).

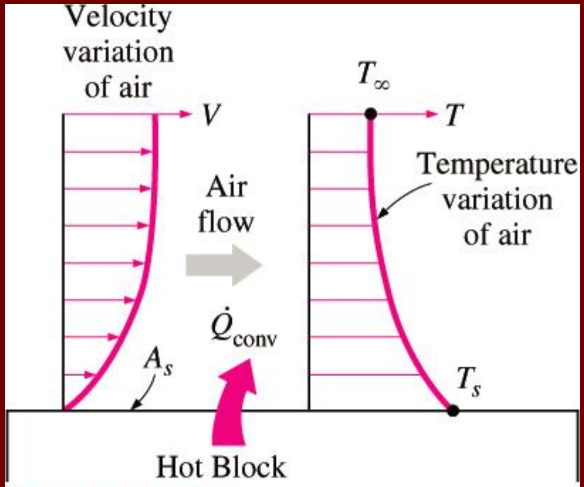


FIGURE 1–32
Heat transfer from a hot surface to air by convection.

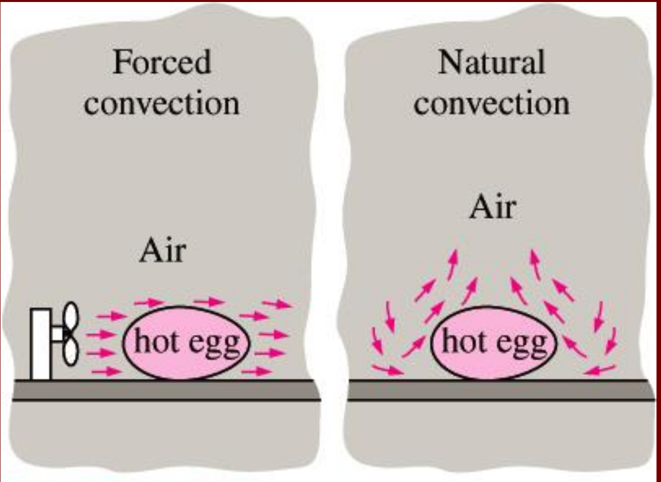


FIGURE 1–33
The cooling of a boiled egg by forced and natural convection.

Forced Convection: where the fluid is forced to flow over the surface by external means such as a fan, pump, or the wind.

Natural (or **Free**) **Convection:** where the fluid motion is caused by buoyancy forces that are induced by density differences due to the variation of temperature in the fluid.

The rate of *convection heat transfer* is observed to be proportional to the temperature difference, and is conveniently expressed by **Newton's law of cooling** as

$$\dot{Q}_{conv} = hA_s(T_s - T_\infty) \qquad \text{(W)}$$

h : Convection heat transfer coefficient in W/m²°C

A_s : Surface area through which convection heat transfer takes place

T_s : Surface temperature

T : Temperature of the fluid sufficiently far from the surface

TABLE 1–5

Typical values of convection heat transfer coefficient

Type of convection	h, W/m² · °C*
Free convection of gases	2–25
Free convection of liquids	10–1000
Forced convection of gases	25–250
Forced convection of liquids	50–20,000
Boiling and condensation	2500–100,000

*Multiply by 0.176 to convert to Btu/h · ft² · °F.

EXAMPLE 1–8 Measuring Convection Heat Transfer Coefficient

A 2-m-long, 0.3-cm-diameter electrical wire extends across a room at 15°C, as shown in Fig. 16–13. Heat is generated in the wire as a result of resistance heating, and the surface temperature of the wire is measured to be 152°C in steady operation. Also, the voltage drop and electric current through the wire are measured to be 60 V and 1.5 A, respectively. Disregarding any heat transfer by radiation, determine the convection heat transfer coefficient for heat transfer between the outer surface of the wire and the air in the room.

SOLUTION The convection heat transfer coefficient for heat transfer from an electrically heated wire to air is to be determined by measuring temperatures when steady operating conditions are reached and the electric power consumed.

Assumptions **1** Steady operating conditions exist since the temperature readings do not change with time. **2** Radiation heat transfer is negligible.

Analysis When steady operating conditions are reached, the rate of heat loss from the wire will equal the rate of heat generation in the wire as a result of resistance heating. That is,

$$\dot{Q} = \dot{E}_{\text{generated}} = VI = (60\text{ V})(1.5\text{ A}) = 90\text{ W}$$

The surface area of the wire is

$$A_s = \pi DL = \pi(0.003\text{ m})(2\text{ m}) = 0.01885\text{ m}^2$$

Newton's law of cooling for convection heat transfer is expressed as

$$\dot{Q}_{\text{conv}} = hA_s\,(T_s - T_\infty)$$

Disregarding any heat transfer by radiation and thus assuming all the heat loss from the wire to occur by convection, the convection heat transfer coefficient is determined to be

$$h = \frac{\dot{Q}_{\text{conv}}}{A_s(T_s - T_\infty)} = \frac{90\text{ W}}{(0.01885\text{ m}^2)(152 - 15)°\text{C}} = 34.9\text{ W/m}^2\cdot°\text{C}$$

Discussion Note that the simple setup described above can be used to determine the average heat transfer coefficients from a variety of surfaces in air. Also, heat transfer by radiation can be eliminated by keeping the surrounding surfaces at the temperature of the wire.

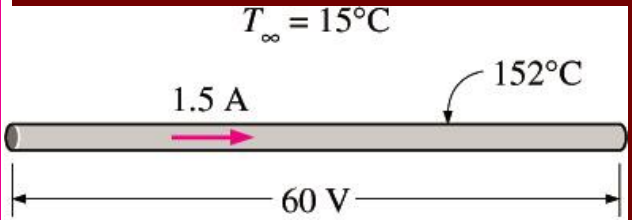


FIGURE 1–34
Schematic for Example 1–8.

RADIATION

Radiation: the energy emitted by matter in the form of *electromagnetic waves* (or *photons*) as a result of the changes in the electronic configurations of the atoms or molecules.

Thermal Radiation: the form of radiation emitted by bodies because of their temperature.

Radiation is a *volumetric phenomenon,* and all solids, liquids, and gases emit, absorb, or transmit radiation to varying degrees.

Radiation is usually considered to be a *surface phenomenon* for solids that are opaque to thermal radiation such as metals, wood, and rocks since the radiation emitted by the interior regions of such material can never reach the surface, and the radiation incident on such bodies is usually absorbed within a few microns from the surface.

The maximum rate of radiation that can be emitted from a surface at an absolute temperature T_s (in K or R) is given by the **Stefan–Boltzmann law** as

$$\dot{Q}_{\text{emit, max}} = \sigma A_s T_s^4 \qquad \text{(W)}$$

σ : The *Stefan–Boltzmann constant* (5.669×10^{-8} W/m^2·K^4)

Blackbody: the idealized surface that emits radiation at this maximum rate.

Blackbody Radiation: the radiation emitted by a blackbody.

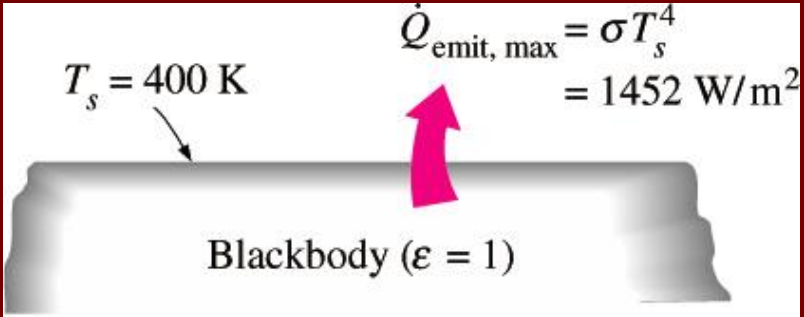


FIGURE 1–35
Blackbody radiation represents the *maximum amount of radiation that can be emitted from a surface at a specified temperature.*

The radiation emitted by all real surfaces is less than the radiation emitted by a blackbody at the same temperature, and is expressed as

$$\dot{Q}_{emit} = \varepsilon \sigma A_s T_s^4 \qquad \text{(W)}$$

where ε is the emissivity of the surface.

Emissivity: a measure of how closely a surface approximates a blackbody for which ε=1.
Its value is in the range $0 \le \varepsilon \le 1$

Absorptivity (α): the fraction of the radiation energy incident on a surface that is absorbed by the surface.
A blackbody absorbs the entire radiation incident on it and is a perfect absorber (α= 1) as it is a perfect emitter.

TABLE 1–6

Emissivities of some materials at 300 K

Material	Emissivity
Aluminum foil	0.07
Anodized aluminum	0.82
Polished copper	0.03
Polished gold	0.03
Polished silver	0.02
Polished stainless steel	0.17
Black paint	0.98
White paint	0.90
White paper	0.92–0.97
Asphalt pavement	0.85–0.93
Red brick	0.93–0.96
Human skin	0.95
Wood	0.82–0.92
Soil	0.93–0.96
Water	0.96
Vegetation	0.92–0.96

Kirchhoff's Law of Radiation: the emissivity and the absorptivity of a surface at a given temperature and wavelength are equal.

$$\dot{Q}_{absorbed} = \alpha \dot{Q}_{incident} \qquad (W)$$

where $\dot{Q}_{incident}$ is the rate at which radiation is incident on the surface and a is the absorptivity of the surface.

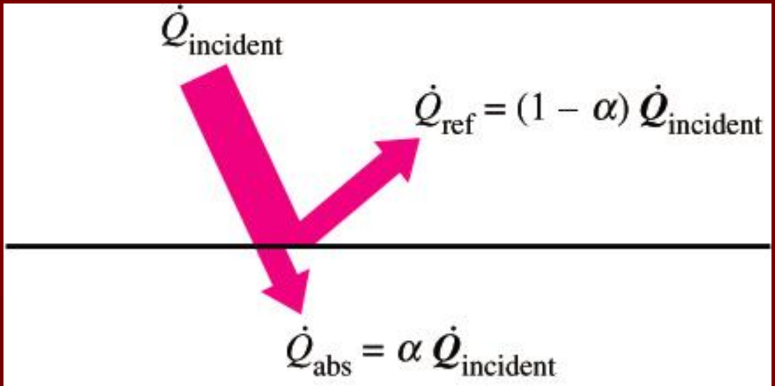
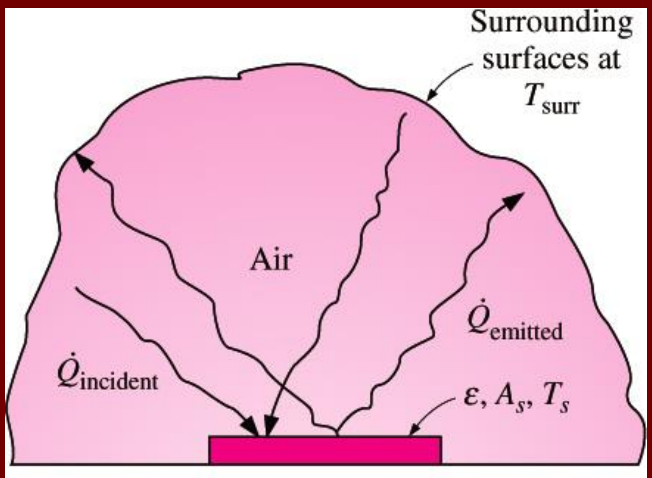


FIGURE 1–36
The absorption of radiation incident on an opaque surface of absorptivity α.

The net rate of radiation heat transfer between two surfaces:

$$\dot{Q}_{rad} = \varepsilon \sigma A_s (T_s^4 - T_{surr}^4) \qquad (W)$$



$$\dot{Q}_{rad} = \varepsilon \sigma A_s (T_s^4 - T_{surr}^4)$$

FIGURE 1–37
Radiation heat transfer between a surface and the surfaces surrounding it.

Combined Heat Transfer Coefficient ($h_{combined}$): including the effects of both convection and radiation.

The *total* heat transfer rate to or from a surface by convection and radiation:

$$\dot{Q}_{total} = h_{combined} A_s (T_s - T_\infty) \qquad \text{(W)}$$

Radiation is usually significant relative to conduction or natural convection, but negligible relative to forced convection. Thus radiation in forced convection applications is usually disregarded, especially when the surfaces involved have low emissivities and low to moderate temperatures.

EXAMPLE 1–9 **Radiation Effect on Thermal Comfort**

It is a common experience to feel "chilly" in winter and "warm" in summer in our homes even when the thermostat setting is kept the same. This is due to the so called "radiation effect" resulting from radiation heat exchange between our bodies and the surrounding surfaces of the walls and the ceiling.

Consider a person standing in a room maintained at 22°C at all times. The inner surfaces of the walls, floors, and the ceiling of the house are observed to be at an average temperature of 10°C in winter and 25°C in summer. Determine the rate of radiation heat transfer between this person and the surrounding surfaces if the exposed surface area and the average outer surface temperature of the person are 1.4 m^2 and 30°C, respectively (Fig. 16–17).

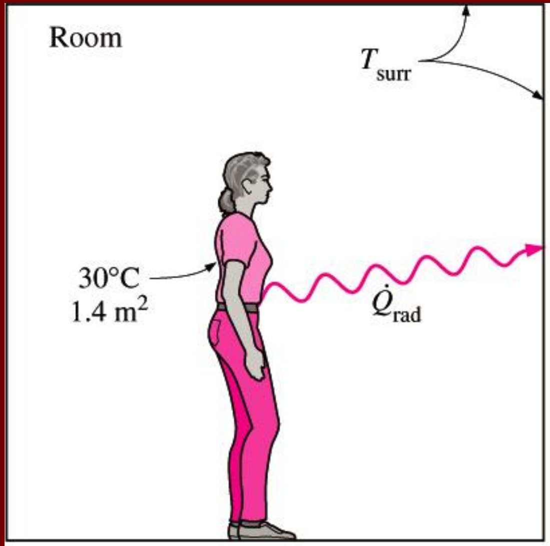


FIGURE 1–38
Schematic for Example 1–9.

SOLUTION The rates of radiation heat transfer between a person and the surrounding surfaces at specified temperatures are to be determined in summer and winter.

Assumptions **1** Steady operating conditions exist. **2** Heat transfer by convection is not considered. **3** The person is completely surrounded by the interior surfaces of the room. **4** The surrounding surfaces are at a uniform temperature.

Properties The emissivity of a person is $\varepsilon = 0.95$ (Table 16–6).

Analysis The net rates of radiation heat transfer from the body to the surrounding walls, ceiling, and floor in winter and summer are

$$\begin{aligned}\dot{Q}_{\text{rad, winter}} &= \varepsilon\sigma A_s (T_s^4 - T_{\text{surr, winter}}^4)\\ &= (0.95)(5.67 \times 10^{-8}\ \text{W/m}^2 \cdot \text{K}^4)(1.4\ \text{m}^2)\\ &\quad \times [(30 + 273)^4 - (10 + 273)^4]\ \text{K}^4\\ &= \mathbf{152\ W}\end{aligned}$$

and

$$\begin{aligned}\dot{Q}_{\text{rad, summer}} &= \varepsilon\sigma A_s (T_s^4 - T_{\text{surr, summer}}^4)\\ &= (0.95)(5.67 \times 10^{-8}\ \text{W/m}^2 \cdot \text{K}^4)(1.4\ \text{m}^2)\\ &\quad \times [(30 + 273)^4 - (25 + 273)^4]\ \text{K}^4\\ &= \mathbf{40.9\ W}\end{aligned}$$

Discussion Note that we must use *absolute temperatures* in radiation calculations. Also note that the rate of heat loss from the person by radiation is almost four times as large in winter than it is in summer, which explains the "chill" we feel in winter even if the thermostat setting is kept the same.

SIMULTANEOUS HEAT TRANSFER MECHANISMS

- Heat transfer is only by conduction in *opaque solids*, but by conduction and radiation in *semitransparent solids.*

- In the absence of radiation, heat transfer through a fluid is either by conduction or convection, depending on the presence of any bulk fluid motion.
- Convection can be viewed as combined conduction and fluid motion, and conduction in a fluid can be viewed as a special case of convection in the absence of any fluid motion.
- Heat transfer through a *vacuum* is by radiation only since conduction or convection requires the presence of a material medium.

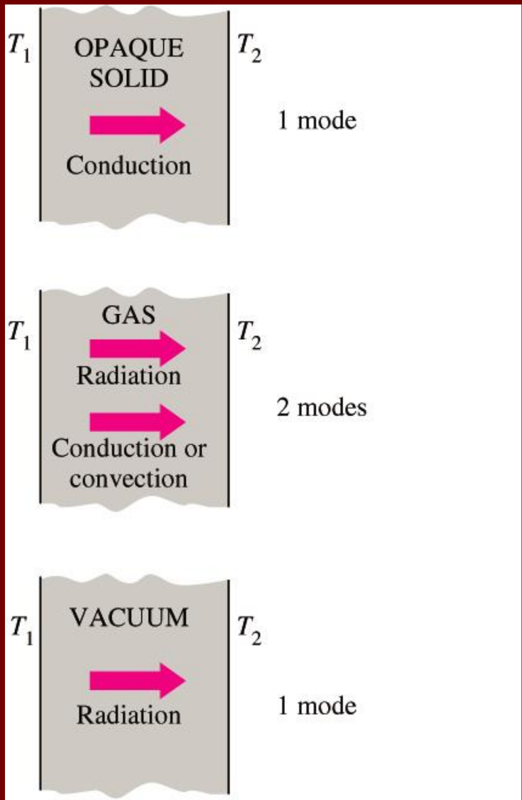


FIGURE 1–39
Although there are three mechanisms of heat transfer, a medium may involve only two of them simultaneously.

EXAMPLE 1–10 **Heat Loss from a Person**

Consider a person standing in a breezy room at 20°C. Determine the total rate of heat transfer from this person if the exposed surface area and the average outer surface temperature of the person are 1.6 m^2 and 29°C, respectively, and the convection heat transfer coefficient is 6 $W/m^2 \cdot °C$ (Fig. 16–19).

SOLUTION The total rate of heat transfer from a person by both convection and radiation to the surrounding air and surfaces at specified temperatures is to be determined.

Assumptions **1** Steady operating conditions exist. **2** The person is completely surrounded by the interior surfaces of the room. **3** The surrounding surfaces are at the same temperature as the air in the room. **4** Heat conduction to the floor through the feet is negligible.

Properties The emissivity of a person is $\varepsilon = 0.95$ (Table 16–6).

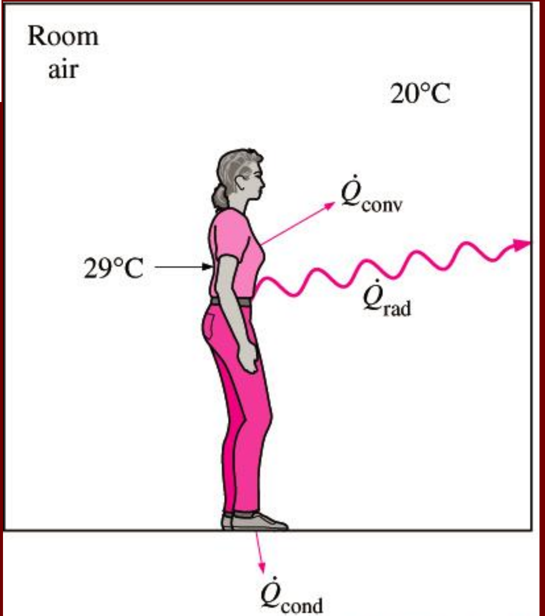


FIGURE 1–40
Heat transfer from the person described in Example 1–10.

Analysis The heat transfer between the person and the air in the room will be by convection (instead of conduction) since it is conceivable that the air in the vicinity of the skin or clothing will warm up and rise as a result of heat transfer from the body, initiating natural convection currents. It appears that the experimentally determined value for the rate of convection heat transfer in this case is 6 W per unit surface area (m^2) per unit temperature difference (in K or °C) between the person and the air away from the person. Thus, the rate of convection heat transfer from the person to the air in the room is

$$\begin{aligned}\dot{Q}_{\text{conv}} &= hA_s(T_s - T_\infty)\\ &= (6\ \text{W/m}^2\cdot{}^\circ\text{C})(1.6\ \text{m}^2)(29-20)^\circ\text{C}\\ &= 86.4\ \text{W}\end{aligned}$$

The person will also lose heat by radiation to the surrounding wall surfaces. We take the temperature of the surfaces of the walls, ceiling, and floor to be equal to the air temperature in this case for simplicity, but we recognize that this does not need to be the case. These surfaces may be at a higher or lower temperature than the average temperature of the room air, depending on the outdoor conditions and the structure of the walls. Considering that air does not intervene with radiation and the person is completely enclosed by the surrounding surfaces, the net rate of radiation heat transfer from the person to the surrounding walls, ceiling, and floor is

$$\begin{aligned}\dot{Q}_{\text{rad}} &= \varepsilon\sigma A_s(T_s^4 - T_{\text{surr}}^4)\\ &= (0.95)(5.67\times10^{-8}\ \text{W/m}^2\cdot\text{K}^4)(1.6\ \text{m}^2)\\ &\quad\times[(29+273)^4-(20+273)^4]\ \text{K}^4\\ &= 81.7\ \text{W}\end{aligned}$$

Note that we must use *absolute* temperatures in radiation calculations. Also note that we used the emissivity value for the skin and clothing at room temperature since the emissivity is not expected to change significantly at a slightly higher temperature.

Then the rate of total heat transfer from the body is determined by adding these two quantities:

$$\dot{Q}_{\text{total}} = \dot{Q}_{\text{conv}} + \dot{Q}_{\text{rad}} = (86.4 + 81.7)\ \text{W} = \mathbf{168.1\ W}$$

Discussion The heat transfer would be much higher if the person were not dressed since the exposed surface temperature would be higher. Thus, an important function of the clothes is to serve as a barrier against heat transfer.

In these calculations, heat transfer through the feet to the floor by conduction, which is usually very small, is neglected. Heat transfer from the skin by perspiration, which is the dominant mode of heat transfer in hot environments, is not considered here.

EXAMPLE 1–11 Heat Transfer between Two Isothermal Plates

Consider steady heat transfer between two large parallel plates at constant temperatures of $T_1 = 300$ K and $T_2 = 200$ K that are $L = 1$ cm apart, as shown in Fig. 16–20. Assuming the surfaces to be black (emissivity $\varepsilon = 1$), determine the rate of heat transfer between the plates per unit surface area assuming the gap between the plates is (*a*) filled with atmospheric air, (*b*) evacuated, (*c*) filled with urethane insulation, and (*d*) filled with superinsulation that has an apparent thermal conductivity of 0.00002 W/m · °C.

SOLUTION The total rate of heat transfer between two large parallel plates at specified temperatures is to be determined for four different cases.

Assumptions **1** Steady operating conditions exist. **2** There are no natural convection currents in the air between the plates. **3** The surfaces are black and thus $\varepsilon = 1$.

Properties The thermal conductivity at the average temperature of 250 K is $k = 0.0219$ W/m · °C for air (Table A-22), 0.026 W/m · °C for urethane insulation (Table A-28), and 0.00002 W/m · °C for the superinsulation.

Analysis (*a*) The rates of conduction and radiation heat transfer between the plates through the air layer are

$$\dot{Q}_{\text{cond}} = kA\frac{T_1 - T_2}{L} = (0.0219\ \text{W/m}\cdot{}^\circ\text{C})(1\ \text{m}^2)\frac{(300 - 200)^\circ\text{C}}{0.01\ \text{m}} = 219\ \text{W}$$

and

$$\begin{aligned}\dot{Q}_{\text{rad}} &= \varepsilon\sigma A(T_1^4 - T_2^4)\\ &= (1)(5.67\times10^{-8}\ \text{W/m}^2\cdot\text{K}^4)(1\ \text{m}^2)[(300\ \text{K})^4 - (200\ \text{K})^4] = 368\ \text{W}\end{aligned}$$

Therefore,

$$\dot{Q}_{\text{total}} = \dot{Q}_{\text{cond}} + \dot{Q}_{\text{rad}} = 219 + 368 = 587\ \text{W}$$

The heat transfer rate in reality will be higher because of the natural convection currents that are likely to occur in the air space between the plates.

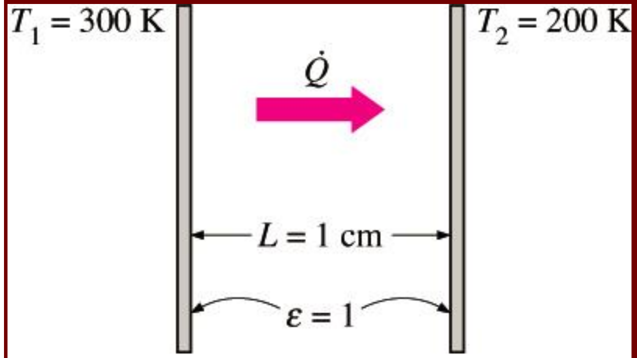


FIGURE 1–41
Schematic for Example 1–11.

(*b*) When the air space between the plates is evacuated, there will be no conduction or convection, and the only heat transfer between the plates will be by radiation. Therefore,

$$\dot{Q}_{\text{total}} = \dot{Q}_{\text{rad}} = \mathbf{368\ W}$$

(*c*) An opaque solid material placed between two plates blocks direct radiation heat transfer between the plates. Also, the thermal conductivity of an insulating material accounts for the radiation heat transfer that may be occurring through the voids in the insulating material. The rate of heat transfer through the urethane insulation is

$$\dot{Q}_{\text{total}} = \dot{Q}_{\text{cond}} = kA\frac{T_1 - T_2}{L} = (0.026\ \text{W/m}\cdot{}^\circ\text{C})(1\ \text{m}^2)\frac{(300 - 200)^\circ\text{C}}{0.01\ \text{m}} = \mathbf{260\ W}$$

Note that heat transfer through the urethane material is less than the heat transfer through the air determined in (*a*), although the thermal conductivity of the insulation is higher than that of air. This is because the insulation blocks the radiation whereas air transmits it.

(*d*) The layers of the superinsulation prevent any direct radiation heat transfer between the plates. However, radiation heat transfer between the sheets of superinsulation does occur, and the apparent thermal conductivity of the superinsulation accounts for this effect. Therefore,

$$\dot{Q}_{\text{total}} = kA\frac{T_1 - T_2}{L} = (0.00002\ \text{W/m}\cdot{}^\circ\text{C})(1\ \text{m}^2)\frac{(300 - 200)^\circ\text{C}}{0.01\ \text{m}} = \mathbf{0.2\ W}$$

which is $\frac{1}{1840}$ of the heat transfer through the vacuum. The results of this example are summarized in Fig. 16–21 to put them into perspective.

Discussion This example demonstrates the effectiveness of superinsulations, which are discussed in Chap. 17, and explains why they are the insulation of choice in critical applications despite their high cost.

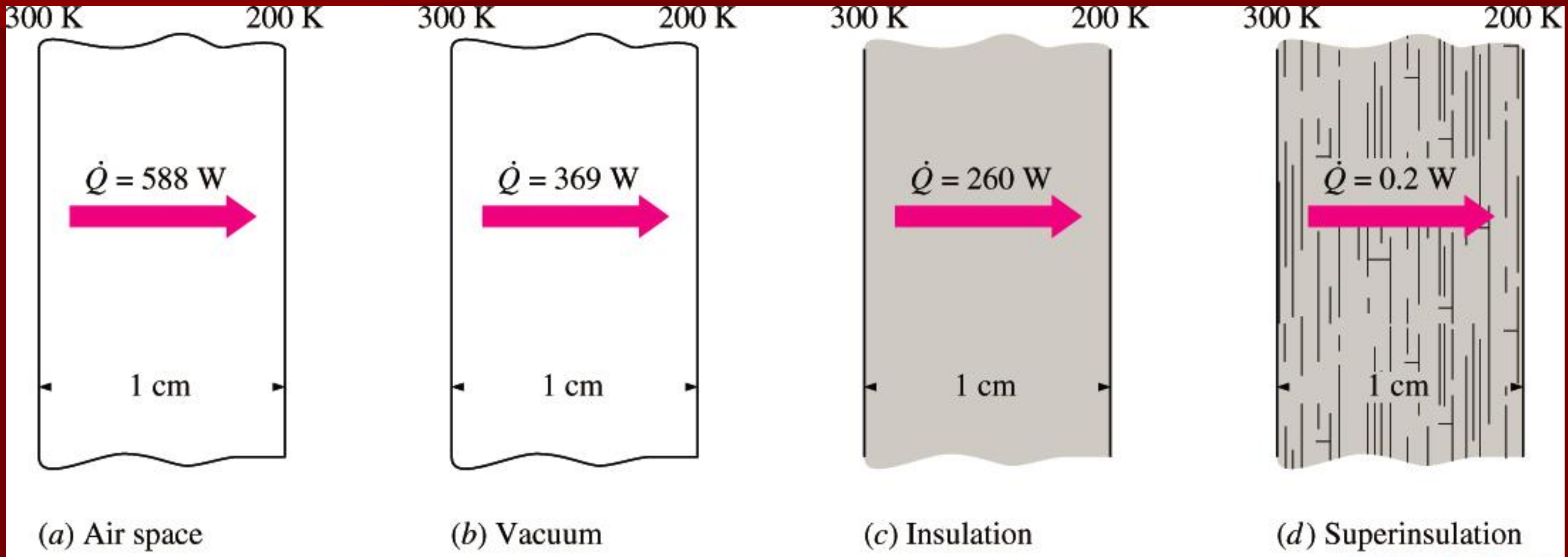


FIGURE 1–42

Different ways of reducing heat transfer between two isothermal plates, and their effectiveness.

EXAMPLE 1–12 Heat Transfer in Conventional and Microwave Ovens

The fast and efficient cooking of microwave ovens made them one of the essential appliances in modern kitchens (Fig. 16–22). Discuss the heat transfer mechanisms associated with the cooking of a chicken in microwave and conventional ovens, and explain why cooking in a microwave oven is more efficient.

SOLUTION Food is cooked in a microwave oven by absorbing the electromagnetic radiation energy generated by the microwave tube, called the magnetron. The radiation emitted by the magnetron is not thermal radiation, since its emission is not due to the temperature of the magnetron; rather, it is due to the conversion of electrical energy into electromagnetic radiation at a specified wavelength. The wavelength of the microwave radiation is such that it is *reflected* by metal surfaces; *transmitted* by the cookware made of glass, ceramic, or plastic; and *absorbed* and converted to internal energy by food (especially the water, sugar, and fat) molecules.

In a microwave oven, the *radiation* that strikes the chicken is absorbed by the skin of the chicken and the outer parts. As a result, the temperature of the chicken at and near the skin rises. Heat is then *conducted* toward the inner parts of the chicken from its outer parts. Of course, some of the heat absorbed by the outer surface of the chicken is lost to the air in the oven by *convection.*

In a conventional oven, the air in the oven is first heated to the desired temperature by the electric or gas heating element. This preheating may take several minutes. The heat is then transferred from the air to the skin of the chicken by *natural convection* in most ovens or by *forced convection* in the newer convection ovens that utilize a fan. The air motion in convection ovens increases the convection heat transfer coefficient and thus decreases the cooking time. Heat is then *conducted* toward the inner parts of the chicken from its outer parts as in microwave ovens.

Microwave ovens replace the slow convection heat transfer process in conventional ovens by the instantaneous radiation heat transfer. As a result, microwave ovens transfer energy to the food at full capacity the moment they are turned on, and thus they cook faster while consuming less energy.

FIGURE 1–43
A chicken being cooked in a microwave oven (Example 1–12).

EXAMPLE 1–13 **Heating of a Plate by Solar Energy**

A thin metal plate is insulated on the back and exposed to solar radiation at the front surface (Fig. 16–23). The exposed surface of the plate has an absorptivity of 0.6 for solar radiation. If solar radiation is incident on the plate at a rate of 700 W/m^2 and the surrounding air temperature is 25°C, determine the surface temperature of the plate when the heat loss by convection and radiation equals the solar energy absorbed by the plate. Assume the combined convection and radiation heat transfer coefficient to be 50 $W/m^2 \cdot °C$.

SOLUTION The back side of the thin metal plate is insulated and the front side is exposed to solar radiation. The surface temperature of the plate is to be determined when it stabilizes.

Assumptions 1 Steady operating conditions exist. **2** Heat transfer through the insulated side of the plate is negligible. **3** The heat transfer coefficient remains constant.

Properties The solar absorptivity of the plate is given to be $\alpha = 0.6$.

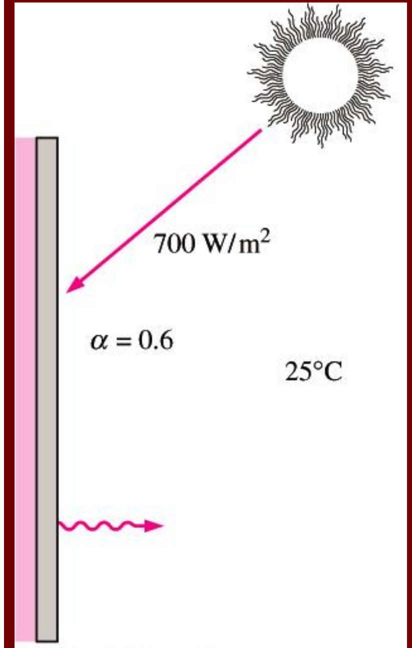


FIGURE 1–44
Schematic for Example 1–13.

Analysis The absorptivity of the plate is 0.6, and thus 60 percent of the solar radiation incident on the plate will be absorbed continuously. As a result, the temperature of the plate will rise, and the temperature difference between the plate and the surroundings will increase. This increasing temperature difference will cause the rate of heat loss from the plate to the surroundings to increase. At some point, the rate of heat loss from the plate will equal the rate of solar energy absorbed, and the temperature of the plate will no longer change. The temperature of the plate when steady operation is established is determined from

$$\dot{E}_{\text{gained}} = \dot{E}_{\text{lost}} \qquad \text{or} \qquad \alpha A_s\, \dot{q}_{\text{incident, solar}} = h_{\text{combined}} A_s\,(T_s - T_\infty)$$

Solving for T_s and substituting, the plate surface temperature is determined to be

$$T_s = T_\infty + \alpha\frac{\dot{q}_{\text{incident, solar}}}{h_{\text{combined}}} = 25°\text{C} + \frac{0.6 \times (700\ \text{W/m}^2)}{50\ \text{W/m}^2 \cdot °\text{C}} = 33.4°\text{C}$$

Discussion Note that the heat losses will prevent the plate temperature from rising above 33.4°C. Also, the combined heat transfer coefficient accounts for the effects of both convection and radiation, and thus it is very convenient to use in heat transfer calculations when its value is known with reasonable accuracy.

Concluding Points

- Differences between Thermodynamics and Heat Transfer?
- Basic Concepts of Thermodynamics
- Heat Transfer Modes?
- Fourier's Law of Heat Conduction?
- Thermal Conductivity and Thermal Diffusivity?
- Natural (or Free) and Forced Convection?
- Convection and Newton's Law of Cooling?
- Radiation and Stefan–Boltzman Law?
- Blackbody and Emissivity?
- Kirchhoff's Law of Radiation?
- Combined Heat Transfer Coefficient?
- Simultaneous Heat Transfer Mechanisms?

HEAT AND MASS TRANSFER

Heat Conduction Equation

Outline

- Introduction
- One-dimensional heat conduction equation
- General heat conduction equation
- Boundary and initial conditions
- Solution of one-dimensional heat conduction problems
- Heat generation in a solid
- Variable thermal conductivity
- Conclusions

Objectives

- To understand multidimensionality and time dependence of heat transfer, and the conditions under which a heat transfer problem can be approximated as being one-dimensional.
- To obtain the differential equation of heat conduction in various coordinate systems, and simplify it for steady one-dimensional case.
- To identify the thermal conditions on surfaces, and express them mathematically as boundary and initial conditions.
- To solve one-dimensional heat conduction problems and obtain the temperature distributions within a medium and the heat flux.
- To analyze one-dimensional heat conduction in solids that involve heat generation.
- To evaluate heat conduction in solids with temperature-dependent thermal conductivity.

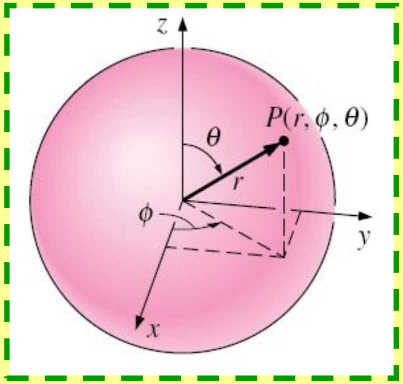
Introduction

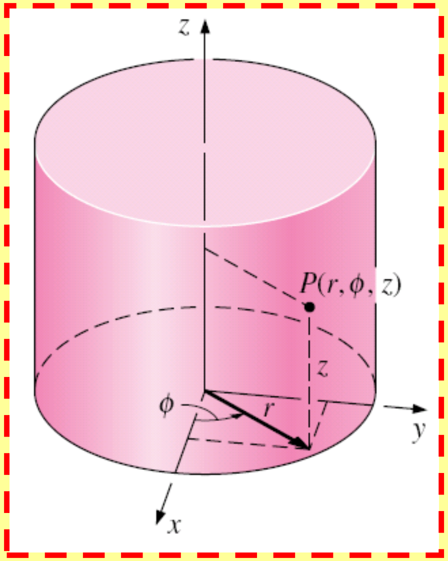
- Although heat transfer and temperature are closely related, they are of a different nature.
- Temperature has only magnitude

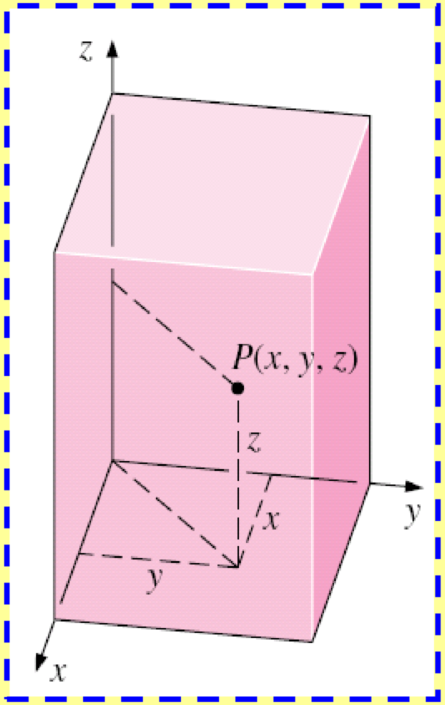
  it is a *scalar* quantity.
- Heat transfer has direction as well as magnitude

  it is a *vector* quantity.
- We work with a coordinate system and indicate direction with plus or minus signs.

- The driving force for any form of heat transfer is the *temperature difference.*
- The larger the temperature difference, the larger the rate of heat transfer.
- Three prime coordinate systems:
  - rectangular (T(x, y, z, t)) ,
  - cylindrical (T(r, f, z, t)),
  - spherical (T(r, f, q, t)).



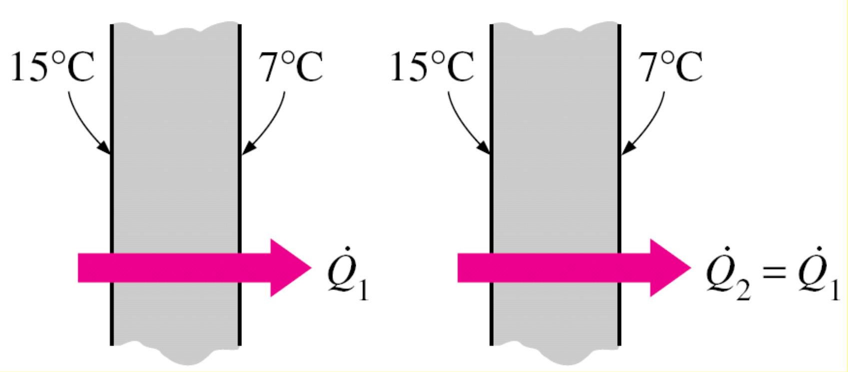




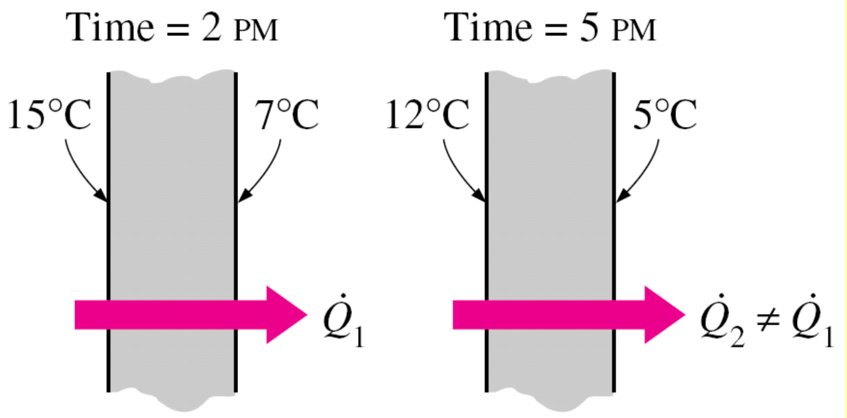
Classification of conduction heat transfer problems:

- steady versus transient heat transfer,
- multidimensional heat transfer,
- heat generation.

- ***Steady*** implies *no change* with time at any point within the medium



- ***Transient*** implies *variation with time* or *time dependence*

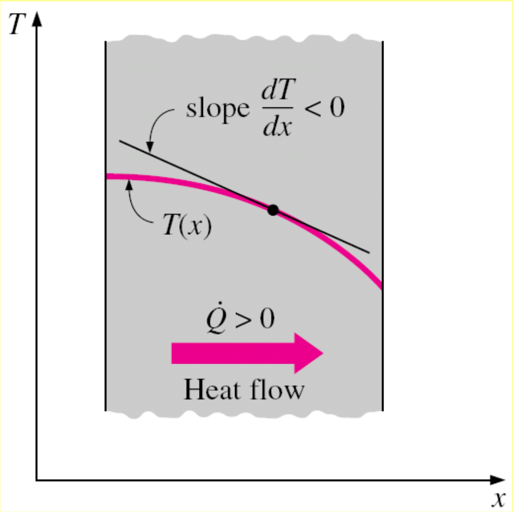


Multidimensional Heat Transfer

- Heat transfer problems are also classified as being:
  - *one-dimensional*,
  - *two dimensional*,
  - *three-dimensional*.
- In the most general case, heat transfer through a medium is **three-dimensional**. However, some problems can be classified as two- or one-dimensional depending on the relative magnitudes of heat transfer rates in different directions and the level of accuracy desired.
- The rate of heat conduction through a medium in a specified direction (say, in the *x*-direction) is expressed by **Fourier's law of heat conduction** for one-dimensional heat conduction as:

$$\dot{Q}_{\text{cond}} = -kA\frac{dT}{dx} \qquad \text{(W)}$$

Heat is conducted in the direction of decreasing temperature, and thus the temperature gradient is negative when heat is conducted in the positive *x* -direction.



Multidimensional Heat Transfer

- **One-dimensional** if the temperature in the medium varies in one direction only and thus heat is transferred in one direction, and the variation of temperature and thus heat transfer in other directions are negligible or zero.
- **Two-dimensional** if the temperature in a medium, in some cases, varies mainly in two primary directions, and the variation of temperature in the third direction (and thus heat transfer in that direction) is negligible.

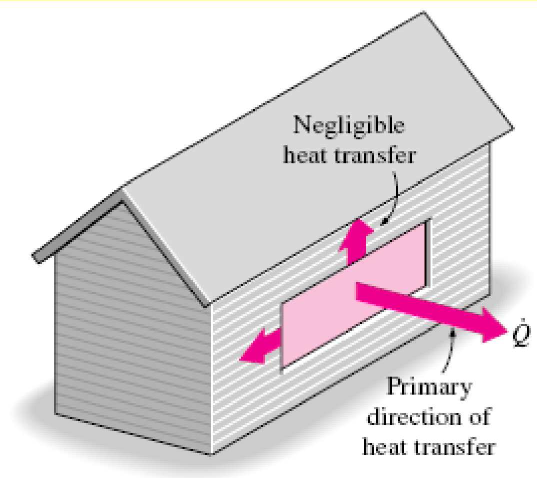


FIGURE 2–6
Heat transfer through the window of a house can be taken to be one-dimensional.

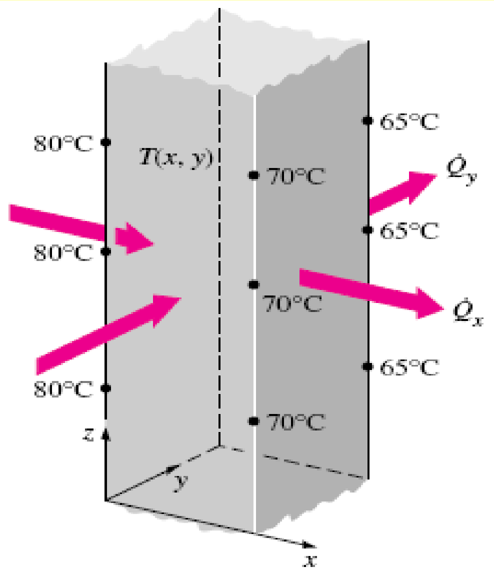


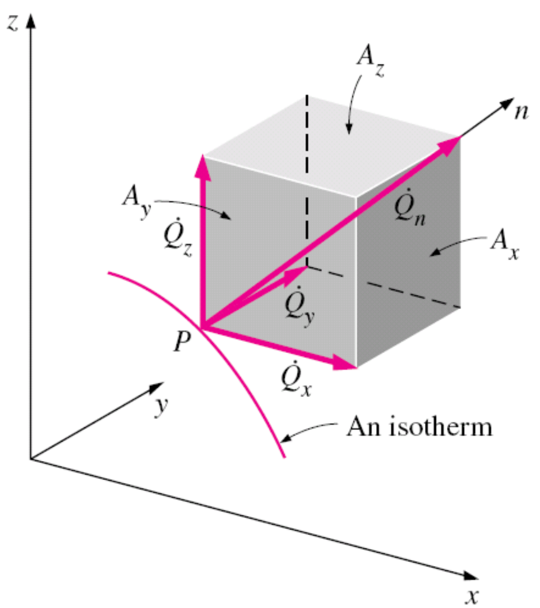
FIGURE 2–5
Two-dimensional heat transfer in a long rectangular bar.

General Relation for Fourier's Law of Heat Conduction

- The heat flux vector at a point *P* on the surface of the figure must be perpendicular to the surface, and it must point in the direction of decreasing temperature
- If *n* is the normal of the isothermal

surface at point *P*, the rate of heat conduction at that point can be expressed by Fourier's law as

$$\dot{Q}_n = -kA\frac{dT}{dn} \qquad \text{(W)}$$



In rectangular coordinates, the heat conduction vector can be expressed in terms of its components as

$$\vec{\dot{Q}}_n = \dot{Q}_x\vec{i} + \dot{Q}_y\vec{j} + \dot{Q}_z\vec{k}$$

which can be determined from Fourier's law as

$$\begin{cases} \dot{Q}_x = -kA_x\dfrac{\partial T}{\partial x} \\ \dot{Q}_y = -kA_y\dfrac{\partial T}{\partial y} \\ \dot{Q}_z = -kA_z\dfrac{\partial T}{\partial z} \end{cases}$$

Heat Generation

- Examples:
  - electrical energy being converted to heat at a rate of I^2R,
  - fuel elements of nuclear reactors,
  - exothermic chemical reactions.
- Heat generation is a *volumetric phenomenon.*
- The rate of heat generation units : W/m^3 or Btu/h · ft^3.
- The rate of heat generation in a medium may vary with time as well as position within the medium.
- The *total* rate of heat generation in a medium of volume *V* can be determined from

$$\dot{E}_{gen} = \int_V \dot{e}_{gen} dV \qquad \text{(W)}$$

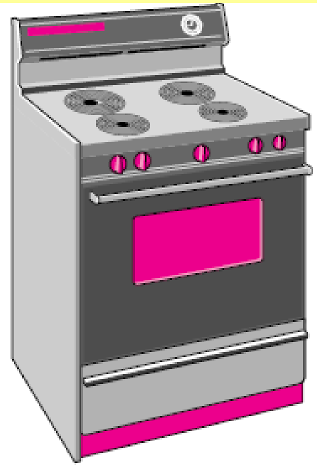


FIGURE 2–9
Heat is generated in the heating coils of an electric range as a result of the conversion of electrical energy to heat.

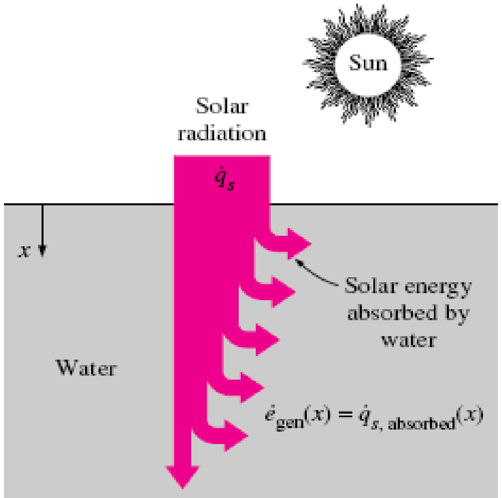


FIGURE 2–10
The absorption of solar radiation by water can be treated as heat generation.

EXAMPLE 2–2 Heat Generation in a Hair Dryer

The resistance wire of a 1200-W hair dryer is 80 cm long and has a diameter of $D = 0.3$ cm (Fig. 2–12). Determine the rate of heat generation in the wire per unit volume, in W/cm³, and the heat flux on the outer surface of the wire as a result of this heat generation.

SOLUTION The power consumed by the resistance wire of a hair dryer is given. The heat generation and the heat flux are to be determined.

Assumptions Heat is generated uniformly in the resistance wire.

Analysis A 1200-W hair dryer converts electrical energy into heat in the wire at a rate of 1200 W. Therefore, the rate of heat generation in a resistance wire is equal to the power consumption of a resistance heater. Then the rate of heat generation in the wire per unit volume is determined by dividing the total rate of heat generation by the volume of the wire,

$$\dot{e}_{gen} = \frac{\dot{E}_{gen}}{V_{wire}} = \frac{\dot{E}_{gen}}{(\pi D^2/4)L} = \frac{1200 \text{ W}}{[\pi(0.3 \text{ cm})^2/4](80 \text{ cm})} = 212 \text{ W/cm}^3$$

Similarly, heat flux on the outer surface of the wire as a result of this heat generation is determined by dividing the total rate of heat generation by the surface area of the wire,

$$\dot{Q}_s = \frac{\dot{E}_{gen}}{A_{wire}} = \frac{\dot{E}_{gen}}{\pi DL} = \frac{1200 \text{ W}}{\pi(0.3 \text{ cm})(80 \text{ cm})} = 15.9 \text{ W/cm}^2$$

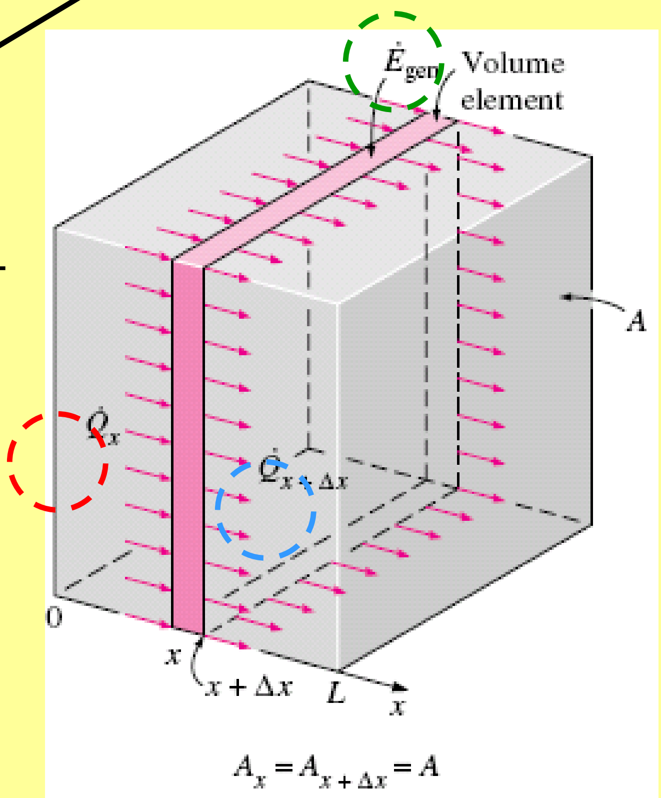
Discussion Note that heat generation is expressed per unit volume in W/cm³ or Btu/h · ft³, whereas heat flux is expressed per unit surface area in W/cm² or Btu/h · ft².

FIGURE 2–12
Schematic for Example 2–2.

One-Dimensional Heat Conduction Equation - Plane Wall

$$\begin{pmatrix}\text{Rate of heat}\\ \text{conduction}\\ \text{at } x\end{pmatrix} - \begin{pmatrix}\text{Rate of heat}\\ \text{conduction}\\ \text{at } x+\Delta x\end{pmatrix} + \begin{pmatrix}\text{Rate of heat}\\ \text{generation inside}\\ \text{the element}\end{pmatrix} = \begin{pmatrix}\text{Rate of change of}\\ \text{the energy content}\\ \text{of the element}\end{pmatrix}$$

$$\dot{Q}_x - \dot{Q}_{x+\Delta x} + \dot{E}_{gen,element} = \frac{\Delta E_{element}}{\Delta t}$$



$$\boxed{\dot{Q}_x - \dot{Q}_{x+\Delta x}} + \boxed{\dot{E}_{gen,element}} = \boxed{\frac{\Delta E_{element}}{\Delta t}}$$

- The change in the energy content and the rate of heat generation can be expressed as

$$\begin{cases} \boxed{\Delta E_{element}} = E_{t+\Delta t} - E_t = mc\left(T_{t+\Delta t} - T_t\right) = \boxed{\rho c A \Delta x\left(T_{t+\Delta t} - T_t\right)} \\ \boxed{\dot{E}_{gen,element}} = \dot{e}_{gen} V_{element} = \boxed{\dot{e}_{gen} A \Delta x} \end{cases}$$

- Substituting into above equation, we get

$$\dot{Q}_x - \dot{Q}_{x+\Delta x} + \dot{e}_{gen} A \Delta x = \rho c A \Delta x \frac{T_{t+\Delta t} - T_t}{\Delta t}$$

- Dividing by $A\Delta x$, taking the limit as $\Delta x \rightarrow 0$ and $\Delta t \rightarrow 0$, and from Fourier's law:

$$\frac{1}{A}\frac{\partial}{\partial x}\left(kA\frac{\partial T}{\partial x}\right) + \dot{e}_{gen} = \rho c \frac{\partial T}{\partial t}$$

The area A is constant for a plane wall → the one dimensional transient heat conduction equation in a plane wall is

Variable conductivity:

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\dot{e}_{gen}=\rho c\frac{\partial T}{\partial t}$$

Constant conductivity:

$$\frac{\partial^2 T}{\partial x^2}+\frac{\dot{e}_{gen}}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t} \quad ; \quad \alpha=\frac{k}{\rho c}$$

The one-dimensional conduction equation may be reduces to the following forms under special conditions

1) Steady-state:

$$\frac{d^2T}{dx^2}+\frac{\dot{e}_{gen}}{k}=0$$

2) Transient, no heat generation:

$$\frac{\partial^2 T}{\partial x^2}=\frac{1}{\alpha}\frac{\partial T}{\partial t}$$

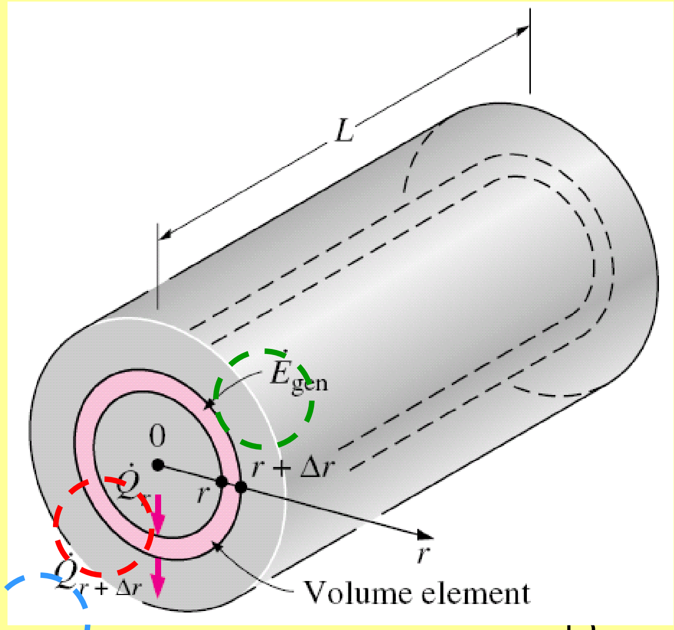
3) Steady-state, no heat generation:

$$\frac{d^2T}{dx^2}=0$$

One-Dimensional Heat Conduction Equation - Long Cylinder

(Rate of heat conduction at *r*) - (Rate of heat conduction at *r*+*Δr*) + (Rate of heat generation inside the element) = (Rate of change of the energy content of the element)

$$\dot{Q}_r - \dot{Q}_{r+\Delta r} + \dot{E}_{gen,element} = \frac{\Delta E_{element}}{\Delta t}$$



$$\dot{Q}_r - \dot{Q}_{r+\Delta r} + \dot{E}_{gen,element} = \frac{\Delta E_{element}}{\Delta t}$$

- The change in the energy content and the rate of heat generation can be expressed as

$$\begin{cases} \Delta E_{element} = E_{t+\Delta t} - E_t = mc\left(T_{t+\Delta t} - T_t\right) = \rho c A \Delta r \left(T_{t+\Delta t} - T_t\right) \\ \dot{E}_{gen,element} = \dot{e}_{gen} V_{element} = \dot{e}_{gen} A \Delta r \end{cases}$$

- Substituting into Eq. 2–18, we get

$$\dot{Q}_r - \dot{Q}_{r+\Delta r} + \dot{e}_{gen} A \Delta r = \rho c A \Delta r \frac{T_{t+\Delta t} - T_t}{\Delta t}$$

- Dividing by $A\Delta r$, taking the limit as $\Delta r \rightarrow 0$ and $\Delta t \rightarrow 0$, and from Fourier's law:

$$\frac{1}{A}\frac{\partial}{\partial r}\left(kA\frac{\partial T}{\partial r}\right) + \dot{e}_{gen} = \rho c \frac{\partial T}{\partial t}$$

Noting that the area varies with the independent variable r according to $A=2\pi rL$, the one dimensional transient heat conduction equation in a long cylinder becomes

Variable conductivity: $$\frac{1}{r}\frac{\partial}{\partial r}\left(rk\frac{\partial T}{\partial r}\right)+\dot{e}_{gen}=\rho c\frac{\partial T}{\partial t}$$

Constant conductivity: $$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{\dot{e}_{gen}}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t}$$

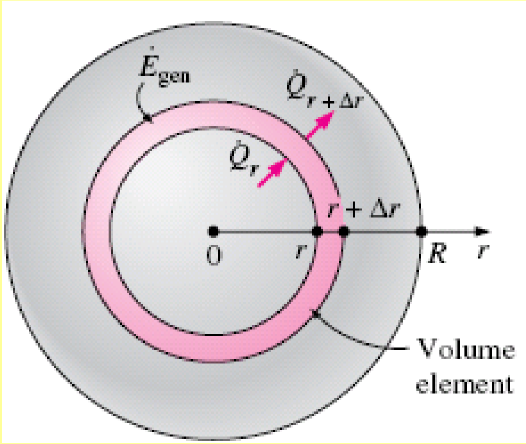
The one-dimensional conduction equation may be reduces to the following forms under special conditions

1) Steady-state: $$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right)+\frac{\dot{e}_{gen}}{k}=0$$

2) Transient, no heat generation: $$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)=\frac{1}{\alpha}\frac{\partial T}{\partial t}$$

3) Steady-state, no heat generation: $$\frac{d}{dr}\left(r\frac{dT}{dr}\right)=0$$

One-Dimensional Heat Conduction Equation - Sphere



Variable conductivity: $$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 k \frac{\partial T}{\partial r}\right) + \dot{e}_{gen} = \rho c \frac{\partial T}{\partial t}$$

Constant conductivity: $$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) + \frac{\dot{e}_{gen}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

EXAMPLE 2–4 Heat Conduction in a Resistance Heater

A 2-kW resistance heater wire with thermal conductivity $k = 15$ W/m · K, diameter $D = 0.4$ cm, and length $L = 50$ cm is used to boil water by immersing it in water (Fig. 2–19). Assuming the variation of the thermal conductivity of the wire with temperature to be negligible, obtain the differential equation that describes the variation of the temperature in the wire during steady operation.

SOLUTION The resistance wire of a water heater is considered. The differential equation for the variation of temperature in the wire is to be obtained.

Analysis The resistance wire can be considered to be a very long cylinder since its length is more than 100 times its diameter. Also, heat is generated uniformly in the wire and the conditions on the outer surface of the wire are uniform. Therefore, it is reasonable to expect the temperature in the wire to vary in the radial r direction only and thus the heat transfer to be one-dimensional. Then we have $T = T(r)$ during steady operation since the temperature in this case depends on r only.

The rate of heat generation in the wire per unit volume can be determined from

$$\dot{e}_{gen} = \frac{\dot{E}_{gen}}{V_{wire}} = \frac{\dot{E}_{gen}}{(\pi D^2/4)L} = \frac{2000 \text{ W}}{[\pi(0.004 \text{ m})^2/4](0.5 \text{ m})} = 0.318 \times 10^9 \text{ W/m}^3$$

Noting that the thermal conductivity is given to be constant, the differential equation that governs the variation of temperature in the wire is simply Eq. 2–27,

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) + \frac{\dot{e}_{gen}}{k} = 0$$

which is the steady one-dimensional heat conduction equation in cylindrical coordinates for the case of constant thermal conductivity.

Discussion Note again that the conditions at the surface of the wire have no effect on the differential equation.

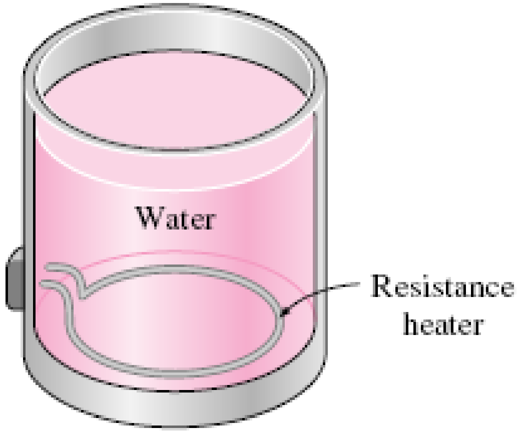
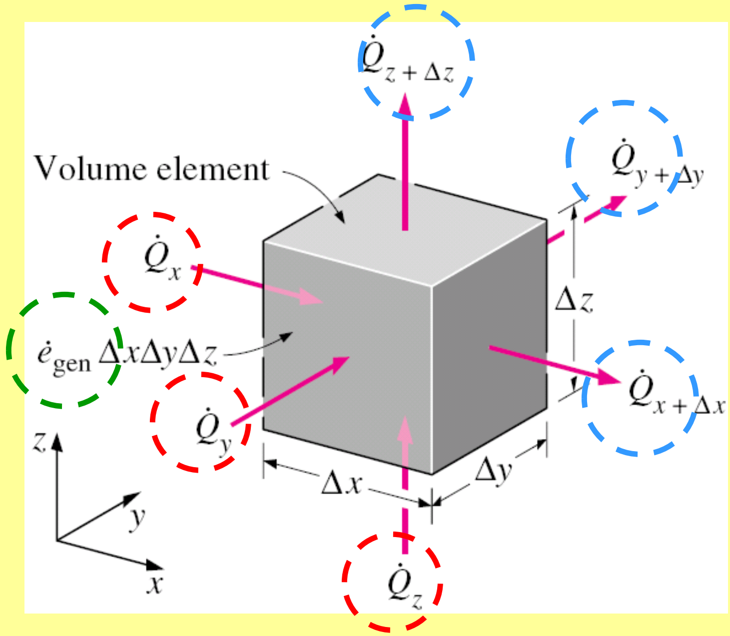


FIGURE 2–19
Schematic for Example 2–4.

General Heat Conduction Equation



Rate of heat conduction at *x*, *y*, and *z* - **Rate of heat conduction at *x+Δx*, *y+Δy*, *and z+Δz*** + **Rate of heat generation inside the element** = **Rate of change of the energy content of the element**

$$\underbrace{\dot{Q}_x + \dot{Q}_y + \dot{Q}_z}_{} - \underbrace{\dot{Q}_{x+\Delta x} - \dot{Q}_{y+\Delta y} - \dot{Q}_{z+\Delta z}}_{} + E_{gen,element} = \frac{\Delta E_{element}}{\Delta t}$$

Repeating the mathematical approach used for the one-dimensional heat conduction the three-dimensional heat conduction equation is determined to be

Constant conductivity:

$$\underbrace{\overbrace{\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}}^{\textbf{Two-dimensional}}+\frac{\partial^2 T}{\partial z^2}}_{\textbf{Three-dimensional}}+\frac{\dot{e}_{gen}}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t}$$

1) Steady-state:

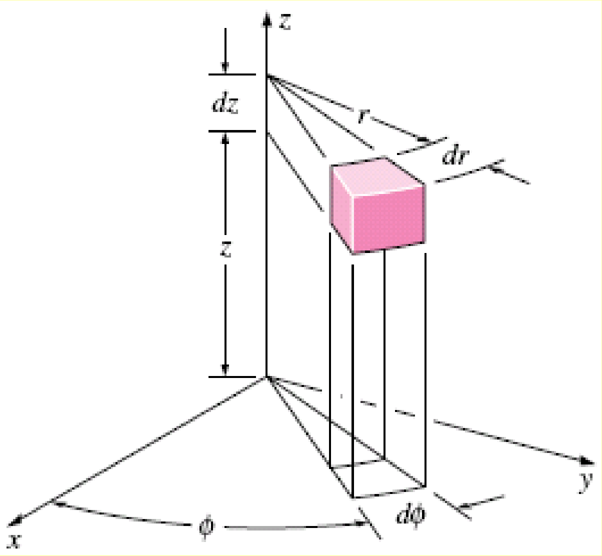
$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}+\frac{\dot{e}_{gen}}{k}=0$$

2) Transient, no heat generation:

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}=\frac{1}{\alpha}\frac{\partial T}{\partial t}$$

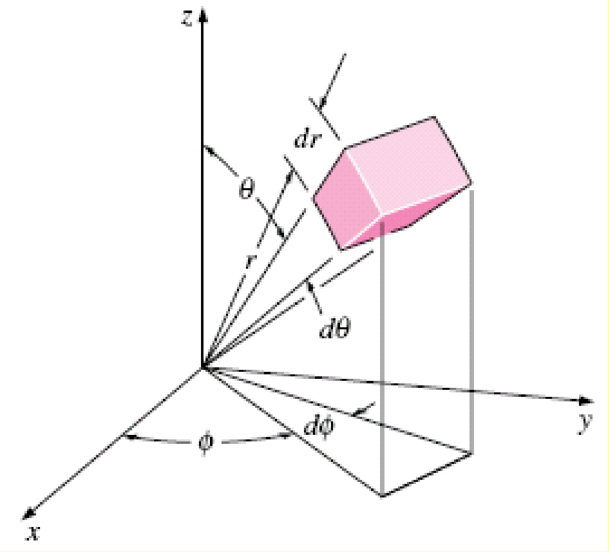
3) Steady-state, no heat generation:

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}=0$$



Cylindrical coordinates

$$\frac{1}{r}\frac{\partial}{\partial r}\left(rk\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial T}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\dot{e}_{gen}=\rho c\frac{\partial T}{\partial t}$$



Spherical coordinates

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(kr^2\frac{\partial T}{\partial r}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(k\sin\theta\frac{\partial T}{\partial \theta}\right)+\dot{e}_{gen}=\rho c\frac{\partial T}{\partial t}$$

EXAMPLE 2–6 Heat Conduction in a Short Cylinder

A short cylindrical metal billet of radius R and height h is heated in an oven to a temperature of 600°F throughout and is then taken out of the oven and allowed to cool in ambient air at $T_\infty = 65°F$ by convection and radiation. Assuming the billet is cooled uniformly from all outer surfaces and the variation of the thermal conductivity of the material with temperature is negligible, obtain the differential equation that describes the variation of the temperature in the billet during this cooling process.

SOLUTION A short cylindrical billet is cooled in ambient air. The differential equation for the variation of temperature is to be obtained.

Analysis The billet shown in Fig. 2–25 is initially at a uniform temperature and is cooled uniformly from the top and bottom surfaces in the z-direction as well as the lateral surface in the radial r-direction. Also, the temperature at any point in the ball changes with time during cooling. Therefore, this is a two-dimensional transient heat conduction problem since the temperature within the billet changes with the radial and axial distances r and z and with time t. That is, $T = T(r, z, t)$.

The thermal conductivity is given to be constant, and there is no heat generation in the billet. Therefore, the differential equation that governs the variation of temperature in the billet in this case is obtained from Eq. 2–43 by setting the heat generation term and the derivatives with respect to ϕ equal to zero. We obtain

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) = \rho c\frac{\partial T}{\partial t}$$

In the case of constant thermal conductivity, it reduces to

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\partial^2 T}{\partial z^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

which is the desired equation.

Discussion Note that the boundary and initial conditions have no effect on the differential equation.

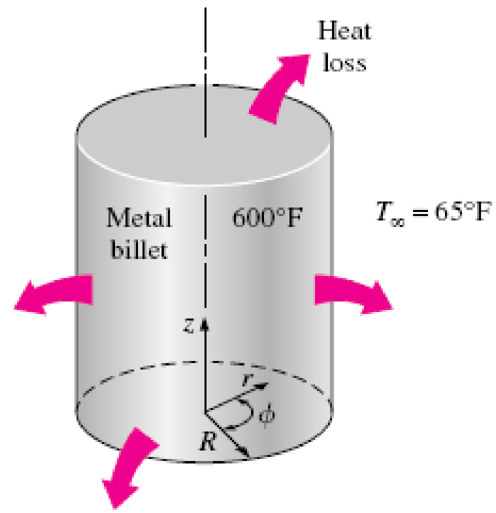


FIGURE 2–25
Schematic for Example 2–6.

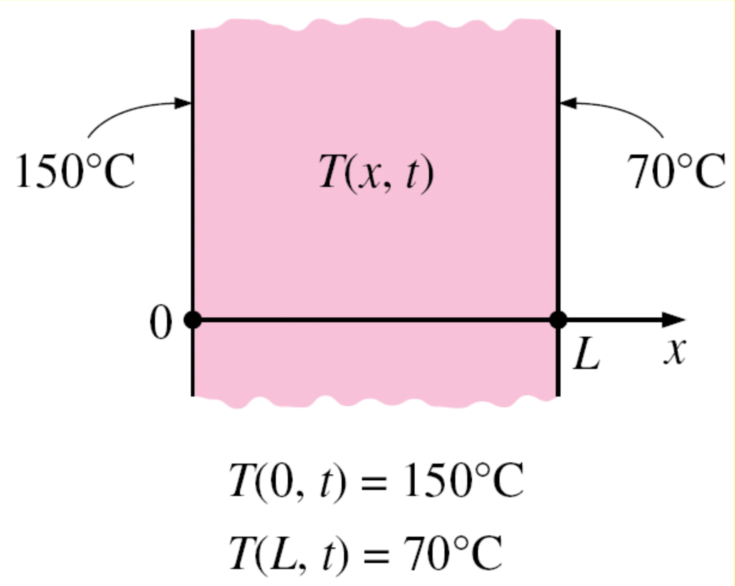
Boundary and Initial Conditions

- Specified Temperature Boundary Condition
- Specified Heat Flux Boundary Condition
- Convection Boundary Condition
- Radiation Boundary Condition
- Interface Boundary Conditions
- Generalized Boundary Conditions

Specified Temperature Boundary Condition

For one-dimensional heat transfer through a plane wall of thickness L, for example, the specified temperature boundary conditions can be expressed as

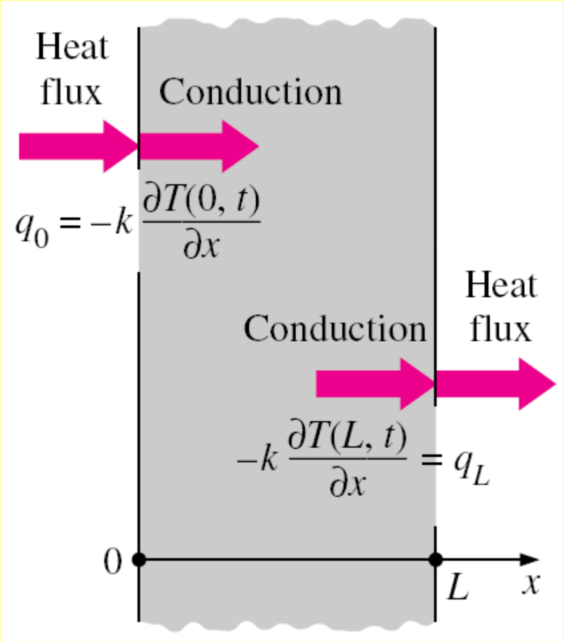
$$T(0, t) = T_1$$
$$T(L, t) = T_2$$



The specified temperatures can be constant, which is the case for steady heat conduction, or may vary with time.

Specified Heat Flux Boundary Condition

The heat flux in the positive *x*-direction anywhere in the medium, including the boundaries, can be expressed by *Fourier's law* of heat conduction as

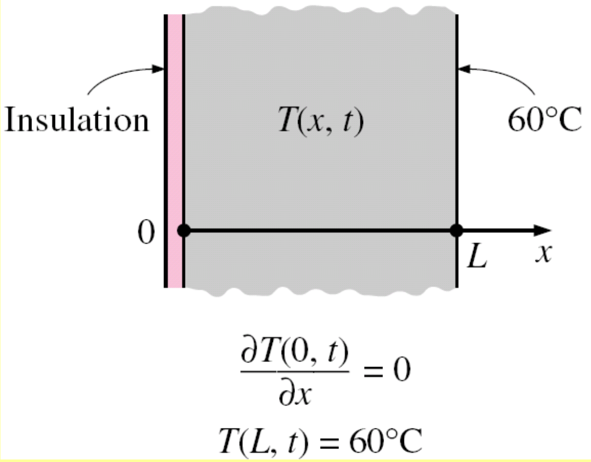


$$\dot{q} = -k \frac{dT}{dx} = \left(\textbf{Heat flux in the positive x-direction} \right)$$

The sign of the specified heat flux is determined by inspection: *positive* if the heat flux is in the positive direction of the coordinate axis, and *negative* if it is in the opposite direction.

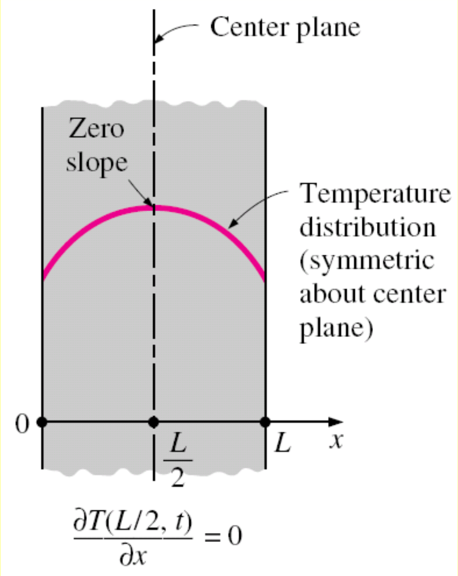
Two Special Cases

Insulated boundary



$$k\frac{\partial T(0,t)}{\partial x} = 0 \quad \text{or} \quad \frac{\partial T(0,t)}{\partial x} = 0$$

Thermal symmetry



$$\frac{\partial T\left(L/2, t\right)}{\partial x} = 0$$

EXAMPLE 2–7 **Heat Flux Boundary Condition**

Consider an aluminum pan used to cook beef stew on top of an electric range. The bottom section of the pan is $L = 0.3$ cm thick and has a diameter of $D = 20$ cm. The electric heating unit on the range top consumes 800 W of power during cooking, and 90 percent of the heat generated in the heating element is transferred to the pan. During steady operation, the temperature of the inner surface of the pan is measured to be 110°C. Express the boundary conditions for the bottom section of the pan during this cooking process.

SOLUTION An aluminum pan on an electric range top is considered. The boundary conditions for the bottom of the pan are to be obtained.

Analysis The heat transfer through the bottom section of the pan is from the bottom surface toward the top and can reasonably be approximated as being one-dimensional. We take the direction normal to the bottom surfaces of the pan as the x axis with the origin at the outer surface, as shown in Fig. 2–32. Then the inner and outer surfaces of the bottom section of the pan can be represented by $x = 0$ and $x = L$, respectively. During steady operation, the temperature will depend on x only and thus $T = T(x)$.

The boundary condition on the outer surface of the bottom of the pan at $x = 0$ can be approximated as being specified heat flux since it is stated that 90 percent of the 800 W (i.e., 720 W) is transferred to the pan at that surface. Therefore,

$$-k\frac{dT(0)}{dx} = \dot{q}_0$$

where

$$\dot{q}_0 = \frac{\text{Heat transfer rate}}{\text{Bottom surface area}} = \frac{0.720\text{ kW}}{\pi(0.1\text{ m})^2} = 22.9\text{ kW/m}^2$$

The temperature at the inner surface of the bottom of the pan is specified to be 110°C. Then the boundary condition on this surface can be expressed as

$$T(L) = 110°\text{C}$$

where $L = 0.003$ m.

Discussion Note that the determination of the boundary conditions may require some reasoning and approximations.

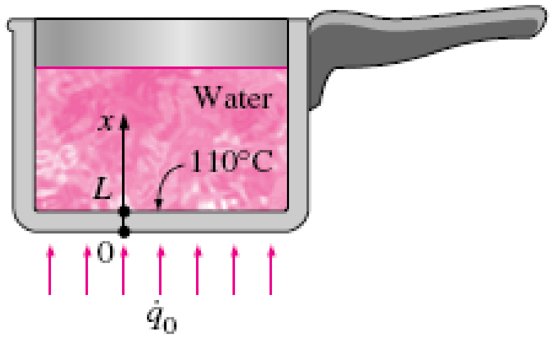


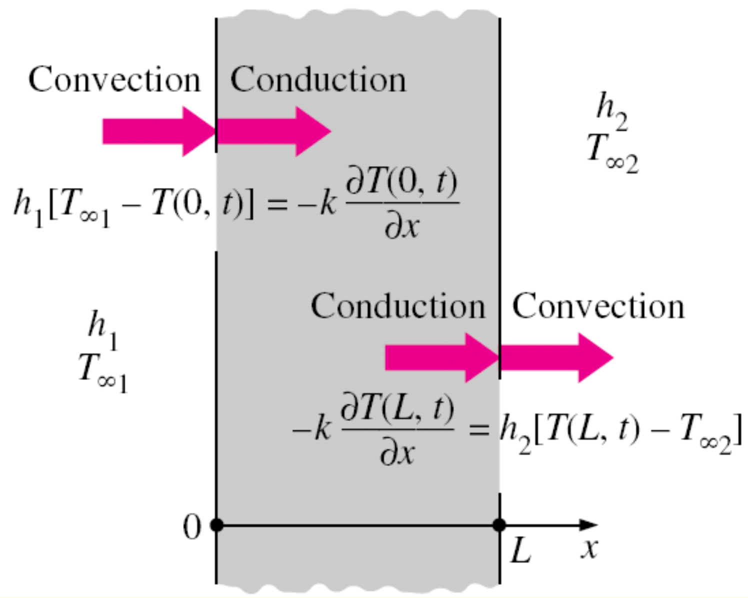
FIGURE 2–32
Schematic for Example 2–7.

Convection Boundary Condition

$$\left(\begin{array}{c} \text{Heat conduction} \\ \text{at the surface in} \\ \text{a selected} \\ \text{direction} \end{array} \right) = \left(\begin{array}{c} \text{Heat convection} \\ \text{at the surface in} \\ \text{the same} \\ \text{direction} \end{array} \right)$$

$$-k\frac{\partial T(0,t)}{\partial x} = h_1\left[T_{\infty 1} - T(0,t)\right]$$

$$-k\frac{\partial T(L,t)}{\partial x} = h_2\left[T(L,t) - T_{\infty 2}\right]$$



EXAMPLE 2–8 Convection and Insulation Boundary Conditions

Steam flows through a pipe shown in Fig. 2–35 at an average temperature of $T_\infty = 200°C$. The inner and outer radii of the pipe are $r_1 = 8$ cm and $r_2 = 8.5$ cm, respectively, and the outer surface of the pipe is heavily insulated. If the convection heat transfer coefficient on the inner surface of the pipe is $h = 65\ W/m^2 \cdot K$, express the boundary conditions on the inner and outer surfaces of the pipe during transient periods.

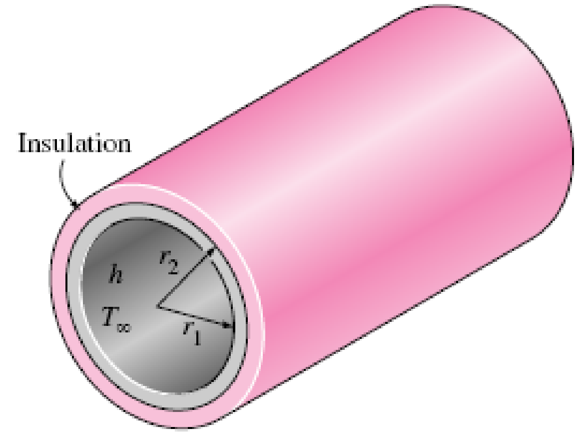


FIGURE 2–35
Schematic for Example 2–8.

SOLUTION The flow of steam through an insulated pipe is considered. The boundary conditions on the inner and outer surfaces of the pipe are to be obtained.

Analysis During initial transient periods, heat transfer through the pipe material predominantly is in the radial direction, and thus can be approximated as being one-dimensional. Then the temperature within the pipe material changes with the radial distance r and the time t. That is, $T = T(r, t)$.

It is stated that heat transfer between the steam and the pipe at the inner surface is by convection. Then taking the direction of heat transfer to be the positive r direction, the boundary condition on that surface can be expressed as

$$-k\frac{\partial T(r_1, t)}{\partial r} = h[T_\infty - T(r_1)]$$

The pipe is said to be well insulated on the outside, and thus heat loss through the outer surface of the pipe can be assumed to be negligible. Then the boundary condition at the outer surface can be expressed as

$$\frac{\partial T(r_2, t)}{\partial r} = 0$$

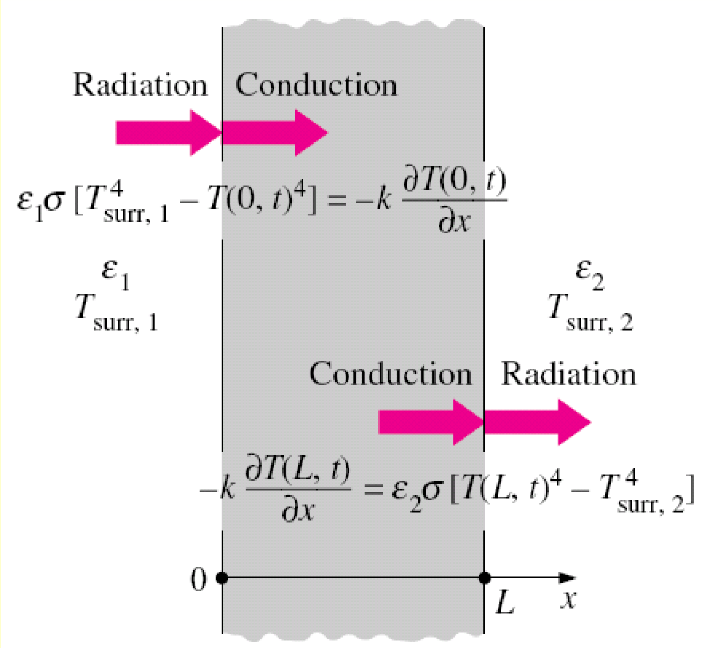
Discussion Note that the temperature gradient must be zero on the outer surface of the pipe at all times.

Radiation Boundary Condition

(Heat conduction at the surface in a selected direction) = (Radiation exchange at the surface in the same direction)

$$-k\frac{\partial T(0,t)}{\partial x} = \varepsilon_1 \sigma \left[T_{surr,1}^4 - T(0,t)^4 \right]$$

$$-k\frac{\partial T(L,t)}{\partial x} = \varepsilon_2 \sigma \left[T(L,t)^4 - T_{surr,2}^4 \right]$$



Interface Boundary Conditions

At the interface the requirements are:

(1) two bodies in contact must have the *same temperature* at the area of contact,

(2) an interface (which is a surface) cannot store any energy, and thus the *heat flux* on the two sides of an interface *must be the same.*

$$T_A(x_0, t) = T_B(x_0, t)$$

$$-k_A \frac{\partial T_A(x_0,t)}{\partial x} = -k_B \frac{\partial T_B(x_0,t)}{\partial x}$$

Generalized boundary condition

$$\left(\begin{array}{c} \text{Heat transfer} \\ \text{to the surface} \\ \text{in all modes} \end{array} \right) = \left(\begin{array}{c} \text{Heat transfer} \\ \text{from the surface} \\ \text{In all modes} \end{array} \right)$$

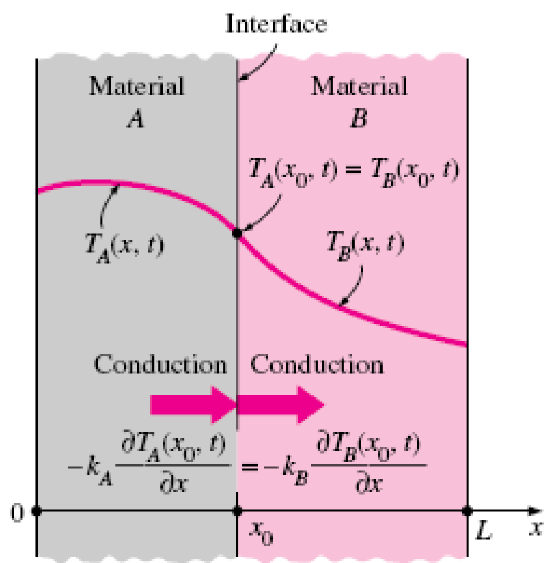


FIGURE 2–37

Boundary conditions at the interface of two bodies in perfect contact.

EXAMPLE 2–11 **Heat Conduction in a Plane Wall**

Consider a large plane wall of thickness $L = 0.2$ m, thermal conductivity $k = 1.2$ W/m · °C, and surface area $A = 15$ m^2. The two sides of the wall are maintained at constant temperatures of $T_1 = 120$°C and $T_2 = 50$°C, respectively, as shown in Fig. 2–41. Determine (*a*) the variation of temperature within the wall and the value of temperature at $x = 0.1$ m and (*b*) the rate of heat conduction through the wall under steady conditions.

SOLUTION A plane wall with specified surface temperatures is given. The variation of temperature and the rate of heat transfer are to be determined.
Assumptions **1** Heat conduction is steady. **2** Heat conduction is one-dimensional since the wall is large relative to its thickness and the thermal conditions on both sides are uniform. **3** Thermal conductivity is constant. **4** There is no heat generation.
Properties The thermal conductivity is given to be $k = 1.2$ W/m · °C.
Analysis (*a*) Taking the direction normal to the surface of the wall to be the *x*-direction, the differential equation for this problem can be expressed as

$$\frac{d^2T}{dx^2} = 0$$

with boundary conditions

$$T(0) = T_1 = 120°\text{C}$$
$$T(L) = T_2 = 50°\text{C}$$

The differential equation is linear and second order, and a quick inspection of it reveals that it has a single term involving derivatives and no terms involving the unknown function T as a factor. Thus, it can be solved by direct integration. Noting that an integration reduces the order of a derivative by one, the general solution of the differential equation above can be obtained by two simple successive integrations, each of which introduces an integration constant.

Integrating the differential equation once with respect to x yields

$$\frac{dT}{dx} = C_1$$

where C_1 is an arbitrary constant. Notice that the order of the derivative went down by one as a result of integration. As a check, if we take the derivative of this equation, we will obtain the original differential equation. This equation is not the solution yet since it involves a derivative.

Integrating one more time, we obtain

$$T(x) = C_1x + C_2$$

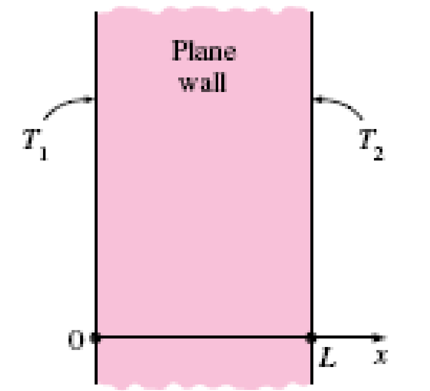


FIGURE 2–41
Schematic for Example 2–11.

Differential equation:

$$\frac{d^2T}{dx^2} = 0$$

Integrate:

$$\frac{dT}{dx} = C_1$$

Integrate again:

$$T(x) = C_1x + C_2$$

General solution

Arbitrary constants

FIGURE 2–42
Obtaining the general solution of a simple second order differential equation by integration.

which is the general solution of the differential equation (Fig. 2–42). The general solution in this case resembles the general formula of a straight line whose slope is C_1 and whose value at $x = 0$ is C_2. This is not surprising since the second derivative represents the change in the slope of a function, and a zero second derivative indicates that the slope of the function remains constant. Therefore, *any straight line* is a solution of this differential equation.

The general solution contains two unknown constants C_1 and C_2, and thus we need two equations to determine them uniquely and obtain the specific solution. These equations are obtained by forcing the general solution to satisfy the specified boundary conditions. The application of each condition yields one equation, and thus we need to specify two conditions to determine the constants C_1 and C_2.

When applying a boundary condition to an equation, *all occurrences of the dependent and independent variables and any derivatives are replaced by the specified values.* Thus the only unknowns in the resulting equations are the arbitrary constants.

The first boundary condition can be interpreted as *in the general solution, replace all the x's by zero and T(x) by T_1.* That is (Fig. 2–43),

$$T(0) = C_1 \times 0 + C_2 \quad \rightarrow \quad C_2 = T_1$$

The second boundary condition can be interpreted as *in the general solution, replace all the x's by L and T(x) by T_2.* That is,

$$T(L) = C_1L + C_2 \quad \rightarrow \quad T_2 = C_1L + T_1 \quad \rightarrow \quad C_1 = \frac{T_2 - T_1}{L}$$

Substituting the C_1 and C_2 expressions into the general solution, we obtain

$$T(x) = \frac{T_2 - T_1}{L}x + T_1 \qquad \text{(2–56)}$$

which is the desired solution since it satisfies not only the differential equation but also the two specified boundary conditions. That is, differentiating Eq. 2–56 with respect to x twice will give d^2T/dx^2, which is the given differential equation, and substituting $x = 0$ and $x = L$ into Eq. 2–56 gives $T(0) = T_1$ and $T(L) = T_2$, respectively, which are the specified conditions at the boundaries.

Substituting the given information, the value of the temperature at $x = 0.1$ m is determined to be

$$T(0.1\text{ m}) = \frac{(50 - 120)°\text{C}}{0.2\text{ m}}(0.1\text{ m}) + 120°\text{C} = 85°\text{C}$$

(*b*) The rate of heat conduction anywhere in the wall is determined from Fourier's law to be

$$\dot{Q}_{\text{wall}} = -kA\frac{dT}{dx} = -kAC_1 = -kA\frac{T_2 - T_1}{L} = kA\frac{T_1 - T_2}{L} \qquad \text{(2–57)}$$

The numerical value of the rate of heat conduction through the wall is determined by substituting the given values to be

$$\dot{Q} = kA\frac{T_1 - T_2}{L} = (1.2\text{ W/m}\cdot°\text{C})(15\text{ m}^2)\frac{(120 - 50)°\text{C}}{0.2\text{ m}} = 6300\text{ W}$$

Discussion Note that under steady conditions, the rate of heat conduction through a plane wall is constant.

Boundary condition:

$$T(0) = T_1$$

General solution:

$$T(x) = C_1x + C_2$$

Applying the boundary condition:

$$\underbrace{T(\underset{\substack{\uparrow\\0}}{x})}_{T_1} = C_1\underset{\substack{\uparrow\\0}}{x} + C_2$$

Substituting:

$$T_1 = C_1 \times 0 + C_2 \quad \rightarrow \quad C_2 = T_1$$

It cannot involve x or $T(x)$ after the boundary condition is applied.

FIGURE 2–43
When applying a boundary condition to the general solution at a specified point, all occurrences of the dependent and independent variables should be replaced by their specified values at that point.

EXAMPLE 2–15 **Heat Loss through a Steam Pipe**

Consider a steam pipe of length $L = 20$ m, inner radius $r_1 = 6$ cm, outer radius $r_2 = 8$ cm, and thermal conductivity $k = 20$ W/m · °C, as shown in Fig. 2–50. The inner and outer surfaces of the pipe are maintained at average temperatures of $T_1 = 150$°C and $T_2 = 60$°C, respectively. Obtain a general relation for the temperature distribution inside the pipe under steady conditions, and determine the rate of heat loss from the steam through the pipe.

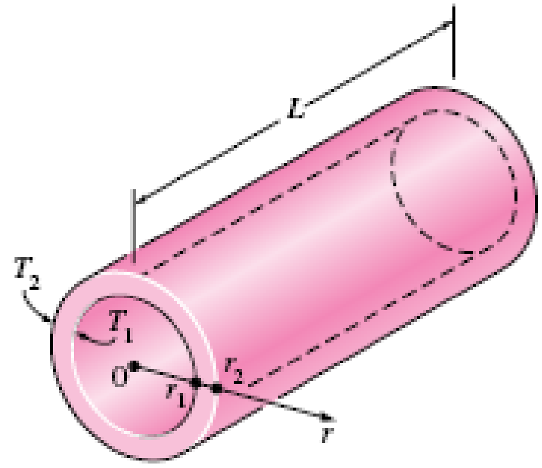


FIGURE 2–50
Schematic for Example 2–15.

SOLUTION A steam pipe is subjected to specified temperatures on its surfaces. The variation of temperature and the rate of heat transfer are to be determined.

Assumptions 1 Heat transfer is steady since there is no change with time. **2** Heat transfer is one-dimensional since there is thermal symmetry about the centerline and no variation in the axial direction, and thus $T = T(r)$. **3** Thermal conductivity is constant. **4** There is no heat generation.

Properties The thermal conductivity is given to be $k = 20$ W/m · °C.

Analysis The mathematical formulation of this problem can be expressed as

$$\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0$$

with boundary conditions

$$T(r_1) = T_1 = 150°\text{C}$$
$$T(r_2) = T_2 = 60°\text{C}$$

Integrating the differential equation once with respect to r gives

$$r\frac{dT}{dr} = C_1$$

where C_1 is an arbitrary constant. We now divide both sides of this equation by r to bring it to a readily integrable form,

$$\frac{dT}{dr} = \frac{C_1}{r}$$

Again integrating with respect to r gives (Fig. 2–51)

$$T(r) = C_1 \ln r + C_2 \qquad (a)$$

We now apply both boundary conditions by replacing all occurrences of r and $T(r)$ in Eq. (a) with the specified values at the boundaries. We get

$$T(r_1) = T_1 \quad \rightarrow \quad C_1 \ln r_1 + C_2 = T_1$$
$$T(r_2) = T_2 \quad \rightarrow \quad C_1 \ln r_2 + C_2 = T_2$$

which are two equations in two unknowns, C_1 and C_2. Solving them simultaneously gives

$$C_1 = \frac{T_2 - T_1}{\ln(r_2/r_1)} \quad \text{and} \quad C_2 = T_1 - \frac{T_2 - T_1}{\ln(r_2/r_1)} \ln r_1$$

Substituting them into Eq. (a) and rearranging, the variation of temperature within the pipe is determined to be

$$T(r) = \frac{\ln(r/r_1)}{\ln(r_2/r_1)}(T_2 - T_1) + T_1 \qquad (2\text{–}58)$$

The rate of heat loss from the steam is simply the total rate of heat conduction through the pipe, and is determined from Fourier's law to be

$$\dot{Q}_{\text{cylinder}} = -kA\frac{dT}{dr} = -k(2\pi rL)\frac{C_1}{r} = -2\pi kLC_1 = 2\pi kL\frac{T_1 - T_2}{\ln(r_2/r_1)} \qquad (2\text{–}59)$$

The numerical value of the rate of heat conduction through the pipe is determined by substituting the given values

$$\dot{Q} = 2\pi(20\ \text{W/m}\cdot{}^\circ\text{C})(20\ \text{m})\frac{(150 - 60)^\circ\text{C}}{\ln(0.08/0.06)} = 786\ \text{kW}$$

Discussion Note that the total rate of heat transfer through a pipe is constant, but the heat flux $\dot{q} = \dot{Q}/(2\pi rL)$ is not since it decreases in the direction of heat transfer with increasing radius.

Differential equation:

$$\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0$$

Integrate:

$$r\frac{dT}{dr} = C_1$$

Divide by r ($r \neq 0$):

$$\frac{dT}{dr} = \frac{C_1}{r}$$

Integrate again:

$$T(r) = C_1 \ln r + C_2$$

which is the general solution.

FIGURE 2–51
Basic steps involved in the solution of the steady one-dimensional heat conduction equation in cylindrical coordinates.

Heat Generation in Solids-
The Surface Temperature

$$\left\{\text{Rate of heat transfer from the solid}\right\} = \left\{\text{Rate of energy generation within the solid}\right\}$$

$$\dot{Q} = \dot{e}_{gen}V \qquad \text{(W)}$$

$$\dot{Q} = hA_s\left(T_s - T_\infty\right) \qquad \text{(W)}$$

$$T_s = T_\infty + \frac{\dot{e}_{gen}V}{hA_s}$$

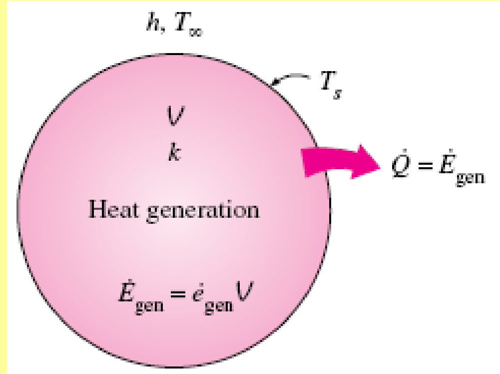


FIGURE 2–55
At steady conditions, the entire heat generated in a solid must leave the solid through its outer surface.

For a large plane wall of thickness $2L$ (A_s=2A_{wall} and V=2LA_{wall})

$$T_{s,plane\ wall} = T_\infty + \frac{\dot{e}_{gen}L}{h}$$

For a long solid cylinder of radius r_0 (A_s=2$\pi r_0 L$ and V=$\pi r_0^2 L$)

$$T_{s,cylinder} = T_\infty + \frac{\dot{e}_{gen}r_0}{2h}$$

For a solid sphere of radius r_0 (A_s=4πr_0^2 and V=$^4/_3\pi r_0^3$)

$$T_{s,sphere} = T_\infty + \frac{\dot{e}_{gen}r_0}{3h}$$

Examples of heat generation

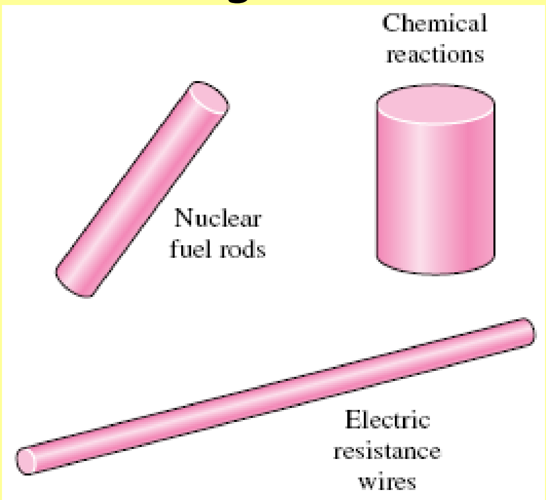


FIGURE 2–54
Heat generation in solids is commonly encountered in practice.

Heat Generation in Solids -The maximum Temperature in a Cylinder (the Centerline)

The *heat generated* within an inner cylinder must be equal to the *heat conducted* through its outer surface.

$$-kA_r \frac{dT}{dr} = \dot{e}_{gen} V_r$$

Substituting these expressions into the above equation and separating the variables, we get

$$-k\left(2\pi rL\right)\frac{dT}{dr} = \dot{e}_{gen}\left(\pi r^2 L\right) \rightarrow dT = -\frac{\dot{e}_{gen}}{2k} r dr$$

Integrating from $r = 0$ where $T(0) = T_0$ to $r = r_o$

Cylinder

$$\Delta T_{\text{max, cylinder}} = T_0 - T_s = \frac{\dot{e}_{\text{gen}} r_o^2}{4k}$$

Plane wall

$$\Delta T_{\text{max, plane wall}} = \frac{\dot{e}_{\text{gen}} L^2}{2k}$$

Sphere

$$\Delta T_{\text{max, sphere}} = \frac{\dot{e}_{\text{gen}} r_o^2}{6k}$$

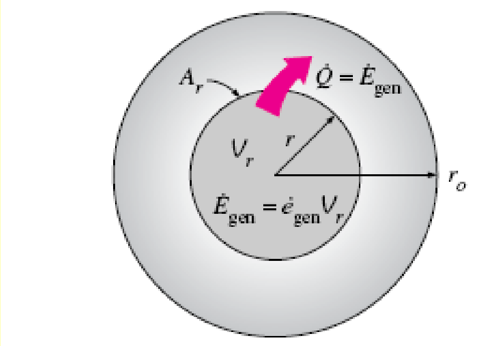


FIGURE 2–56
Heat conducted through a cylindrical shell of radius r is equal to the heat generated within a shell.

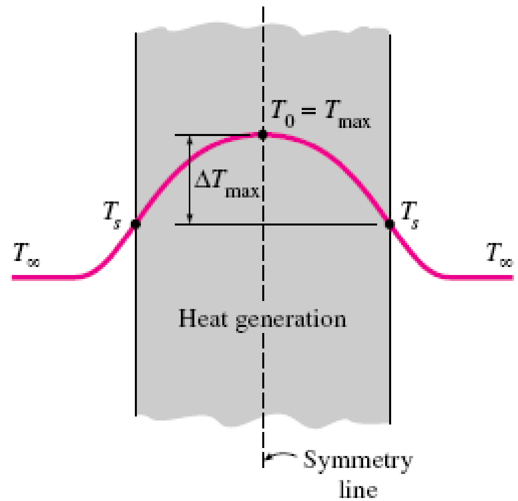


FIGURE 2–57
The maximum temperature in a symmetrical solid with uniform heat generation occurs at its center.

EXAMPLE 2–19 **Heat Conduction in a Two-Layer Medium**

Consider a long resistance wire of radius $r_1 = 0.2$ cm and thermal conductivity $k_{wire} = 15$ W/m · °C in which heat is generated uniformly as a result of resistance heating at a constant rate of $\dot{e}_{gen} = 50$ W/cm³ (Fig. 2–61). The wire is embedded in a 0.5-cm-thick layer of ceramic whose thermal conductivity is $k_{ceramic} = 1.2$ W/m · °C. If the outer surface temperature of the ceramic layer is measured to be $T_s = 45°C$, determine the temperatures at the center of the resistance wire and the interface of the wire and the ceramic layer under steady conditions.

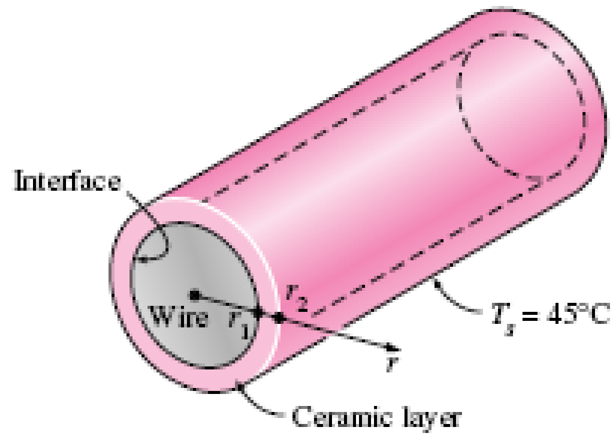


FIGURE 2–61
Schematic for Example 2–19.

SOLUTION The surface and interface temperatures of a resistance wire covered with a ceramic layer are to be determined.

Assumptions **1** Heat transfer is steady since there is no change with time. **2** Heat transfer is one-dimensional since this two-layer heat transfer problem possesses symmetry about the centerline and involves no change in the axial direction, and thus $T = T(r)$. **3** Thermal conductivities are constant. **4** Heat generation in the wire is uniform.

Properties It is given that $k_{wire} = 15$ W/m · °C and $k_{ceramic} = 1.2$ W/m · ° C.

Analysis Letting T_I denote the unknown interface temperature, the heat transfer problem in the wire can be formulated as

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT_{wire}}{dr}\right) + \frac{\dot{e}_{gen}}{k} = 0$$

with

$$T_{wire}(r_1) = T_I$$

$$\frac{dT_{wire}(0)}{dr} = 0$$

This problem was solved in Example 2–18, and its solution was determined to be

$$T_{wire}(r) = T_I + \frac{\dot{e}_{gen}}{4k_{wire}}(r_1^2 - r^2) \qquad (a)$$

Noting that the ceramic layer does not involve any heat generation and its outer surface temperature is specified, the heat conduction problem in that layer can be expressed as

$$\frac{d}{dr}\left(r\frac{dT_{\text{ceramic}}}{dr}\right) = 0$$

with

$$T_{\text{ceramic}}(r_1) = T_I$$
$$T_{\text{ceramic}}(r_2) = T_s = 45°\text{C}$$

This problem was solved in Example 2–15, and its solution was determined to be

$$T_{\text{ceramic}}(r) = \frac{\ln(r/r_1)}{\ln(r_2/r_1)}(T_s - T_I) + T_I \qquad (b)$$

We have already utilized the first interface condition by setting the wire and ceramic layer temperatures equal to T_I at the interface $r = r_1$. The interface temperature T_I is determined from the second interface condition that the heat flux in the wire and the ceramic layer at $r = r_1$ must be the same:

$$-k_{\text{wire}}\frac{dT_{\text{wire}}(r_1)}{dr} = -k_{\text{ceramic}}\frac{dT_{\text{ceramic}}(r_1)}{dr} \quad \rightarrow \quad \frac{\dot{e}_{\text{gen}}r_1}{2} = -k_{\text{ceramic}}\frac{T_s - T_I}{\ln(r_2/r_1)}\left(\frac{1}{r_1}\right)$$

Solving for T_I and substituting the given values, the interface temperature is determined to be

$$\begin{aligned} T_I &= \frac{\dot{e}_{\text{gen}}r_1^2}{2k_{\text{ceramic}}}\ln\frac{r_2}{r_1} + T_s \\ &= \frac{(50 \times 10^6\ \text{W/m}^3)(0.002\ \text{m})^2}{2(1.2\ \text{W/m}\cdot°\text{C})}\ln\frac{0.007\ \text{m}}{0.002\ \text{m}} + 45°\ \text{C} = 149.4°\text{C} \end{aligned}$$

Knowing the interface temperature, the temperature at the centerline ($r = 0$) is obtained by substituting the known quantities into Eq. (a),

$$T_{\text{wire}}(0) = T_I + \frac{\dot{e}_{\text{gen}}r_1^2}{4k_{\text{wire}}} = 149.4°\text{C} + \frac{(50 \times 10^6\ \text{W/m}^3)(0.002\ \text{m})^2}{4 \times (15\ \text{W/m}\cdot°\text{C})} = 152.7°\text{C}$$

Variable Thermal Conductivity, $k(T)$

- The thermal conductivity of a material, in general, varies with temperature.
- An average value for the thermal conductivity is commonly used when the variation is mild.
- This is also common practice for other temperature-dependent properties such as the density and specific heat.

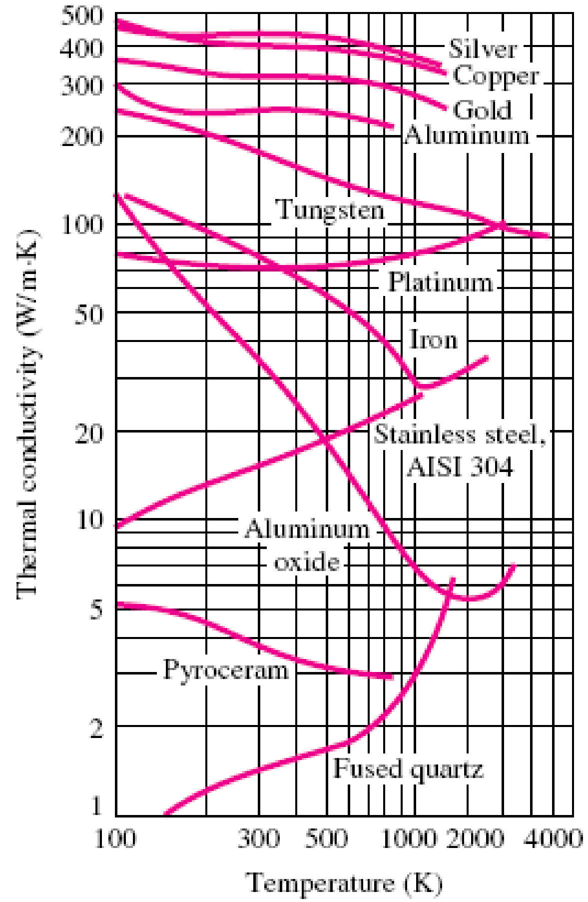


FIGURE 2–62
Variation of the thermal conductivity of some solids with temperature.

Variable Thermal Conductivity for One-Dimensional Cases

When the variation of thermal conductivity with temperature $k(T)$ is known, the average value of the thermal conductivity in the temperature range between T_1 and T_2 can be determined from

$$k_{ave} = \frac{\int_{T_1}^{T_2} k(T)dT}{T_2 - T_1}$$

The variation in thermal conductivity of a material with can often be approximated as a linear function and expressed as

$$k(T) = k_0(1 + \beta T)$$

β is the **temperature coefficient of thermal conductivity**.

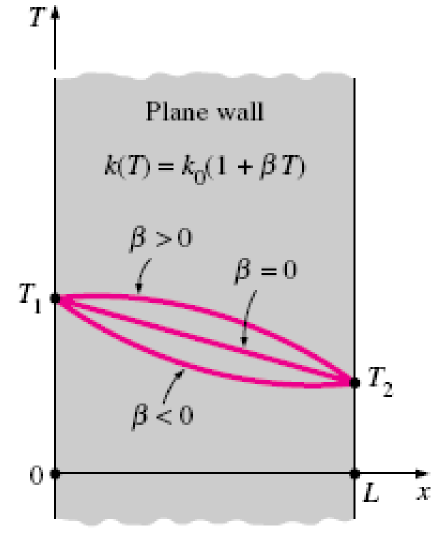


FIGURE 2–63
The variation of temperature in a plane wall during steady one-dimensional heat conduction for the cases of constant and variable thermal conductivity.

For a plane wall the temperature varies linearly during steady one-dimensional heat conduction when the thermal conductivity is constant. This is no longer the case when the thermal conductivity changes with temperature (even linearly).

Concluding Points

- One-Dimensional Heat Conduction
- General Heat Conduction Equation
- Boundary and Initial Conditions
- Solution of Steady One-Dimensional Heat Conduction Problems
- Heat Generation in a Solid
- Variable Thermal Conductivity $k(T)$

HEAT AND MASS TRANSFER

Steady Heat Conduction

OUTLINE

- Steady Heat Conduction in Plane Walls
- Thermal Contact Resistance
- Generalized Thermal Resistance Networks
- Heat Conduction in Cylinders and Spheres
- Critical Radius of Insulation
- Heat Transfer from Finned Surfaces
- Heat Transfer in Common Configurations
- Conclusions

Steady Heat Conduction In Plane Walls

Heat transfer through the wall is in the *normal direction* to the wall surface, and no significant heat transfer takes place in the wall in other directions.

Heat transfer in a certain direction is driven by the *temperature gradient* in that direction.

There will be no heat transfer in a direction in which there is no change in temperature.

If the air temperatures in and outside the house remain constant, then heat transfer through the wall of a house can be modeled as *steady and one-dimensional.*

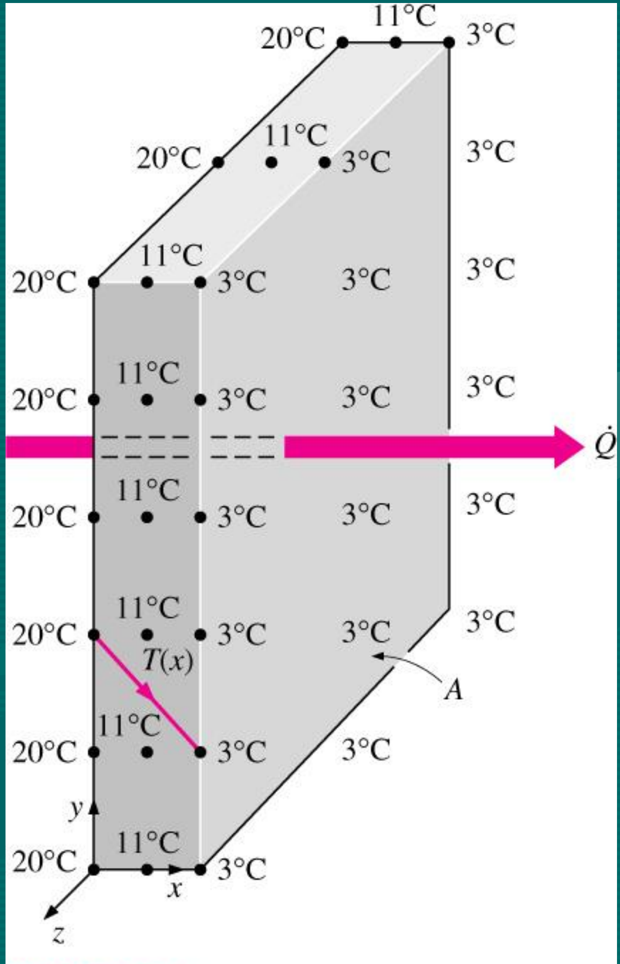


FIGURE 3–1
Heat transfer through a wall is one-dimensional when the temperature of the wall varies in one direction only.

$$\int_{x=0}^{L} \dot{Q}_{\text{cond, wall}}\, dx = -\int_{T=T_1}^{T_2} kA\, dT$$

Integrating and rearranging

$$\dot{Q}_{\text{cond, wall}} = kA \frac{T_1 - T_2}{L} \qquad \text{(W)}$$

- Energy balance:

$$\begin{pmatrix}\text{Rate of}\\ \text{heat transfer}\\ \text{into the wall}\end{pmatrix} - \begin{pmatrix}\text{Rate of}\\ \text{heat transfer}\\ \text{out of the wall}\end{pmatrix} = \begin{pmatrix}\text{Rate of change}\\ \text{of the energy}\\ \text{of the wall}\end{pmatrix}$$

or

$$\dot{Q}_{\text{in}} - \dot{Q}_{\text{out}} = \frac{dE_{\text{wall}}}{dt}$$

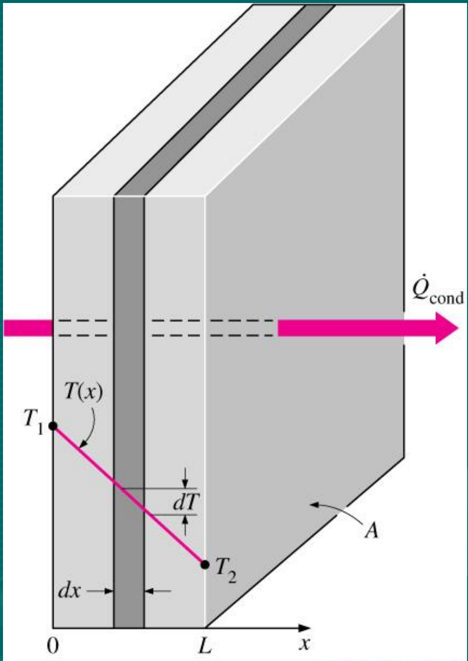


FIGURE 3–2
Under steady conditions, the temperature distribution in a plane wall is a straight line.

$dE_{\text{wall}}/dt = 0$ for *steady* operation (no change in the temperature of the wall with time at any point) and $\dot{Q}_{\text{cond, wall}} = \text{constant}$

- The Fourier's law of heat conduction for the wall:

$$\dot{Q}_{\text{cond, wall}} = -kA \frac{dT}{dx} \qquad \text{(W)}$$

where *dT/dx= constant* and *T varies linearly with x.*

The Thermal Resistance Concept

Heat conduction through a plane wall is

$$\dot{Q}_{\text{cond, wall}} = \frac{T_1 - T_2}{R_{\text{wall}}} \qquad \text{(W)}$$

where

$$R_{\text{wall}} = \frac{L}{kA} \qquad (^\circ\text{C/W})$$

is the *thermal resistance* of the wall against heat conduction (**conduction resistance**). The thermal resistance of a medium depends on the *geometry* and the *thermal properties* of the medium.

Taking into account analogous to the relation for *electric current flow I:*

$$I = \frac{\mathbf{V}_1 - \mathbf{V}_2}{R_e}$$

where $R_e = L/\sigma_e A$ is the *electric resistance* and $V_1 - V_2$ is the *voltage difference across* the resistance (σ_e is the electrical conductivity).

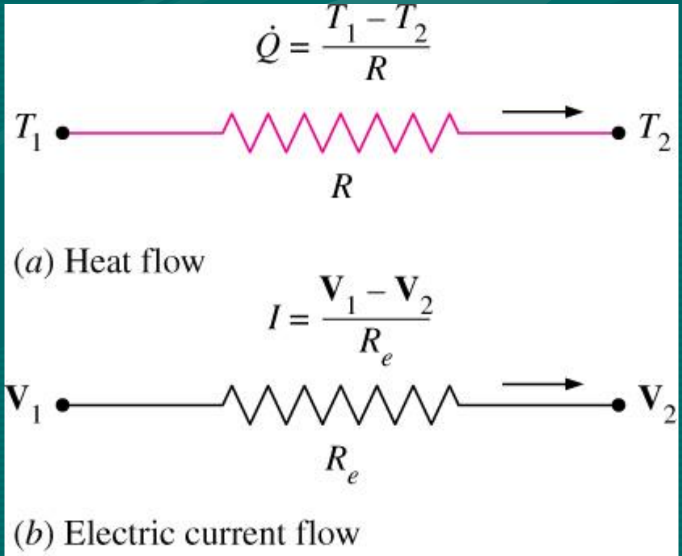


FIGURE 3–3
Analogy between thermal and electrical resistance concepts.

Newton's law of cooling for convection heat transfer rate:

$$\dot{Q}_{\text{conv}} = hA_s(T_s - T_\infty)$$

can be rearranged as

$$\dot{Q}_{\text{conv}} = \frac{T_s - T_\infty}{R_{\text{conv}}} \qquad \text{(W)}$$

with

$$R_{\text{conv}} = \frac{1}{hA_s} \qquad \text{(°C/W)}$$

which is the *thermal resistance* of the surface against heat convection, or simply the **convection resistance** of the surface.

When the convection heat transfer coefficient is very large $(h \to \infty)$, the convection resistance becomes *zero* and $T_s \approx T_\infty$. That is, the surface offers *no resistance to convection*, and thus it does not slow down the heat transfer process.

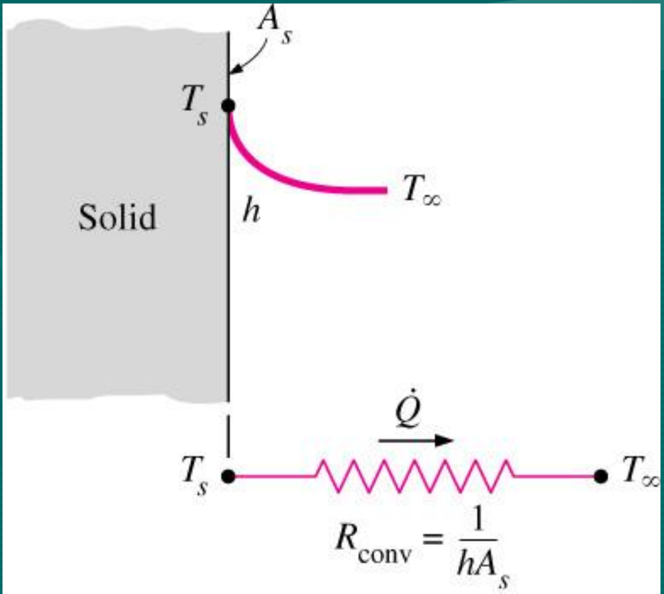


FIGURE 3–4
Schematic for convection resistance at a surface.

The rate of radiation heat transfer between a surface of emissivity ε and area A_s at temperature T_s and the surrounding surfaces at some average temperature T_{surr} can be expressed as

$$\dot{Q}_{\text{rad}} = \varepsilon\sigma\, A_s(T_s^4 - T_{\text{surr}}^4) = h_{\text{rad}}\, A_s(T_s - T_{\text{surr}}) = \frac{T_s - T_{\text{surr}}}{R_{\text{rad}}} \qquad \text{(W)}$$

with $$R_{\text{rad}} = \frac{1}{h_{\text{rad}}\, A_s} \qquad \text{(K/W)}$$ which is the *radiation resistance.*

$$h_{\text{rad}} = \frac{\dot{Q}_{\text{rad}}}{A_s(T_s - T_{\text{surr}})} = \varepsilon\sigma(T_s^2 + T_{\text{surr}}^2)(T_s + T_{\text{surr}}) \qquad (\text{W/m}^2\cdot\text{K})$$

is the radiation heat transfer coefficient.

Both T_s and T_{surr} *must* be in K in the evaluation of h_{rad}.

When $T_{\text{surr}} \approx T_\infty$, the radiation effect can properly be accounted for by replacing *h* in the convection resistance relation by

$$h_{\text{combined}} = h_{\text{conv}} + h_{\text{rad}} \qquad (\text{W/m}^2\cdot\text{K})$$

where $h_{combined}$ is the combined heat transfer coefficient.

Thermal Resistance Network

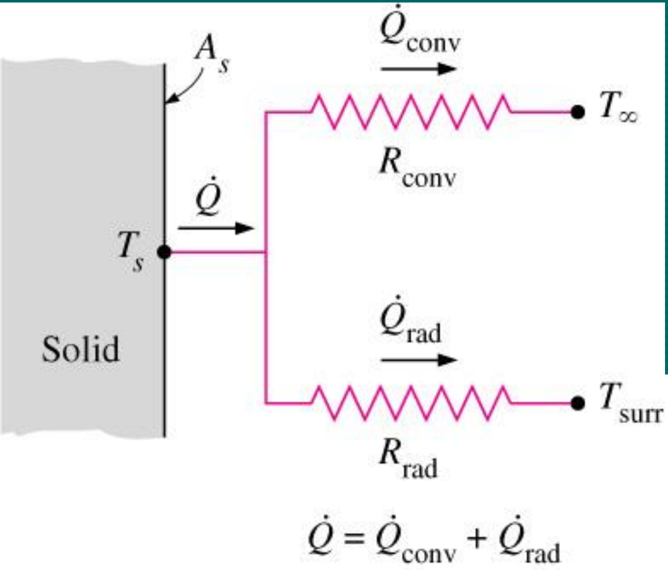


FIGURE 3–5
Schematic for convection and radiation resistances at a surface.

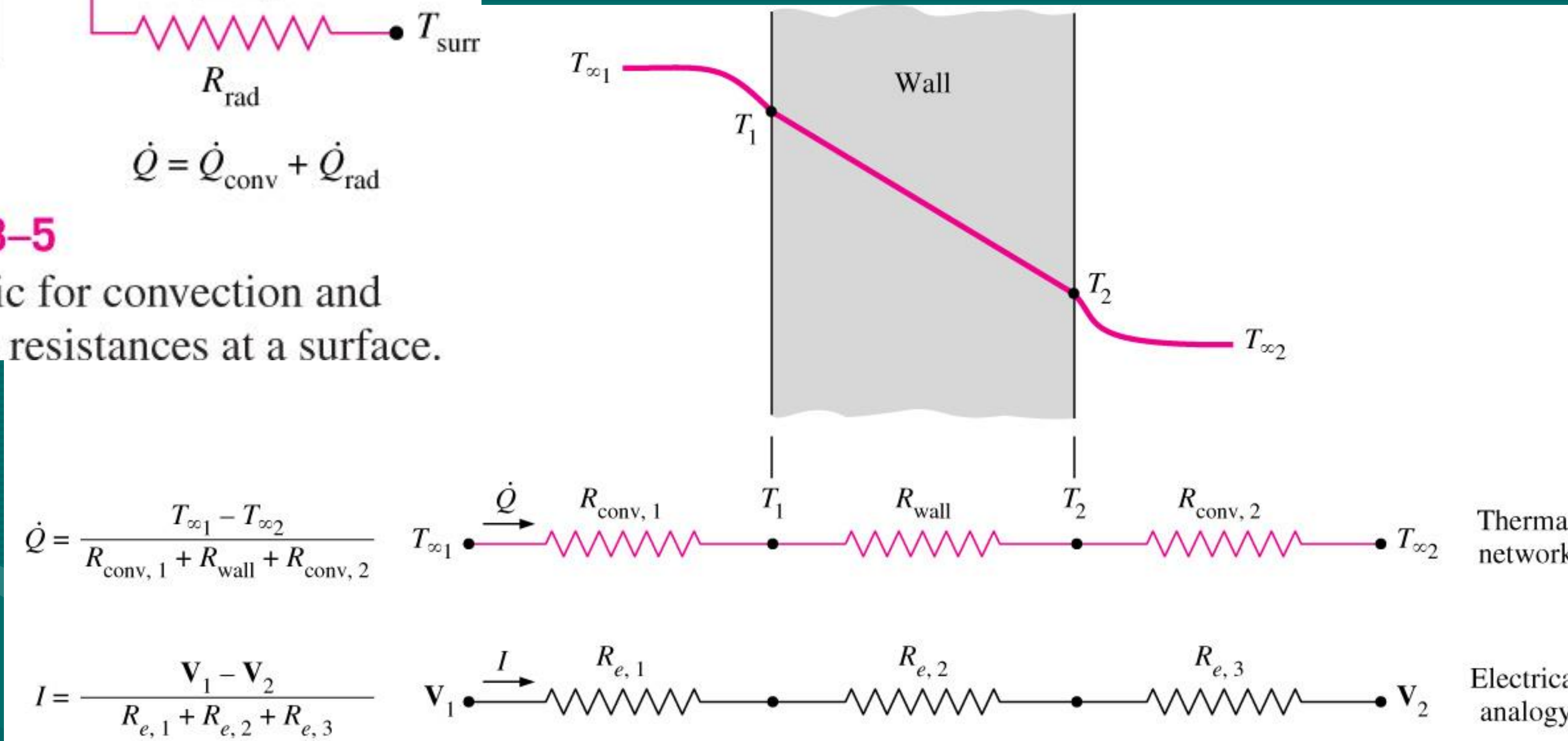


FIGURE 3–6
The thermal resistance network for heat transfer through a plane wall subjected to convection on both sides, and the electrical analogy.

Under steady conditions

$$\begin{pmatrix}\text{Rate of} \\ \textit{heat convection} \\ \text{into the wall}\end{pmatrix} = \begin{pmatrix}\text{Rate of} \\ \textit{heat conduction} \\ \text{through the wall}\end{pmatrix} = \begin{pmatrix}\text{Rate of} \\ \textit{heat convection} \\ \text{from the wall}\end{pmatrix}$$

or

$$\dot{Q} = h_1 A(T_{\infty 1} - T_1) = kA\frac{T_1 - T_2}{L} = h_2 A(T_2 - T_{\infty 2})$$

which can be rearranged as

$$\dot{Q} = \frac{T_{\infty 1} - T_1}{1/h_1 A} = \frac{T_1 - T_2}{L/kA} = \frac{T_2 - T_{\infty 2}}{1/h_2 A}$$

$$= \frac{T_{\infty 1} - T_1}{R_{\text{conv, 1}}} = \frac{T_1 - T_2}{R_{\text{wall}}} = \frac{T_2 - T_{\infty 2}}{R_{\text{conv, 2}}}$$

Adding the numerators and denominators yields

$$\dot{Q} = \frac{T_{\infty} - T_{\infty 2}}{R_{\text{total}}} \qquad \text{(W)}$$

where

$$R_{\text{total}} = R_{\text{conv, 1}} + R_{\text{wall}} + R_{\text{conv, 2}} = \frac{1}{h_1 A} + \frac{L}{kA} + \frac{1}{h_2 A} \qquad (°\text{C/W})$$

The thermal resistances are in *series*, and the equivalent thermal resistance is determined by simply *adding* the individual resistances, just like the electrical resistances connected in series.

The equation $\dot{Q} = \Delta T/R$ can be rearranged as $\Delta T = \dot{Q}R \qquad (^\circ\text{C})$

Here, the *temperature drop* across any layer is equal to the *rate of heat transfer* times the *thermal resistance* across that layer.

If

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_n}{b_n} = c$$

then

$$\frac{a_1 + a_2 + \cdots + a_n}{b_1 + b_2 + \cdots + b_n} = c$$

For example,

$$\frac{1}{4} = \frac{2}{8} = \frac{5}{20} = 0.25$$

and

$$\frac{1+2+5}{4+8+20} = 0.25$$

FIGURE 3–7
A useful mathematical identity.

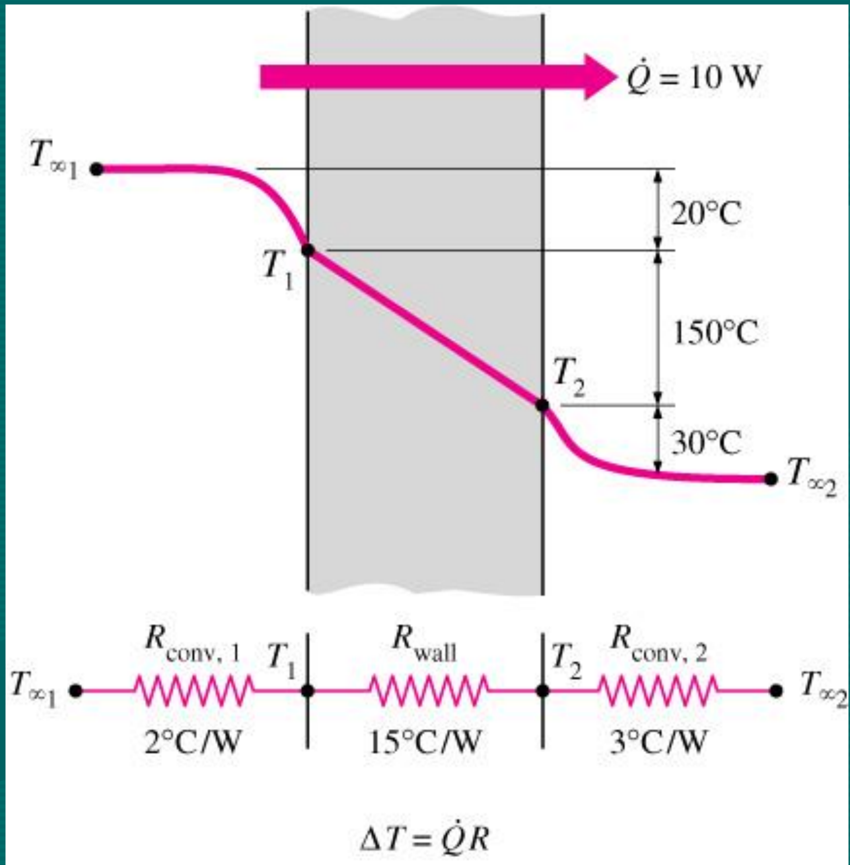


FIGURE 3–8
The temperature drop across a layer is proportional to its thermal resistance.

Analogous to Newton's law of cooling as $\dot{Q} = UA\,\Delta T \qquad \text{(W)}$

U: the overall heat transfer coefficient

$$UA = \frac{1}{R_{\text{total}}}$$

$$\dot{Q} = \frac{T_{\infty 1} - T_1}{R_{\text{conv, 1}}} = \frac{T_{\infty 1} - T_1}{1/h_1 A}$$

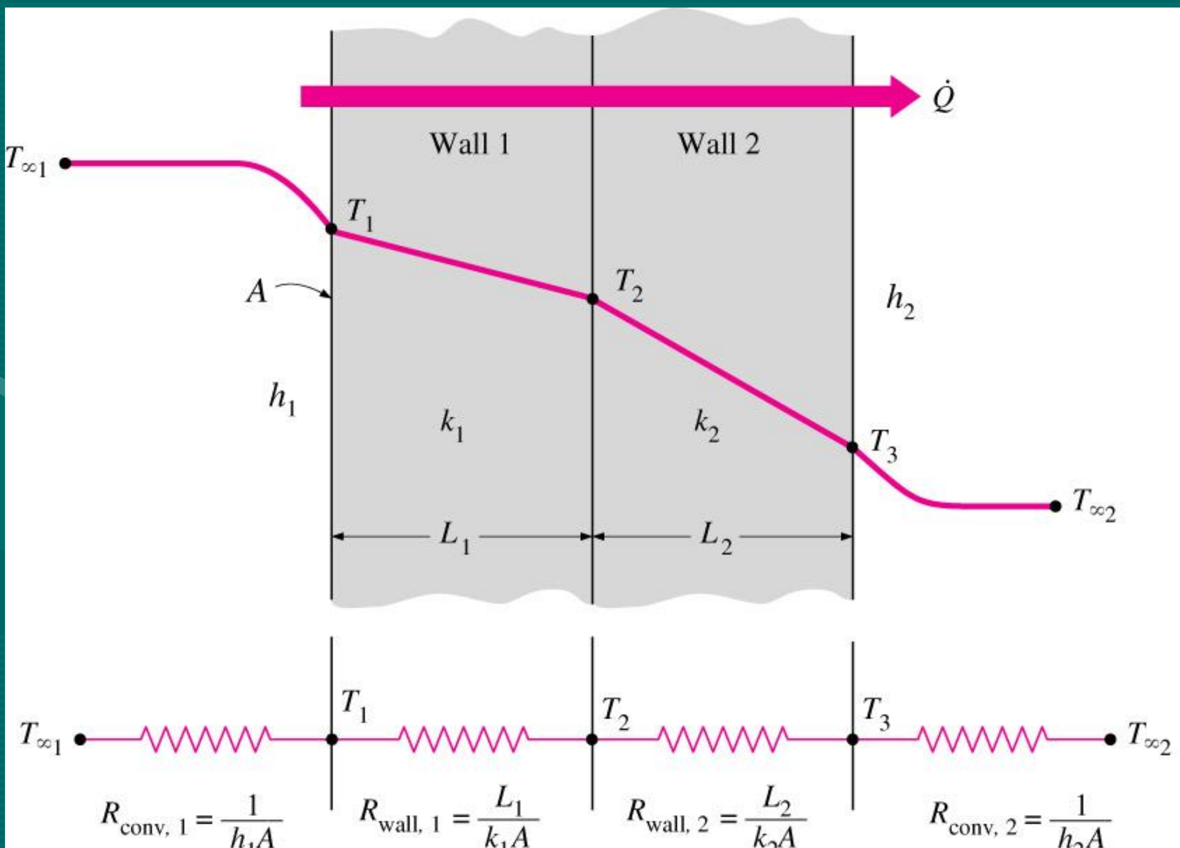


FIGURE 3–9
The thermal resistance network for heat transfer through a two-layer plane wall subjected to convection on both sides.

Multilayer Plane Walls

The rate of steady heat transfer through a plane wall consisting of two layers

$$\dot{Q} = \frac{T_{\infty 1} - T_{\infty 2}}{R_{\text{total}}}$$

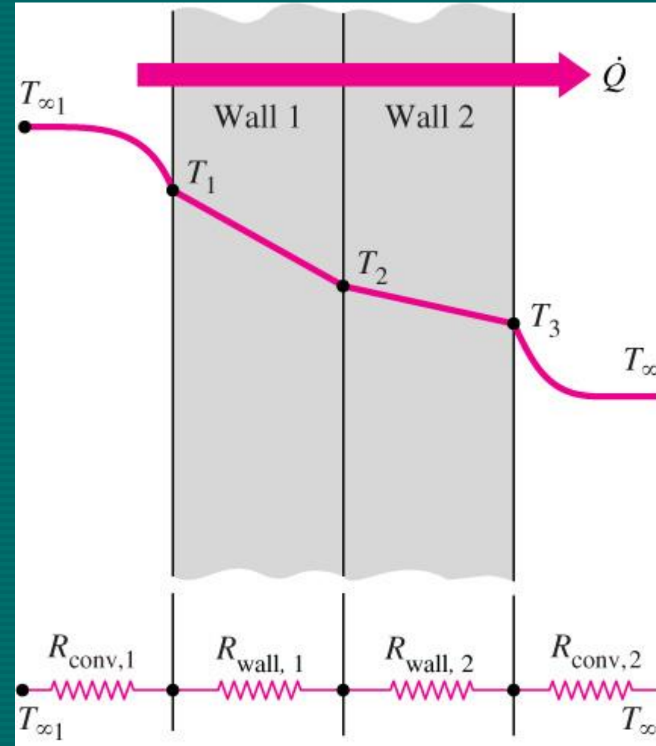
R_{total}: the *total thermal resistance*

$$R_{\text{total}} = R_{\text{conv, 1}} + R_{\text{wall, 1}} + R_{\text{wall, 2}} + R_{\text{conv, 2}} = \frac{1}{h_1 A} + \frac{L_1}{k_1 A} + \frac{L_2}{k_2 A} + \frac{1}{h_2 A}$$

for the resistances *in series*.

$$\dot{Q} = \frac{T_{\infty 1} - T_2}{R_{\text{conv, 1}} + R_{\text{wall, 1}}} = \frac{T_{\infty 1} - T_2}{\dfrac{1}{h_1 A} + \dfrac{L_1}{k_1 A}}$$

☞ It is limited to systems involving *steady* heat transfer with *no heat generation*.



To find T_1: $\dot{Q} = \dfrac{T_{\infty 1} - T_1}{R_{\text{conv},1}}$

To find T_2: $\dot{Q} = \dfrac{T_{\infty 1} - T_2}{R_{\text{conv},1} + R_{\text{wall},1}}$

To find T_3: $\dot{Q} = \dfrac{T_3 - T_{\infty 2}}{R_{\text{conv},2}}$

FIGURE 3–10
The evaluation of the surface and interface temperatures when $T_{\infty 1}$ and $T_{\infty 2}$ are given and $\dot{Q}$ is calculated.

EXAMPLE 3–1 **Heat Loss through a Wall**

Consider a 17-m-high, 5-m-wide, and 0.17-m-thick wall whose thermal conductivity is $k = 0.9$ W/m · °C (Fig. 17–11). On a certain day, the temperatures of the inner and the outer surfaces of the wall are measured to be 16°C and 2°C, respectively. Determine the rate of heat loss through the wall on that day.

SOLUTION The two surfaces of a wall are maintained at specified temperatures. The rate of heat loss through the wall is to be determined.

Assumptions **1** Heat transfer through the wall is steady since the surface temperatures remain constant at the specified values. **2** Heat transfer is one-dimensional since any significant temperature gradients will exist in the direction from the indoors to the outdoors. **3** Thermal conductivity is constant.

Properties The thermal conductivity is given to be $k = 0.9$ W/m · °C.

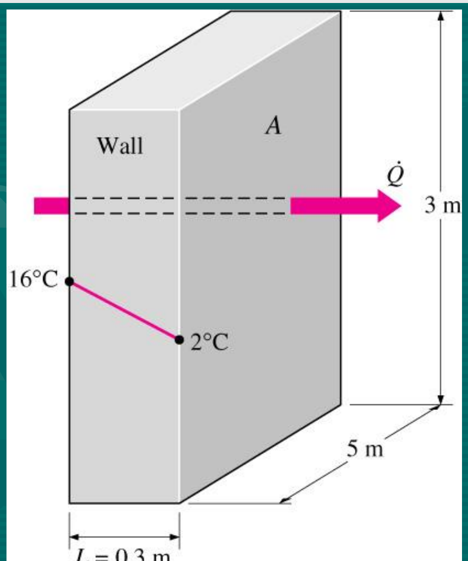


FIGURE 3–11
Schematic for Example 3–1.

Analysis Noting that the heat transfer through the wall is by conduction and the area of the wall is $A = 3\text{ m} \times 5\text{ m} = 15\text{ m}^2$, the steady rate of heat transfer through the wall can be determined from Eq. 17–3 to be

$$\dot{Q} = kA\frac{T_1 - T_2}{L} = (0.9\text{ W/m}\cdot{}^\circ\text{C})(15\text{ m}^2)\frac{(16-2)^\circ\text{C}}{0.3\text{ m}} = 630\text{ W}$$

We could also determine the steady rate of heat transfer through the wall by making use of the thermal resistance concept from

$$\dot{Q} = \frac{\Delta T_{\text{wall}}}{R_{\text{wall}}}$$

where

$$R_{\text{wall}} = \frac{L}{kA} = \frac{0.3\text{ m}}{(0.9\text{ W/m}\cdot{}^\circ\text{C})(15\text{ m}^2)} = 0.02222^\circ\text{C/ W}$$

Substituting, we get

$$\dot{Q} = \frac{(16-2)^\circ\text{C}}{0.02222^\circ\text{C/ W}} = 630\text{ W}$$

Discussion This is the same result obtained earlier. Note that heat conduction through a plane wall with specified surface temperatures can be determined directly and easily without utilizing the thermal resistance concept. However, the thermal resistance concept serves as a valuable tool in more complex heat transfer problems, as you will see in the following examples.

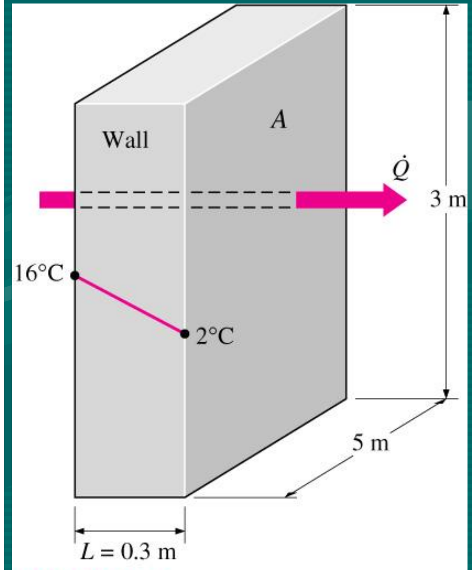


FIGURE 3–11
Schematic for Example 3–1.

EXAMPLE 3–2 Heat Loss through a Single-Pane Window

Consider a 0.8-m-high and 1.5-m-wide glass window with a thickness of 8 mm and a thermal conductivity of $k = 0.78$ W/m · °C. Determine the steady rate of heat transfer through this glass window and the temperature of its inner surface for a day during which the room is maintained at 20°C while the temperature of the outdoors is −10°C. Take the heat transfer coefficients on the inner and outer surfaces of the window to be $h_1 = 10$ W/m² · °C and $h_2 = 40$ W/m² · °C, which includes the effects of radiation.

SOLUTION Heat loss through a window glass is considered. The rate of heat transfer through the window and the inner surface temperature are to be determined.

Assumptions **1** Heat transfer through the window is steady since the surface temperatures remain constant at the specified values. **2** Heat transfer through the wall is one-dimensional since any significant temperature gradients will exist in the direction from the indoors to the outdoors. **3** Thermal conductivity is constant.

Properties The thermal conductivity is given to be $k = 0.78$ W/m · °C.

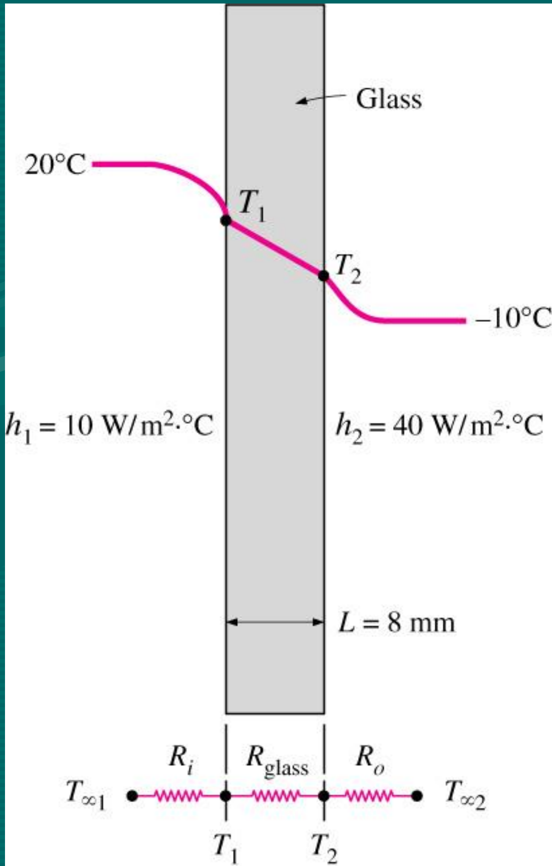


FIGURE 3–12
Schematic for Example 3–2.

Analysis This problem involves conduction through the glass window and convection at its surfaces, and can best be handled by making use of the thermal resistance concept and drawing the thermal resistance network, as shown in Fig. 17–12. Noting that the area of the window is A = 0.8 m × 1.5 m = 1.2 m², the individual resistances are evaluated from their definitions to be

$$R_i = R_{\text{conv, 1}} = \frac{1}{h_1 A} = \frac{1}{(10\ \text{W/m}^2\cdot{}^\circ\text{C})(1.2\ \text{m}^2)} = 0.08333^\circ\text{C/W}$$

$$R_{\text{glass}} = \frac{L}{kA} = \frac{0.008\ \text{m}}{(0.78\ \text{W/m}\cdot{}^\circ\text{C})(1.2\ \text{m}^2)} = 0.00855^\circ\text{C/W}$$

$$R_o = R_{\text{conv, 2}} = \frac{1}{h_2 A} = \frac{1}{(40\ \text{W/m}^2\cdot{}^\circ\text{C})(1.2\ \text{m}^2)} = 0.02083^\circ\text{C/W}$$

Noting that all three resistances are in series, the total resistance is

$$R_{\text{total}} = R_{\text{conv, 1}} + R_{\text{glass}} + R_{\text{conv, 2}} = 0.08333 + 0.00855 + 0.02083$$
$$= 0.1127^\circ\text{C/W}$$

Then the steady rate of heat transfer through the window becomes

$$\dot{Q} = \frac{T_{\infty 1} - T_{\infty 2}}{R_{\text{total}}} = \frac{[20 - (-10)]^\circ\text{C}}{0.1127^\circ\text{C/W}} = 266\ \text{W}$$

Knowing the rate of heat transfer, the inner surface temperature of the window glass can be determined from

$$\dot{Q} = \frac{T_{\infty 1} - T_1}{R_{\text{conv, 1}}} \longrightarrow T_1 = T_{\infty 1} - \dot{Q}R_{\text{conv, 1}}$$
$$= 20^\circ\text{C} - (266\ \text{W})(0.08333^\circ\text{C/W})$$
$$= -2.2^\circ\text{C}$$

Discussion Note that the inner surface temperature of the window glass will be −2.2°C even though the temperature of the air in the room is maintained at 20°C. Such low surface temperatures are highly undesirable since they cause the formation of fog or even frost on the inner surfaces of the glass when the humidity in the room is high.

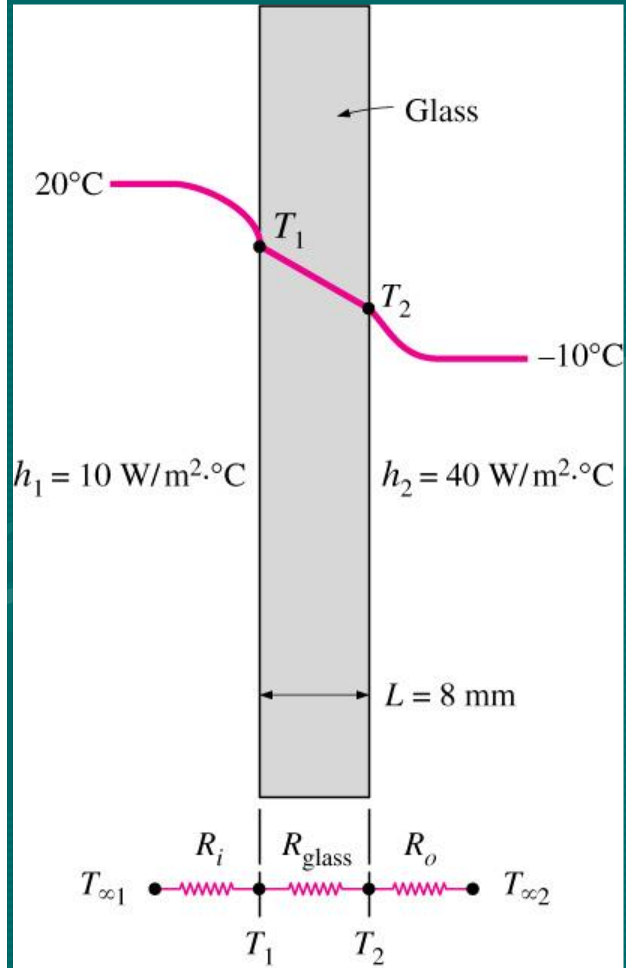


FIGURE 3–12
Schematic for Example 3–2.

EXAMPLE 3–3 **Heat Loss through Double-Pane Windows**

Consider a 0.8-m-high and 1.5-m-wide double-pane window consisting of two 4-mm-thick layers of glass (k = 0.78 W/m · °C) separated by a 10-mm-wide stagnant air space (k = 0.026 W/m · °C). Determine the steady rate of heat transfer through this double-pane window and the temperature of its inner surface for a day during which the room is maintained at 20°C while the temperature of the outdoors is −10°C. Take the convection heat transfer coefficients on the inner and outer surfaces of the window to be h_1 = 10 W/m² · °C and h_2 = 40 W/m² · °C, which includes the effects of radiation.

SOLUTION A double-pane window is considered. The rate of heat transfer through the window and the inner surface temperature are to be determined.

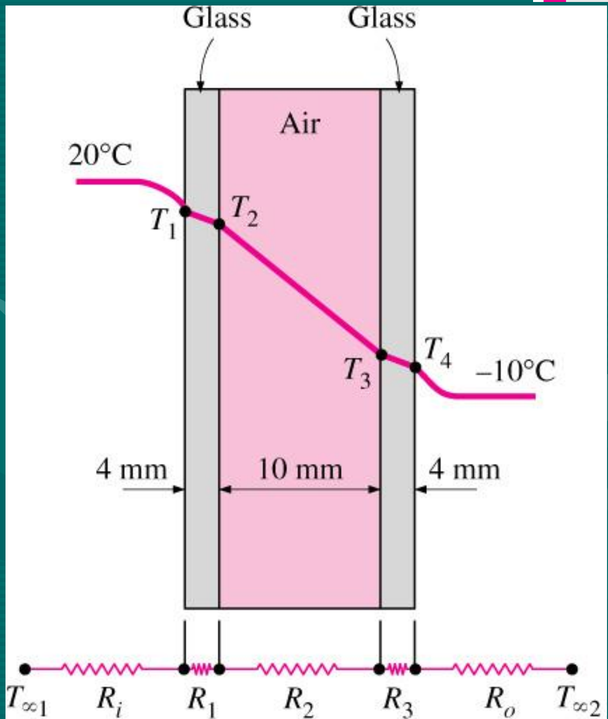


FIGURE 3–13
Schematic for Example 3–3.

Analysis This example problem is identical to the previous one except that the single 8-mm-thick window glass is replaced by two 4-mm-thick glasses that enclose a 10-mm-wide stagnant air space. Therefore, the thermal resistance network of this problem will involve two additional conduction resistances corresponding to the two additional layers, as shown in Fig. 17–13. Noting that the area of the window is again $A = 0.8\text{ m} \times 1.5\text{ m} = 1.2\text{ m}^2$, the individual resistances are evaluated from their definitions to be

$$R_i = R_{\text{conv},1} = \frac{1}{h_1 A} = \frac{1}{(10\text{ W/m}^2\cdot{}^\circ\text{C})(1.2\text{ m}^2)} = 0.08333^\circ\text{C/W}$$

$$R_1 = R_3 = R_{\text{glass}} = \frac{L_1}{k_1 A} = \frac{0.004\text{ m}}{(0.78\text{ W/m}\cdot{}^\circ\text{C})(1.2\text{ m}^2)} = 0.00427^\circ\text{C/W}$$

$$R_2 = R_{\text{air}} = \frac{L_2}{k_2 A} = \frac{0.01\text{ m}}{(0.026\text{ W/m}\cdot{}^\circ\text{C})(1.2\text{ m}^2)} = 0.3205^\circ\text{C/W}$$

$$R_o = R_{\text{conv},2} = \frac{1}{h_2 A} = \frac{1}{(40\text{ W/m}^2\cdot{}^\circ\text{C})(1.2\text{ m}^2)} = 0.02083^\circ\text{C/W}$$

Noting that all three resistances are in series, the total resistance is

$$\begin{aligned} R_{\text{total}} &= R_{\text{conv},1} + R_{\text{glass},1} + R_{\text{air}} + R_{\text{glass},2} + R_{\text{conv},2} \\ &= 0.08333 + 0.00427 + 0.3205 + 0.00427 + 0.02083 \\ &= 0.4332^\circ\text{C/W} \end{aligned}$$

Then the steady rate of heat transfer through the window becomes

$$\dot{Q} = \frac{T_{\infty 1} - T_{\infty 2}}{R_{\text{total}}} = \frac{[20 - (-10)]^\circ\text{C}}{0.4332^\circ\text{C/W}} = \mathbf{69.2\ W}$$

which is about one-fourth of the result obtained in the previous example. This explains the popularity of the double- and even triple-pane windows in cold climates. The drastic reduction in the heat transfer rate in this case is due to the large thermal resistance of the air layer between the glasses.

The inner surface temperature of the window in this case will be

$$T_1 = T_{\infty 1} - \dot{Q}R_{\text{conv},1} = 20^\circ\text{C} - (69.2\text{ W})(0.08333^\circ\text{C/W}) = \mathbf{14.2^\circ C}$$

which is considerably higher than the -2.2°C obtained in the previous example. Therefore, a double-pane window will rarely get fogged. A double-pane window will also reduce the heat gain in summer, and thus reduce the air-conditioning costs.

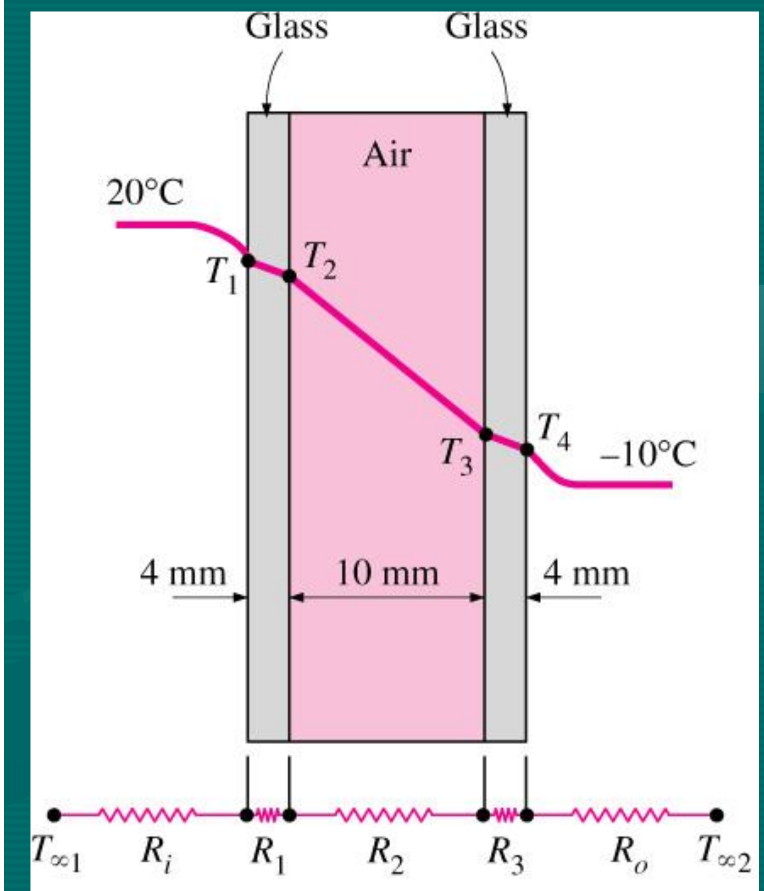


FIGURE 3–13
Schematic for Example 3–3.

THERMAL CONTACT RESISTANCE

In the analysis of heat conduction through multilayer solids, we assumed "perfect contact" at the interface of two layers, and thus no temperature drop at the interface.

Thermal Contact Resistance (*Rc.*): the resistance per unit interface area

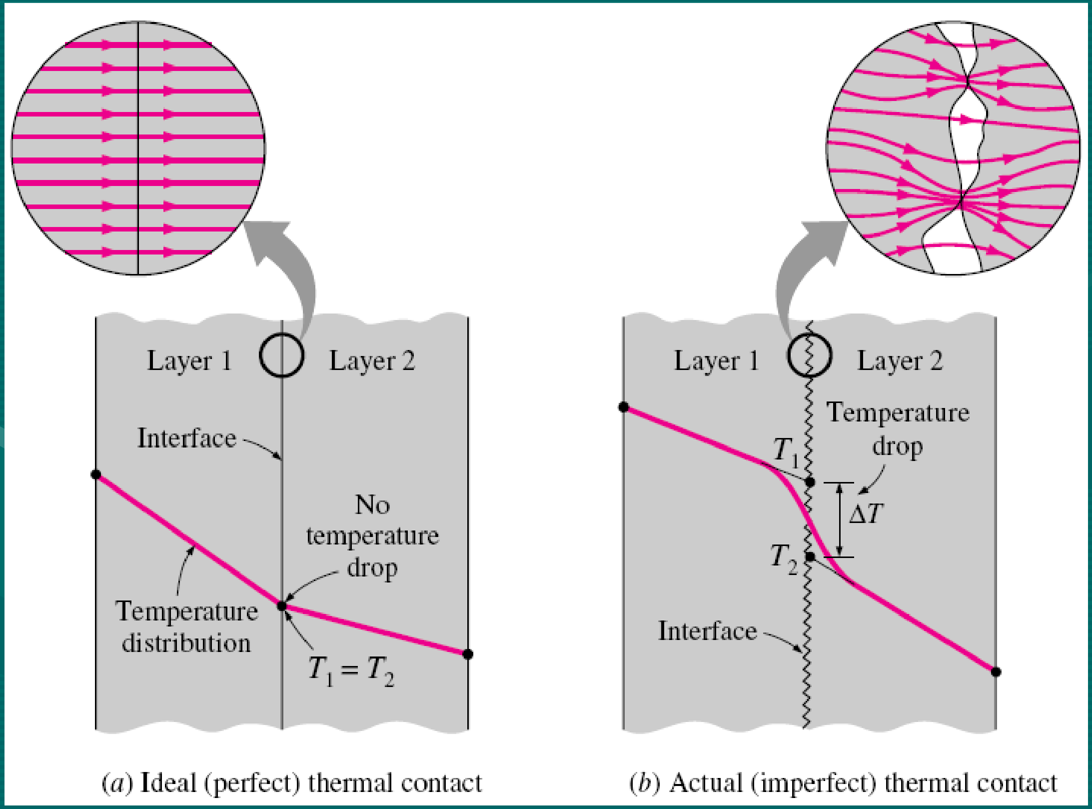


FIGURE 3–14
Temperature distribution and heat flow lines along two solid plates pressed against each other for the case of perfect and imperfect contact.

Heat transfer through the interface of two metal rods of cross-sectional area *A* is the sum of the heat transfers through the *solid contact spots* and the *gaps* in the noncontact areas and can be expressed as

$$\dot{Q} = \dot{Q}_{\text{contact}} + \dot{Q}_{\text{gap}}$$

An analogous manner to Newton's law of cooling:

$$\dot{Q} = h_c A \, \Delta T_{\text{interface}}$$

A: the apparent interface area (which is the same as the cross-sectional area of the rods

$T_{interface}$: the effective temperature difference at the interface

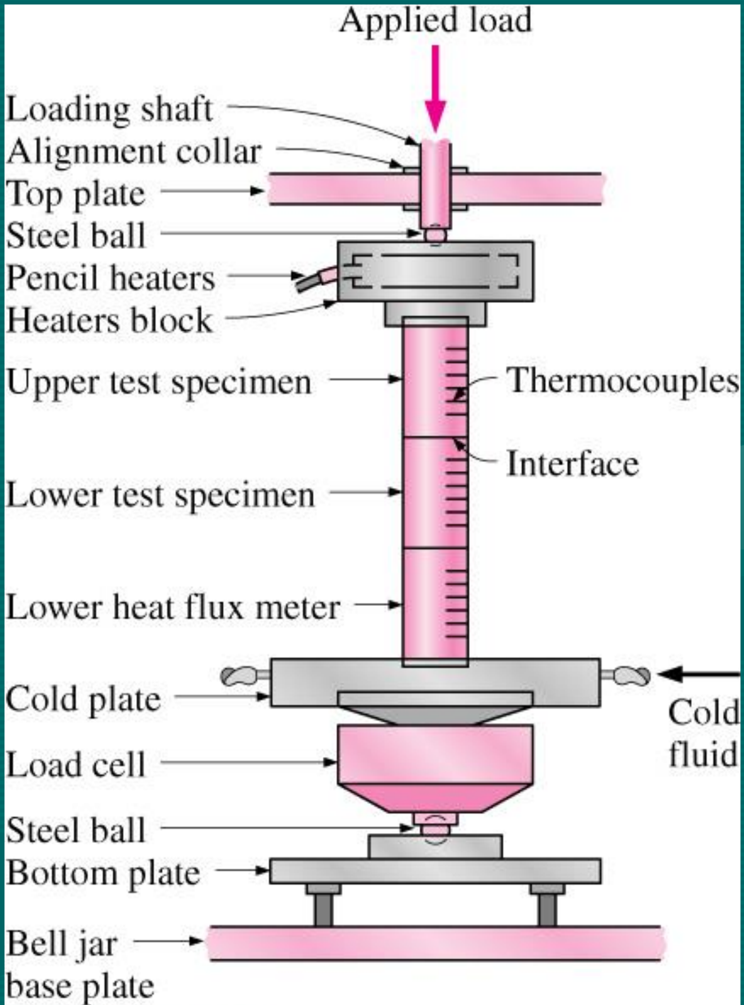


FIGURE 3–15

A typical experimental setup for the determination of thermal contact resistance (from Song et al.).

The thermal contact conductance is expressed as

$$h_c = \frac{\dot{Q}/A}{\Delta T_{\text{interface}}} \qquad (\text{W/m}^2 \cdot {}^\circ\text{C})$$

It is related to thermal contact resistance by

$$R_c = \frac{1}{h_c} = \frac{\Delta T_{\text{interface}}}{\dot{Q}/A} \qquad (\text{m}^2 \cdot {}^\circ\text{C/W})$$

The thermal resistance of a 1-cm-thick layer of an insulating material per unit surface area is

$$R_{c,\text{ insulation}} = \frac{L}{k} = \frac{0.01\text{ m}}{0.04\text{ W/m} \cdot {}^\circ\text{C}} = 0.25\text{ m}^2 \cdot {}^\circ\text{C/W}$$

whereas for a 1-cm-thick layer of copper, it is

$$R_{c,\text{ copper}} = \frac{L}{k} = \frac{0.01\text{ m}}{386\text{ W/m} \cdot {}^\circ\text{C}} = 0.000026\text{ m}^2 \cdot {}^\circ\text{C/W}$$

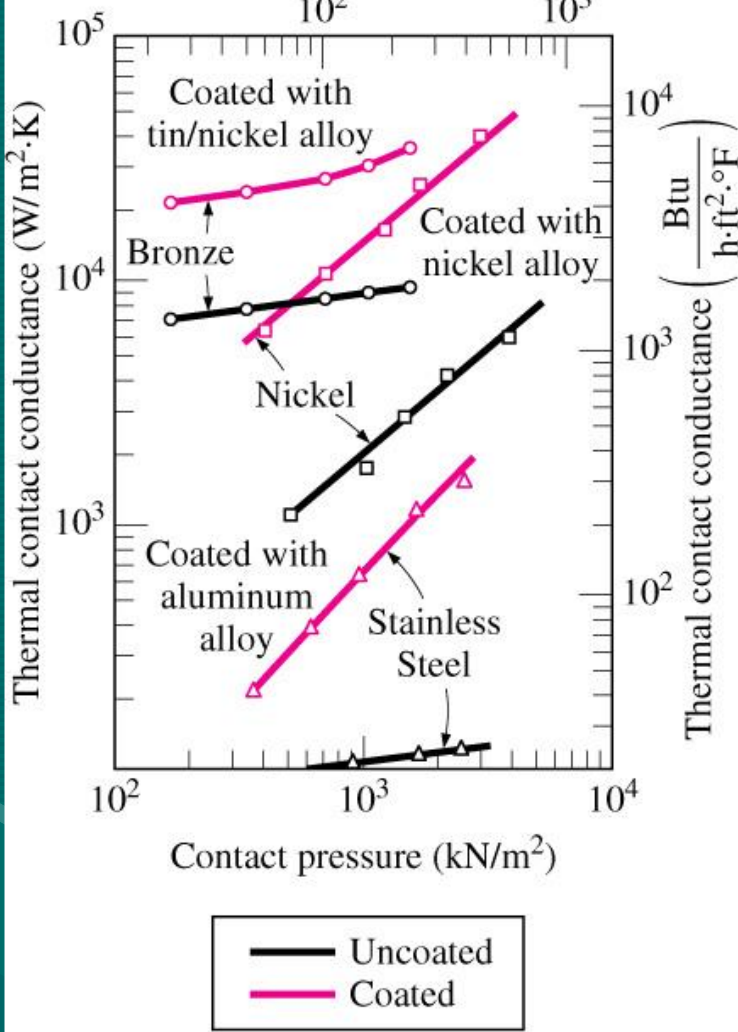


FIGURE 3–16

Effect of metallic coatings on thermal contact conductance (from Peterson).

TABLE 3–1

Thermal contact conductance for aluminum plates with different fluids at the interface for a surface roughness of 10 μm and interface pressure of 1 atm (from Fried).

Fluid at the interface	Contact conductance, h_c, W/m² · K
Air	3640
Helium	9520
Hydrogen	13,900
Silicone oil	19,000
Glycerin	37,700

TABLE 3–2

Thermal contact conductance of some metal surfaces in air (from various sources)

Material	Surface condition	Roughness, μm	Temperature, °C	Pressure, MPa	h_c,* W/m^2 · °C
Identical Metal Pairs					
416 Stainless steel	Ground	2.54	90–200	0.17–2.5	3800
304 Stainless steel	Ground	1.14	20	4–7	1900
Aluminum	Ground	2.54	150	1.2–2.5	11,400
Copper	Ground	1.27	20	1.2–20	143,000
Copper	Milled	3.81	20	1–5	55,500
Copper (vacuum)	Milled	0.25	30	0.17–7	11,400
Dissimilar Metal Pairs					
Stainless steel–				10	2900
Aluminum		20–30	20	20	3600
Stainless steel–				10	16,400
Aluminum		1.0–2.0	20	20	20,800
Steel Ct-30–				10	50,000
Aluminum	Ground	1.4–2.0	20	15–35	59,000
Steel Ct-30–				10	4800
Aluminum	Milled	4.5–7.2	20	30	8300
				5	42,000
Aluminum-Copper	Ground	1.17–1.4	20	15	56,000
				10	12,000
Aluminum-Copper	Milled	4.4–4.5	20	20–35	22,000

*Divide the given values by 5.678 to convert to Btu/h · ft^2 · °F.

The thermal contact conductance is highest (with the lowest contact resistance) for soft metals with smooth surfaces at high pressure.

EXAMPLE 3–4 **Equivalent Thickness for Contact Resistance**

The thermal contact conductance at the interface of two 1-cm-thick aluminum plates is measured to be 11,000 $W/m^2 \cdot °C$. Determine the thickness of the aluminum plate whose thermal resistance is equal to the thermal resistance of the interface between the plates (Fig. 17–17).

SOLUTION The thickness of the aluminum plate whose thermal resistance is equal to the thermal contact resistance is to be determined.

Properties The thermal conductivity of aluminum at room temperature is $k = 237$ W/m · °C (Table A–25).

Analysis Noting that thermal contact resistance is the inverse of thermal contact conductance, the thermal contact resistance is

$$R_c = \frac{1}{h_c} = \frac{1}{11{,}000 \text{ W/m}^2 \cdot °\text{C}} = 0.909 \times 10^{-4} \text{ m}^2 \cdot °\text{C/W}$$

For a unit surface area, the thermal resistance of a flat plate is defined as

$$R = \frac{L}{k}$$

where L is the thickness of the plate and k is the thermal conductivity. Setting $R = R_c$, the equivalent thickness is determined from the relation above to be

$$L = kR_c = (237 \text{ W/m} \cdot °\text{C})(0.909 \times 10^{-4} \text{ m}^2 \cdot °\text{C/W}) = 0.0215 \text{ m} = 2.15 \text{ cm}$$

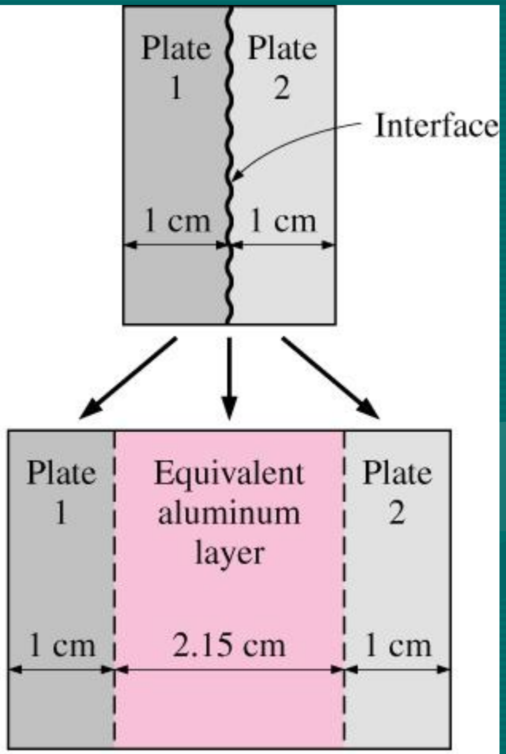


FIGURE 3–17
Schematic for Example 3–4.

EXAMPLE 3–5 Contact Resistance of Transistors

Four identical power transistors with aluminum casing are attached on one side of a 1-cm-thick 20-cm × 20-cm square copper plate (k = 386 W/m · °C) by screws that exert an average pressure of 6 MPa (Fig. 17–18). The base area of each transistor is 8 cm^2, and each transistor is placed at the center of a 10-cm × 10-cm quarter section of the plate. The interface roughness is estimated to be about 1.5 μm. All transistors are covered by a thick Plexiglas layer, which is a poor conductor of heat, and thus all the heat generated at the junction of the transistor must be dissipated to the ambient at 20°C through the back surface of the copper plate. The combined convection/radiation heat transfer coefficient at the back surface can be taken to be 25 W/m^2 · °C. If the case temperature of the transistor is not to exceed 70°C, determine the maximum power each transistor can dissipate safely, and the temperature jump at the case-plate interface.

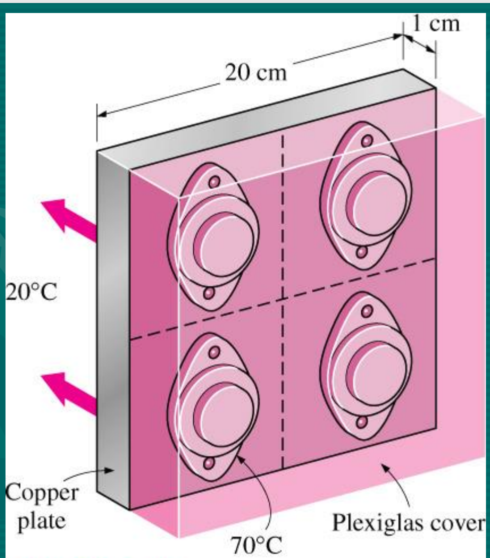


FIGURE 3–18
Schematic for Example 3–5.

SOLUTION Four identical power transistors are attached on a copper plate. For a maximum case temperature of 70°C, the maximum power dissipation and the temperature jump at the interface are to be determined.

Assumptions **1** Steady operating conditions exist. **2** Heat transfer can be approximated as being one-dimensional, although it is recognized that heat conduction in some parts of the plate will be two-dimensional since the plate area is much larger than the base area of the transistor. But the large thermal conductivity of copper will minimize this effect. **3** All the heat generated at the junction is dissipated through the back surface of the plate since the transistors are covered by a thick Plexiglas layer. **4** Thermal conductivities are constant.

Properties The thermal conductivity of copper is given to be $k = 386$ W/m · °C. The contact conductance is obtained from Table 17–2 to be $h_c = 42{,}000$ W/m² · °C, which corresponds to copper-aluminum interface for the case of 1.17–1.4 μm roughness and 5 MPa pressure, which is sufficiently close to what we have.

Analysis The contact area between the case and the plate is given to be 8 cm², and the plate area for each transistor is 100 cm². The thermal resistance network of this problem consists of three resistances in series (interface, plate, and convection), which are determined to be

$$R_{\text{interface}} = \frac{1}{h_c A_c} = \frac{1}{(42{,}000\ \text{W/m}^2 \cdot {}^\circ\text{C})(8 \times 10^{-4}\ \text{m}^2)} = 0.030^\circ\text{C/W}$$

$$R_{\text{plate}} = \frac{L}{kA} = \frac{0.01\ \text{m}}{(386\ \text{W/m} \cdot {}^\circ\text{C})(0.01\ \text{m}^2)} = 0.0026^\circ\text{C/W}$$

$$R_{\text{conv}} = \frac{1}{h_o A} = \frac{1}{(25\ \text{W/m}^2 \cdot {}^\circ\text{C})(0.01\ \text{m}^2)} = 4.0^\circ\text{C/W}$$

The total thermal resistance is then

$$R_{\text{total}} = R_{\text{interface}} + R_{\text{plate}} + R_{\text{ambient}} = 0.030 + 0.0026 + 4.0 = 4.0326°\text{C/W}$$

Note that the thermal resistance of a copper plate is very small and can be ignored altogether. Then the rate of heat transfer is determined to be

$$\dot{Q} = \frac{\Delta T}{R_{\text{total}}} = \frac{(70 - 20)°\text{C}}{4.0326°\text{C/W}} = \mathbf{12.4\ W}$$

Therefore, the power transistor should not be operated at power levels greater than 12.4 W if the case temperature is not to exceed 70°C.

The temperature jump at the interface is determined from

$$\Delta T_{\text{interface}} = \dot{Q}R_{\text{interface}} = (12.4\ \text{W})(0.030°\text{C/W}) = \mathbf{0.37°C}$$

which is not very large. Therefore, even if we eliminate the thermal contact resistance at the interface completely, we will lower the operating temperature of the transistor in this case by less than 0.4°C.

GENERALIZED THERMAL RESISTANCE NETWORKS

For the composite wall consisting of two parallel layers, the total heat transfer is the sum of the heat transfers through each layer.

$$\dot{Q} = \dot{Q}_1 + \dot{Q}_2 = \frac{T_1 - T_2}{R_1} + \frac{T_1 - T_2}{R_2} = (T_1 - T_2)\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

With the electrical analogy

$$\dot{Q} = \frac{T_1 - T_2}{R_{\text{total}}}$$

with

$$\frac{1}{R_{\text{total}}} = \frac{1}{R_1} + \frac{1}{R_2} \longrightarrow R_{\text{total}} = \frac{R_1 R_2}{R_1 + R_2}$$

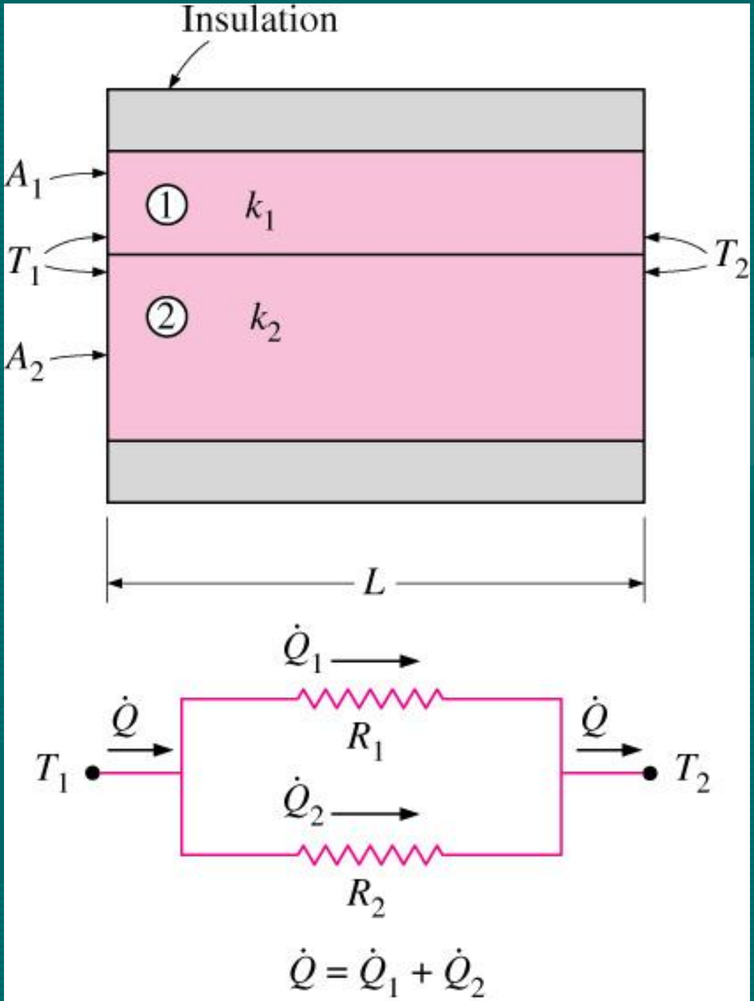


FIGURE 3–19
Thermal resistance network for two parallel layers.

For the combined series-parallel arrangement, the total rate of heat transfer through this composite system is

$$\dot{Q} = \frac{T_1 - T_\infty}{R_{\text{total}}}$$

with

$$R_{\text{total}} = R_{12} + R_3 + R_{\text{conv}} = \frac{R_1 R_2}{R_1 + R_2} + R_3 + R_{\text{conv}}$$

and

$$R_1 = \frac{L_1}{k_1 A_1}, \qquad R_2 = \frac{L_2}{k_2 A_2}, \qquad R_3 = \frac{L_3}{k_3 A_3}, \qquad R_{\text{conv}} = \frac{1}{hA_3}$$

Two assumptions:

(i) any plane wall normal to the *x*-axis is *isothermal* and

(ii) any plane parallel to the *x*-axis is *adiabatic.*

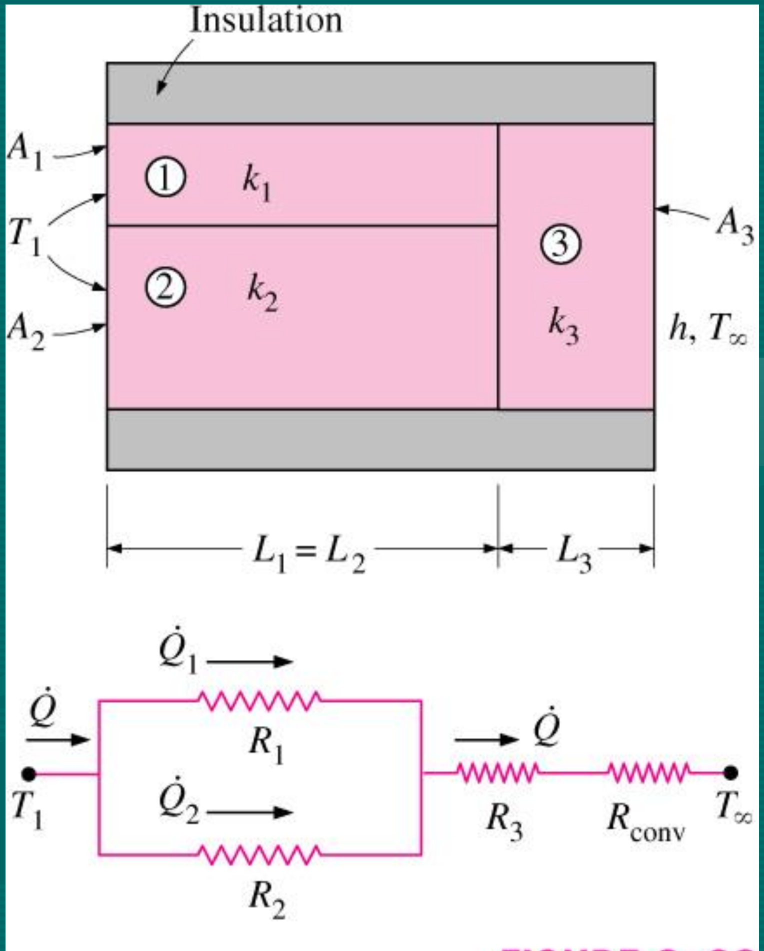


FIGURE 3–20
Thermal resistance network for combined series-parallel arrangement.

These assumptions result in different resistance networks, while the actual result lies between two assumptions.

HEAT CONDUCTION IN CYLINDERS AND SPHERES

The Fourier's law of heat conduction for heat transfer through the cylindrical layer is

$$\dot{Q}_{\text{cond, cyl}} = -kA\frac{dT}{dr} \qquad \text{(W)}$$

Here, $A = 2\pi rL$ is the heat transfer area at location *r*

$$\int_{r=r_1}^{r_2} \frac{\dot{Q}_{\text{cond, cyl}}}{A}\, dr = -\int_{T=T_1}^{T_2} k\, dT$$

We obtain

$$\dot{Q}_{\text{cond, cyl}} = \frac{T_1 - T_2}{R_{\text{cyl}}} \qquad \text{(W)}$$

since $\dot{Q}_{\text{cond, cyl}}$ = constant.

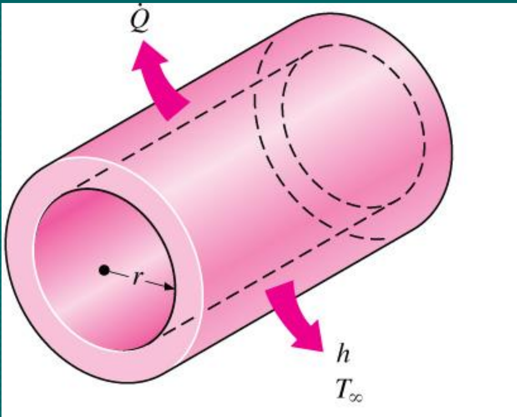


FIGURE 3–23
Heat is lost from a hot-water pipe to the air outside in the radial direction, and thus heat transfer from a long pipe is one-dimensional.

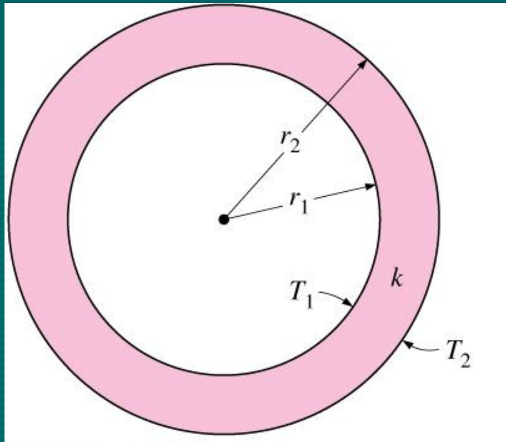


FIGURE 3–24
A long cylindrical pipe (or spherical shell) with specified inner and outer surface temperatures T_1 and T_2.

The *thermal resistance* of the cylindrical layer against heat conduction, or simply the **conduction resistance** of the cylinder layer.

$$R_{cyl} = \frac{\ln(r_2/r_1)}{2\pi Lk} = \frac{\ln(\text{Outer radius/Inner radius})}{2\pi \times (\text{Length}) \times (\text{Thermal conductivity})}$$

Repeating the analysis for *a spherical layer* by taking $A = 4\pi r^2$

$$\dot{Q}_{cond,\ sph} = \frac{T_1 - T_2}{R_{sph}}$$

with

$$R_{sph} = \frac{r_2 - r_1}{4\pi r_1 r_2 k} = \frac{\text{Outer radius} - \text{Inner radius}}{4\pi(\text{Outer radius})(\text{Inner radius})(\text{Thermal conductivity})}$$

which is the *thermal resistance* of the spherical layer against heat conduction, or simply the **conduction resistance** of the spherical layer.

The rate of heat transfer through a cylindrical or spherical layer under steady conditions:

$$\dot{Q} = \frac{T_{\infty 1} - T_{\infty 2}}{R_{\text{total}}}$$

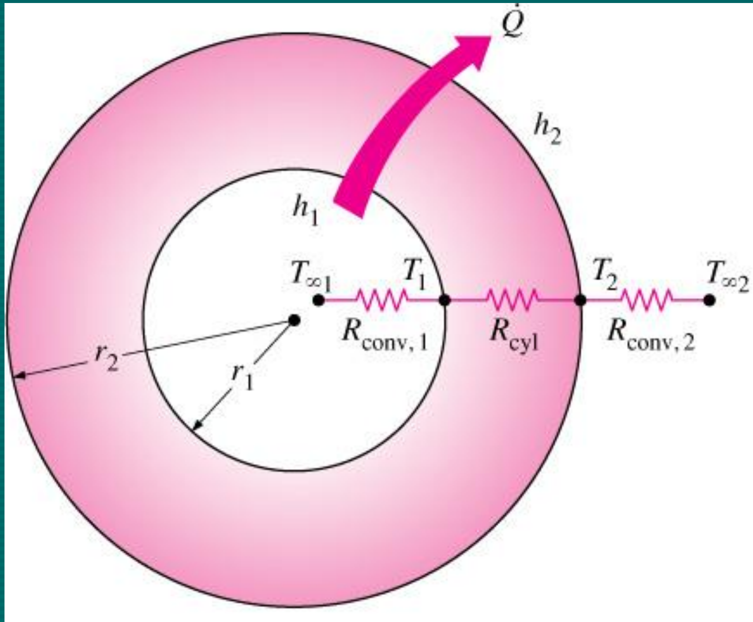
where

$$R_{\text{total}} = R_{\text{conv},1} + R_{\text{cyl}} + R_{\text{conv},2} = \frac{1}{(2\pi r_1 L)h_1} + \frac{\ln(r_2/r_1)}{2\pi L k} + \frac{1}{(2\pi r_2 L)h_2}$$

for a *cylindrical* layer, and

$$R_{\text{total}} = R_{\text{conv},1} + R_{\text{sph}} + R_{\text{conv},2} = \frac{1}{(4\pi r_1^2)h_1} + \frac{r_2 - r_1}{4\pi r_1 r_2 k} + \frac{1}{(4\pi r_2^2)h_2}$$

for a *spherical* layer.



$R_{\text{total}} = R_{\text{conv},1} + R_{\text{cyl}} + R_{\text{conv},2}$

FIGURE 3–25

The thermal resistance network for a cylindrical (or spherical) shell subjected to convection from both the inner and the outer sides.

A in the convection resistance relation $R_{conv} = 1/hA$ is the *surface area at which convection occurs.*

It is equal to $A = 2\pi rL$ for a cylindrical surface and $A = 4\pi r^2$ for a spherical surface of radius *r*.

Multilayered Cylinders and Spheres

Steady heat transfer through multilayered cylindrical or spherical shells is treated like multilayered plane walls.

$$\dot{Q} = \frac{T_{\infty 1} - T_{\infty 2}}{R_{\text{total}}}$$

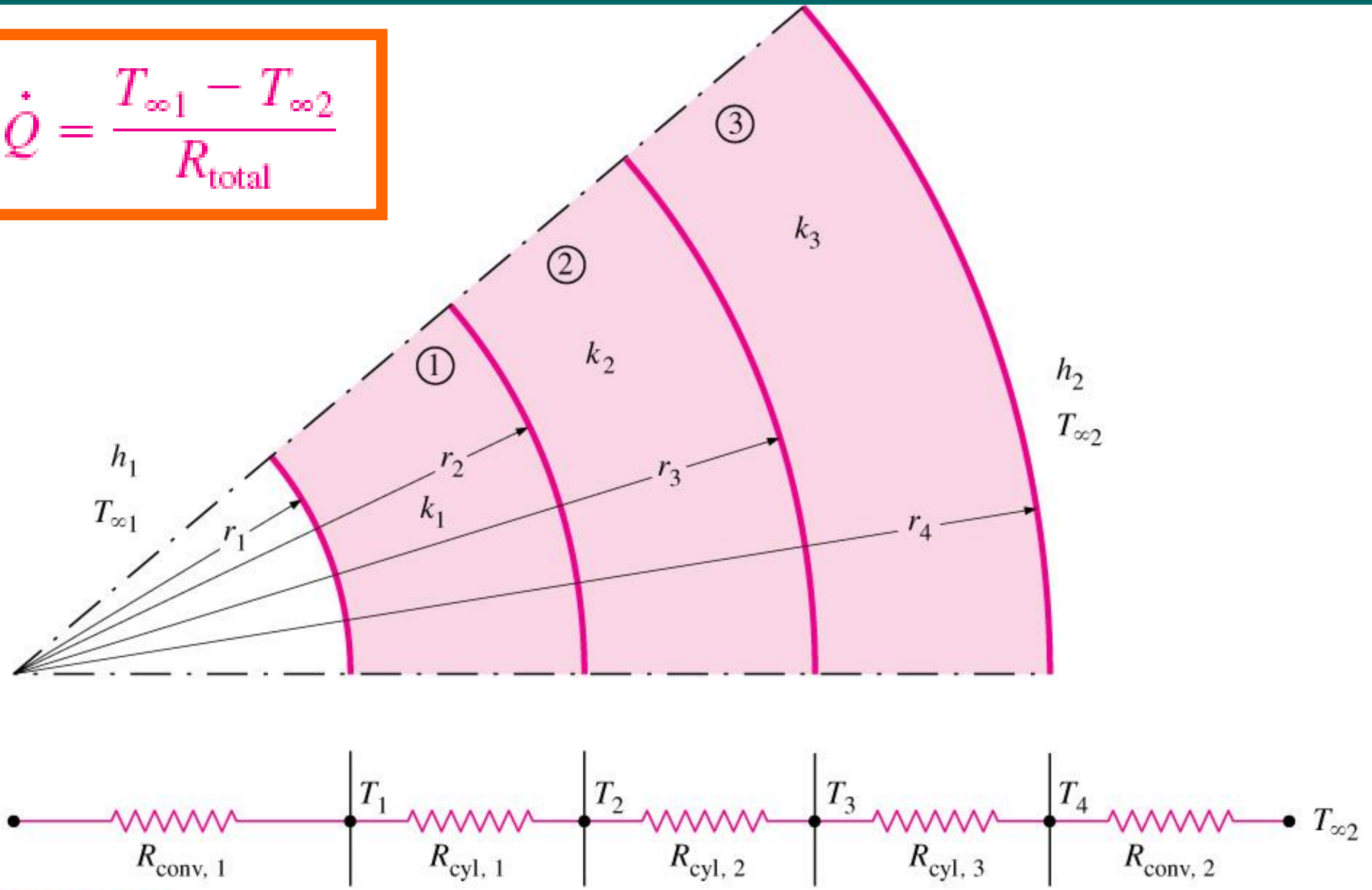


FIGURE 3–26

The thermal resistance network for heat transfer through a three-layered composite cylinder subjected to convection on both sides.

R_{total} is the *total thermal resistance*, expressed as

$$R_{\text{total}} = R_{\text{conv},1} + R_{\text{cyl},1} + R_{\text{cyl},2} + R_{\text{cyl},3} + R_{\text{conv},2}$$
$$= \frac{1}{h_1 A_1} + \frac{\ln(r_2/r_1)}{2\pi L k_1} + \frac{\ln(r_3/r_2)}{2\pi L k_2} + \frac{\ln(r_4/r_3)}{2\pi L k_3} + \frac{1}{h_2 A_4}$$

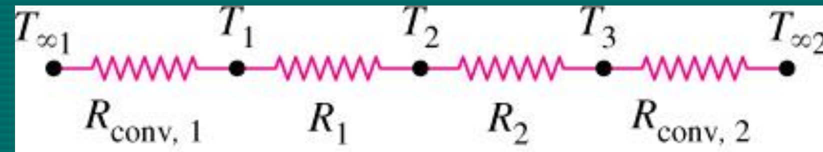
Here, $A_1 = 2\,\pi r_1 L$ and $A_4 = 2\,\pi r_4 L$

The total thermal resistance is simply the *arithmetic sum* of the individual thermal resistances in the path of heat flow

$$\dot{Q} = \frac{T_{\infty 1} - T_2}{R_{\text{conv},1} + R_{\text{cyl},1}} = \frac{T_{\infty 1} - T_2}{\dfrac{1}{h_1(2\pi r_1 L)} + \dfrac{\ln(r_2/r_1)}{2\pi L k_1}}$$

We can also calculate T_2 from

$$\dot{Q} = \frac{T_2 - T_{\infty 2}}{R_2 + R_3 + R_{\text{conv},2}} = \frac{T_2 - T_{\infty 2}}{\dfrac{\ln(r_3/r_2)}{2\pi L k_2} + \dfrac{\ln(r_4/r_3)}{2\pi L k_3} + \dfrac{1}{h_o(2\pi r_4 L)}}$$



$$\dot{Q} = \frac{T_{\infty 1} - T_1}{R_{\text{conv},1}}$$
$$= \frac{T_{\infty 1} - T_2}{R_{\text{conv},1} + R_1}$$
$$= \frac{T_1 - T_3}{R_1 + R_2}$$
$$= \frac{T_2 - T_3}{R_2}$$
$$= \frac{T_2 - T_{\infty 2}}{R_2 + R_{\text{conv},2}}$$
$$= \cdots$$

FIGURE 3–27

The ratio $\Delta T/R$ across any layer is equal to $\dot{Q}$, which remains constant in one-dimensional steady conduction.

EXAMPLE 3–7 Heat Transfer to a Spherical Container

A 17-m internal diameter spherical tank made of 2-cm-thick stainless steel (k = 15 W/m · °C) is used to store iced water at $T_{\infty 1}$ = 0°C. The tank is located in a room whose temperature is $T_{\infty 2}$ = 22°C. The walls of the room are also at 22°C. The outer surface of the tank is black and heat transfer between the outer surface of the tank and the surroundings is by natural convection and radiation. The convection heat transfer coefficients at the inner and the outer surfaces of the tank are h_1 = 80 W/m^2 · °C and h_2 = 10 W/m^2 · °C, respectively. Determine (*a*) the rate of heat transfer to the iced water in the tank and (*b*) the amount of ice at 0°C that melts during a 24-h period.

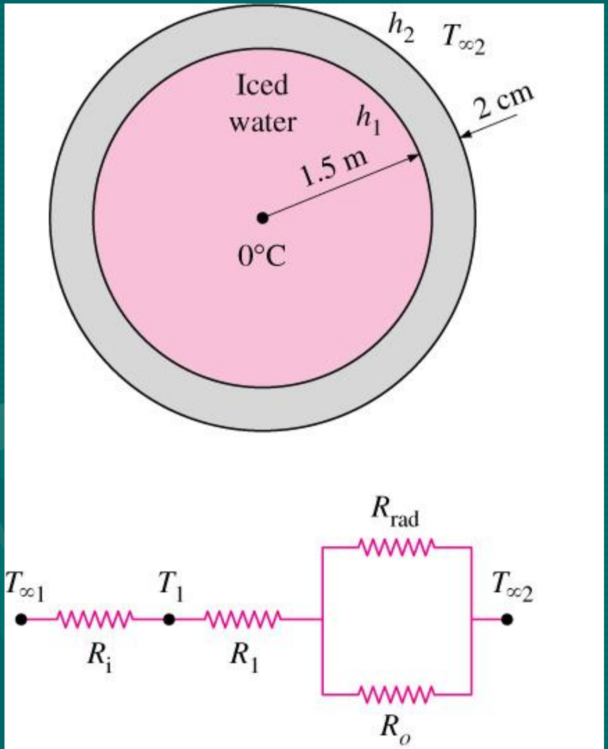


FIGURE 3–28
Schematic for Example 3–7.

SOLUTION A spherical container filled with iced water is subjected to convection and radiation heat transfer at its outer surface. The rate of heat transfer and the amount of ice that melts per day are to be determined.

Assumptions **1** Heat transfer is steady since the specified thermal conditions at the boundaries do not change with time. **2** Heat transfer is one-dimensional since there is thermal symmetry about the midpoint. **3** Thermal conductivity is constant.

Properties The thermal conductivity of steel is given to be $k = 15$ W/m · °C. The heat of fusion of water at atmospheric pressure is $h_{if} = 333.7$ kJ/kg. The outer surface of the tank is black and thus its emissivity is $\varepsilon = 1$.

Analysis (*a*) The thermal resistance network for this problem is given in Fig. 17–28. Noting that the inner diameter of the tank is $D_1 = 3$ m and the outer diameter is $D_2 = 3.04$ m, the inner and the outer surface areas of the tank are

$$A_1 = \pi D_1^2 = \pi(3\text{ m})^2 = 28.3\text{ m}^2$$
$$A_2 = \pi D_2^2 = \pi(3.04\text{ m})^2 = 29.0\text{ m}^2$$

Also, the radiation heat transfer coefficient is given by

$$h_{\text{rad}} = \varepsilon\sigma(T_2^2 + T_{\infty 2}^2)(T_2 + T_{\infty 2})$$

But we do not know the outer surface temperature T_2 of the tank, and thus we cannot calculate h_{rad}. Therefore, we need to assume a T_2 value now and check the accuracy of this assumption later. We will repeat the calculations if necessary using a revised value for T_2.

We note that T_2 must be between 0°C and 22°C, but it must be closer to 0°C, since the heat transfer coefficient inside the tank is much larger. Taking T_2 = 5°C = 278 K, the radiation heat transfer coefficient is determined to be

$$h_{rad} = (1)(5.67 \times 10^{-8}\ \mathrm{W/m^2 \cdot K^4})[(295\ \mathrm{K})^2 + (278\ \mathrm{K})^2][(295 + 278)\ \mathrm{K}]$$
$$= 5.34\ \mathrm{W/m^2 \cdot K} = 5.34\ \mathrm{W/m^2 \cdot °C}$$

Then the individual thermal resistances become

$$R_i = R_{\mathrm{conv},1} = \frac{1}{h_1 A_1} = \frac{1}{(80\ \mathrm{W/m^2 \cdot °C})(28.3\ \mathrm{m^2})} = 0.000442\mathrm{°C/W}$$

$$R_1 = R_{\mathrm{sphere}} = \frac{r_2 - r_1}{4\pi k r_1 r_2} = \frac{(1.52 - 1.50)\ \mathrm{m}}{4\pi\ (15\ \mathrm{W/m \cdot °C})(1.52\ \mathrm{m})(1.50\ \mathrm{m})}$$
$$= 0.000047\mathrm{°C/W}$$

$$R_o = R_{\mathrm{conv},2} = \frac{1}{h_2 A_2} = \frac{1}{(10\ \mathrm{W/m^2 \cdot °C})(29.0\ \mathrm{m^2})} = 0.00345\mathrm{°C/W}$$

$$R_{rad} = \frac{1}{h_{rad} A_2} = \frac{1}{(5.34\ \mathrm{W/m^2 \cdot °C})(29.0\ \mathrm{m^2})} = 0.00646\mathrm{°C/W}$$

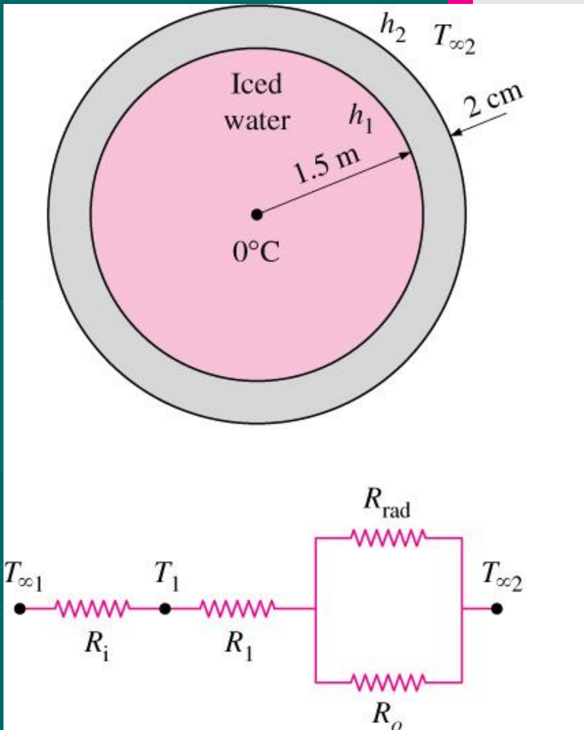


FIGURE 3–28
Schematic for Example 3–7.

The two parallel resistances R_o and R_{rad} can be replaced by an equivalent resistance R_{equiv} determined from

$$\frac{1}{R_{\text{equiv}}} = \frac{1}{R_o} + \frac{1}{R_{\text{rad}}} = \frac{1}{0.00345} + \frac{1}{0.00646} = 444.7\ \text{W/°C}$$

which gives

$$R_{\text{equiv}} = 0.00225\text{°C/W}$$

Now all the resistances are in series, and the total resistance is determined to be

$$R_{\text{total}} = R_i + R_1 + R_{\text{equiv}} = 0.000442 + 0.000047 + 0.00225 = 0.00274\text{°C/W}$$

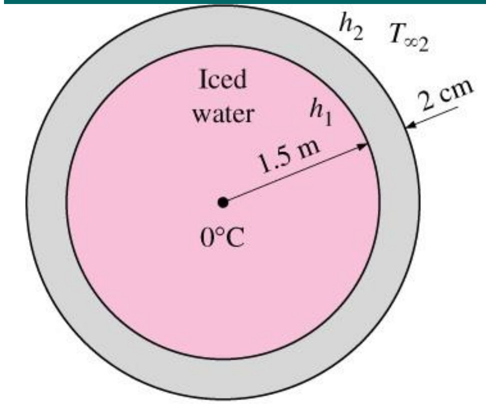
Then the steady rate of heat transfer to the iced water becomes

$$\dot{Q} = \frac{T_{\infty 2} - T_{\infty 1}}{R_{\text{total}}} = \frac{(22-0)\text{°C}}{0.00274\text{°C/W}} = \mathbf{8029\ W} \qquad (\text{or } \dot{Q} = 8.027\ \text{kJ/s})$$

To check the validity of our original assumption, we now determine the outer surface temperature from

$$\dot{Q} = \frac{T_{\infty 2} - T_2}{R_{\text{equiv}}} \longrightarrow T_2 = T_{\infty 2} - \dot{Q}R_{\text{equiv}}$$
$$= 22\text{°C} - (8029\ \text{W})(0.00225\text{°C/W}) = 4\text{°C}$$

which is sufficiently close to the 5°C assumed in the determination of the radiation heat transfer coefficient. Therefore, there is no need to repeat the calculations using 4°C for T_2.



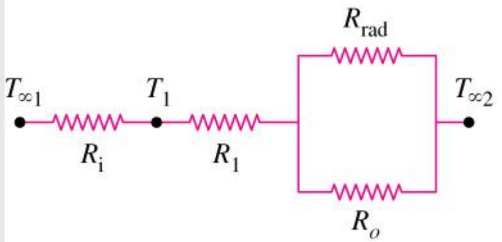


FIGURE 3–28
Schematic for Example 3–7.

(*b*) The total amount of heat transfer during a 24-h period is

$$Q = \dot{Q}\,\Delta t = (8.029 \text{ kJ/s})(24 \times 3600 \text{ s}) = 673{,}700 \text{ kJ}$$

Noting that it takes 333.7 kJ of energy to melt 1 kg of ice at 0°C, the amount of ice that will melt during a 24-h period is

$$m_{\text{ice}} = \frac{Q}{h_{if}} = \frac{673{,}700 \text{ kJ}}{333.7 \text{ kJ/kg}} = \mathbf{2079 \text{ kg}}$$

Therefore, about 2 metric tons of ice will melt in the tank every day.

Discussion An easier way to deal with combined convection and radiation at a surface when the surrounding medium and surfaces are at the same temperature is to add the radiation and convection heat transfer coefficients and to treat the result as the convection heat transfer coefficient. That is, to take $h = 10 + 5.34 = 15.34 \text{ W/m}^2 \cdot °\text{C}$ in this case. This way, we can ignore radiation since its contribution is accounted for in the convection heat transfer coefficient. The convection resistance of the outer surface in this case would be

$$R_{\text{combined}} = \frac{1}{h_{\text{combined}} A_2} = \frac{1}{(15.34 \text{ W/m}^2 \cdot °\text{C})(29.0 \text{ m}^2)} = 0.00225°\text{C/ W}$$

which is identical to the value obtained for the equivalent resistance for the parallel convection and the radiation resistances.

CRITICAL RADIUS OF INSULATION

The rate of heat transfer from the insulated pipe to the surrounding air is

$$\dot{Q} = \frac{T_1 - T_\infty}{R_{\text{ins}} + R_{\text{conv}}} = \frac{T_1 - T_\infty}{\dfrac{\ln(r_2/r_1)}{2\pi Lk} + \dfrac{1}{h(2\pi r_2 L)}}$$

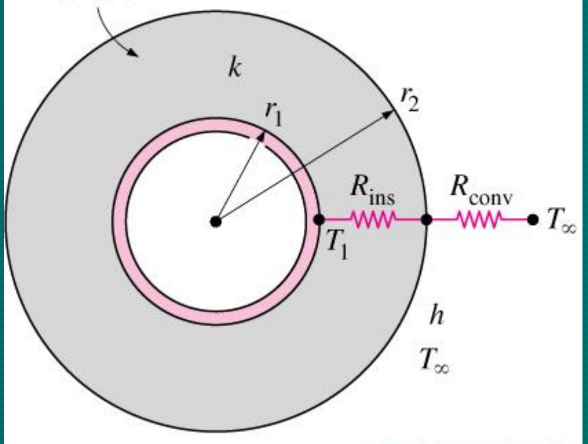


FIGURE 3–30
An insulated cylindrical pipe exposed to convection from the outer surface and the thermal resistance network associated with it.

Performing the differentiation and solving for r_2 yields the critical radius of insulation for a cylindrical body to be

$$r_{\text{cr, cylinder}} = \frac{k}{h} \qquad \text{(m)}$$

The critical radius of insulation for a spherical shell is

$$r_{\text{cr, sphere}} = \frac{2k}{h}$$

k : the thermal conductivity of the insulation
h : the convection heat transfer coefficient on the outer surface

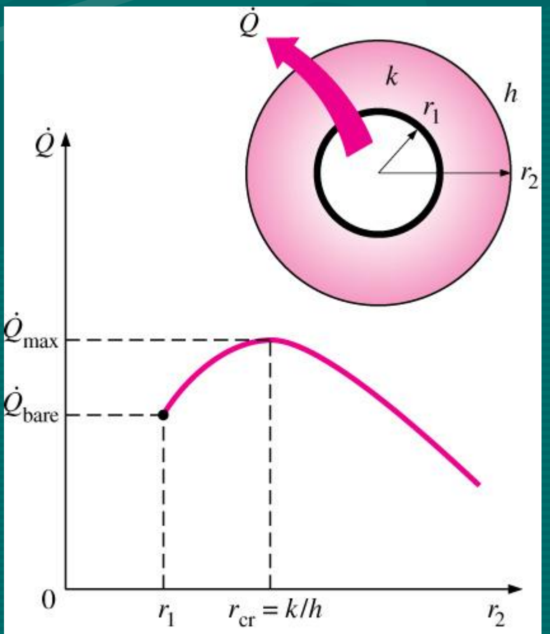


FIGURE 3–31

EXAMPLE 3–9 Heat Loss from an Insulated Electric Wire

A 17-mm-diameter and 5-m-long electric wire is tightly wrapped with a 2-mm-thick plastic cover whose thermal conductivity is $k = 0.15$ W/m · °C. Electrical measurements indicate that a current of 10 A passes through the wire and there is a voltage drop of 8 V along the wire. If the insulated wire is exposed to a medium at $T_\infty = 30°C$ with a heat transfer coefficient of $h = 12$ W/m^2 · °C, determine the temperature at the interface of the wire and the plastic cover in steady operation. Also determine whether doubling the thickness of the plastic cover will increase or decrease this interface temperature.

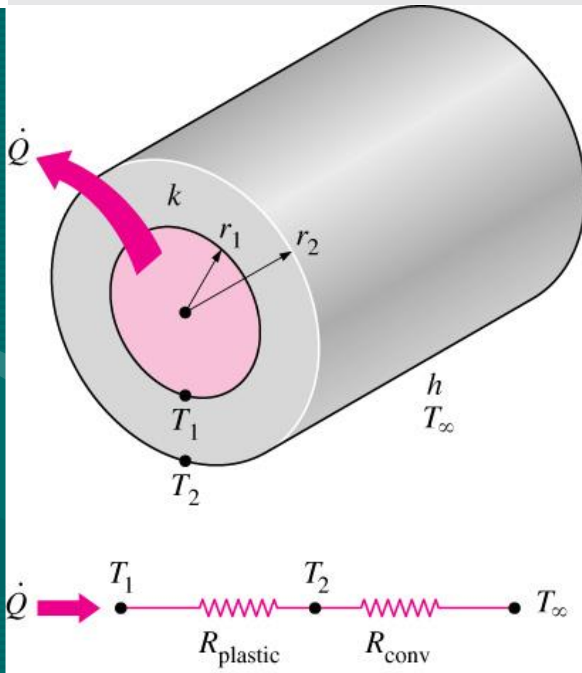


FIGURE 3–32
Schematic for Example 3–9.

SOLUTION An electric wire is tightly wrapped with a plastic cover. The interface temperature and the effect of doubling the thickness of the plastic cover on the interface temperature are to be determined.

Assumptions **1** Heat transfer is steady since there is no indication of any change with time. **2** Heat transfer is one-dimensional since there is thermal symmetry about the centerline and no variation in the axial direction. **3** Thermal conductivities are constant. **4** The thermal contact resistance at the interface is negligible. **5** Heat transfer coefficient incorporates the radiation effects, if any.

Properties The thermal conductivity of plastic is given to be $k = 0.15$ W/m · °C.

Analysis Heat is generated in the wire and its temperature rises as a result of resistance heating. We assume heat is generated uniformly throughout the wire and is transferred to the surrounding medium in the radial direction. In steady operation, the rate of heat transfer becomes equal to the heat generated within the wire, which is determined to be

$$\dot{Q} = \dot{W}_e = VI = (8\text{ V})(10\text{ A}) = 80\text{ W}$$

The thermal resistance network for this problem involves a conduction resistance for the plastic cover and a convection resistance for the outer surface in series, as shown in Fig. 17–32. The values of these two resistances are determined to be

$$A_2 = (2\pi r_2)L = 2\pi(0.0035\text{ m})(5\text{ m}) = 0.110\text{ m}^2$$

$$R_{\text{conv}} = \frac{1}{hA_2} = \frac{1}{(12\text{ W/m}^2\cdot{}^\circ\text{C})(0.110\text{ m}^2)} = 0.76^\circ\text{C/W}$$

$$R_{\text{plastic}} = \frac{\ln(r_2/r_1)}{2\pi kL} = \frac{\ln(3.5/1.5)}{2\pi(0.15\text{ W/m}\cdot{}^\circ\text{C})(5\text{ m})} = 0.18^\circ\text{C/W}$$

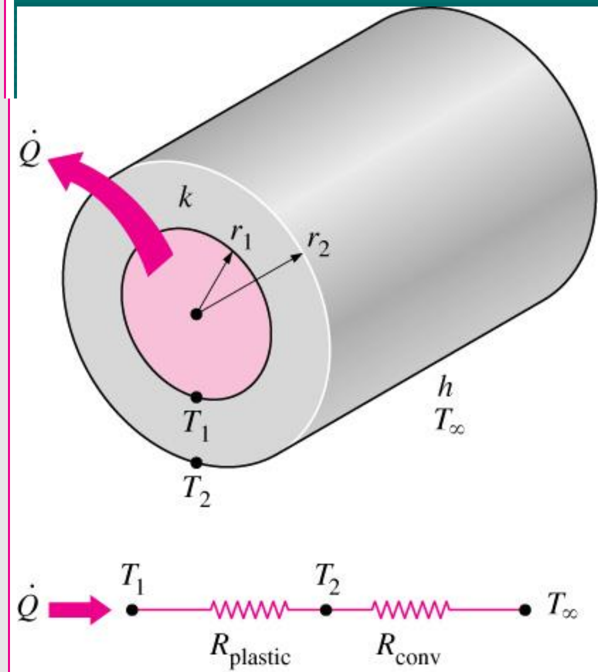


FIGURE 3–32
Schematic for Example 3–9.

and therefore

$$R_{\text{total}} = R_{\text{plastic}} + R_{\text{conv}} = 0.76 + 0.18 = 0.94°\text{C/W}$$

Then the interface temperature can be determined from

$$\dot{Q} = \frac{T_1 - T_\infty}{R_{\text{total}}} \longrightarrow \quad T_1 = T_\infty + \dot{Q}R_{\text{total}}$$
$$= 30°\text{C} + (80\text{ W})(0.94°\text{C/W}) = \mathbf{105°C}$$

Note that we did not involve the electrical wire directly in the thermal resistance network, since the wire involves heat generation.

To answer the second part of the question, we need to know the critical radius of insulation of the plastic cover. It is determined from Eq. 17–50 to be

$$r_{\text{cr}} = \frac{k}{h} = \frac{0.15\text{ W/m}\cdot°\text{C}}{12\text{ W/m}^2\cdot°\text{C}} = 0.0125\text{ m} = 12.5\text{ mm}$$

which is larger than the radius of the plastic cover. Therefore, increasing the thickness of the plastic cover will *enhance* heat transfer until the outer radius of the cover reaches 12.5 mm. As a result, the rate of heat transfer $\dot{Q}$ will *increase* when the interface temperature T_1 is held constant, or T_1 will *decrease* when $\dot{Q}$ is held constant, which is the case here.

Discussion It can be shown by repeating the calculations above for a 4-mm-thick plastic cover that the interface temperature drops to 90.6°C when the thickness of the plastic cover is doubled. It can also be shown in a similar manner that the interface reaches a minimum temperature of 83°C when the outer radius of the plastic cover equals the critical radius.

HEAT TRANSFER FROM FINNED SURFACES

The rate of heat transfer from a surface at a temperature T_s to the surrounding medium at T is given by Newton's law of cooling as

$$\dot{Q}_{conv} = hA_s(T_s - T_\infty)$$

As : the heat transfer surface area
h : the convection heat transfer coefficient

There are *two ways* to increase the rate of heat transfer:
1) to increase the *convection heat transfer coefficient h*
2) to increase the *surface area* A_s

Increasing *h* may require the installation of a pump or fan, or replacing the existing one with a larger one, but this approach may or may not be practical. Besides, it may not be adequate.

The alternative is to increase the surface area by attaching to the surface *extended surfaces* called *fins* made of highly conductive materials such as aluminum.

Consider *steady* operation with *no heat generation* in the fin with the following assumptions:

- The thermal conductivity *k* of the material remains constant.
- The convection heat transfer coefficient *h is constant* and *uniform* over the entire surface of the fin for convenience in the analysis.

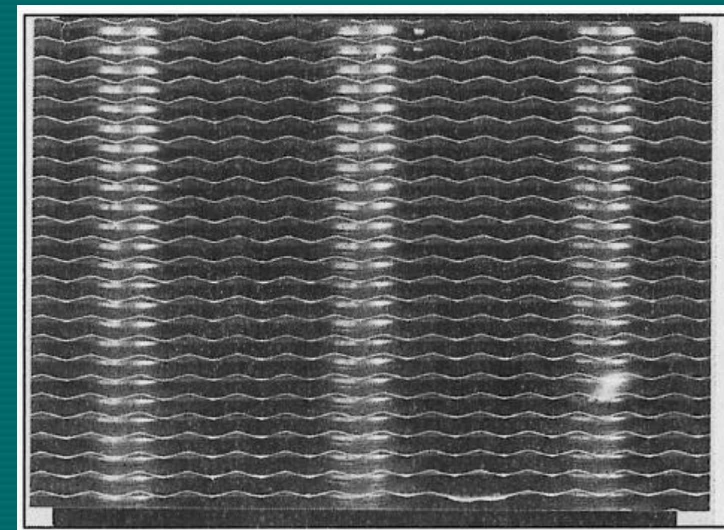


FIGURE 3–33
The thin plate fins of a car radiator greatly increase the rate of heat transfer to the air. (© Yunus Çengel, photo by James Kleiser.)

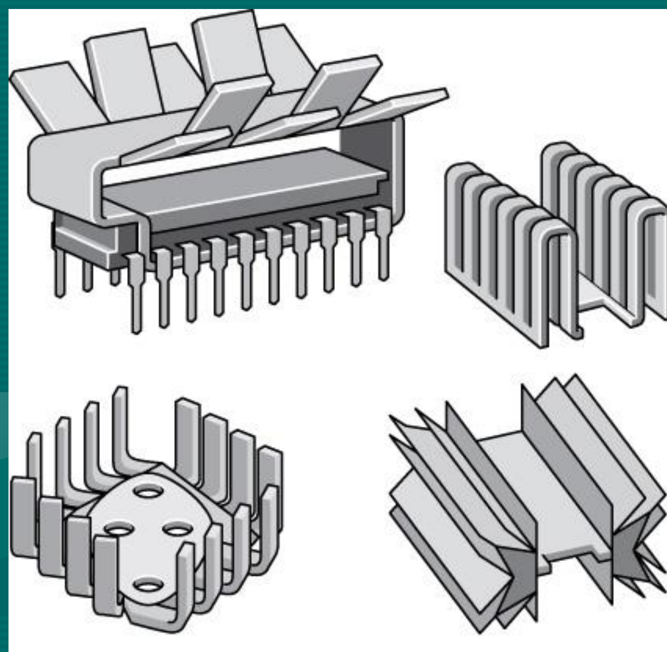


FIGURE 3–34
Some innovative fin designs.

Fin Equation

Under steady conditions, the energy balance on this volume element can be expressed as

$$\begin{pmatrix}\text{Rate of } \textit{heat} \\ \textit{conduction} \text{ into} \\ \text{the element at } x\end{pmatrix} = \begin{pmatrix}\text{Rate of } \textit{heat} \\ \textit{conduction} \text{ from the} \\ \text{element at } x + \Delta x\end{pmatrix} + \begin{pmatrix}\text{Rate of } \textit{heat} \\ \textit{convection} \text{ from} \\ \text{the element}\end{pmatrix}$$

or $$\dot{Q}_{\text{cond}, x} = \dot{Q}_{\text{cond}, x+\Delta x} + \dot{Q}_{\text{conv}}$$

with $$\dot{Q}_{\text{conv}} = h(p\,\Delta x)(T - T_\infty)$$

Substituting and dividing by Δx, we obtain

$$\frac{\dot{Q}_{\text{cond}, x+\Delta x} - \dot{Q}_{\text{cond}, x}}{\Delta x} + hp(T - T_\infty) = 0$$

Taking the limit as $\Delta x \to 0$ gives

$$\frac{d\dot{Q}_{\text{cond}}}{dx} + hp(T - T_\infty) = 0$$

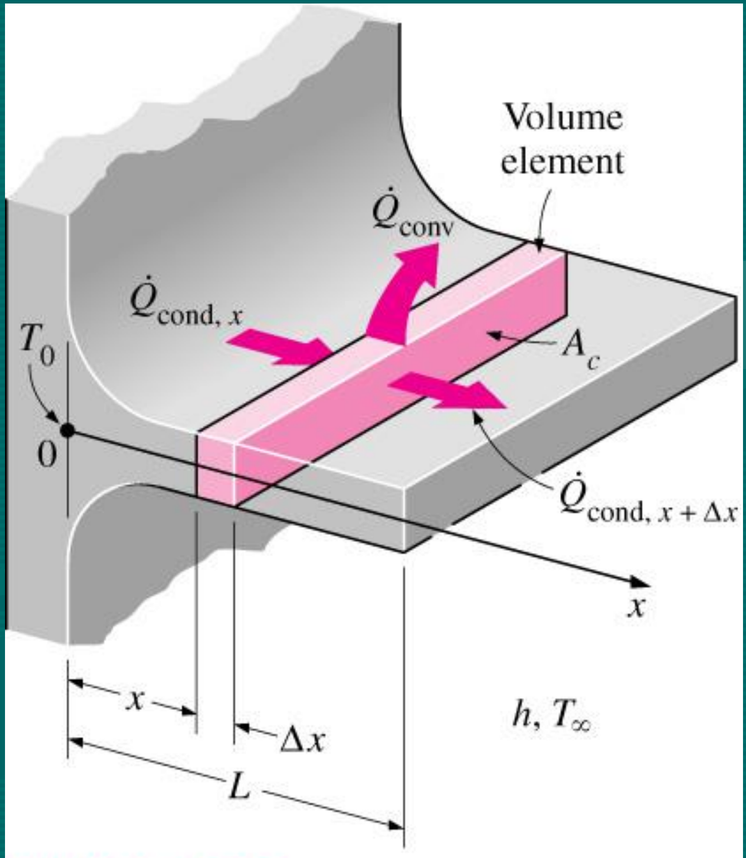


FIGURE 3–35
Volume element of a fin at location x having a length of Δx, cross-sectional area of A_c, and perimeter of p.

From Fourier's law of heat conduction we have $\dot{Q}_{\text{cond}} = -kA_c \dfrac{dT}{dx}$

where A_c: the cross-sectional area of the fin at location x

$$\frac{d}{dx}\left(kA_c \frac{dT}{dx}\right) - hp(T - T_\infty) = 0$$

In the special case (with *constant cross section* and *thermal conductivity*):

$$\frac{d^2\theta}{dx^2} - a^2\theta = 0 \qquad \text{with} \qquad a^2 = \frac{hp}{kA_c}$$

and $\theta = T - T_\infty$ is the *temperature excess.*

At the fin base we have $\theta_b = T_b - T_\infty$.

The function u and its second derivative must be *constant multiples* of each other.

$$\theta(x) = C_1 e^{ax} + C_2 e^{-ax}$$

where C_1 and C_2 are arbitrary constants.

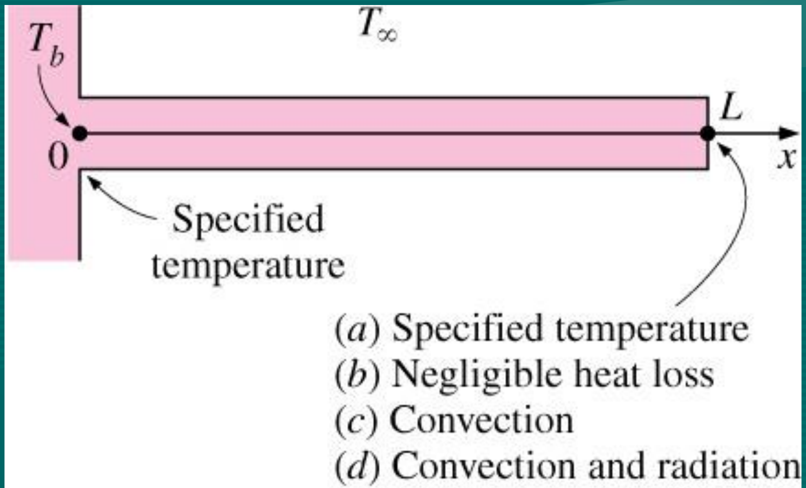


FIGURE 3–36
Boundary conditions at the fin base and the fin tip.

Boundary condition at fin base: $\theta(0) = \theta_b = T_b - T_\infty$

Infinitely Long Fin ($T_{\text{fin tip}} = T_\infty$)

For a sufficiently long fin of *uniform* cross section (*Ac* constant):

Boundary condition at fin tip: $\theta(L) = T(L) - T_\infty = 0 \quad \text{as} \quad L \rightarrow \infty$

Very long fin: $$\frac{T(x) - T_\infty}{T_b - T_\infty} = e^{-ax} = e^{-x\sqrt{hp/kA_c}}$$

Very long fin: $$\dot{Q}_{\text{long fin}} = -kA_c \left.\frac{dT}{dx}\right|_{x=0} = \sqrt{hpkA_c}\,(T_b - T_\infty)$$

p : the perimeter
A_c : the cross-sectional area of the fin
x : the distance from the fin base

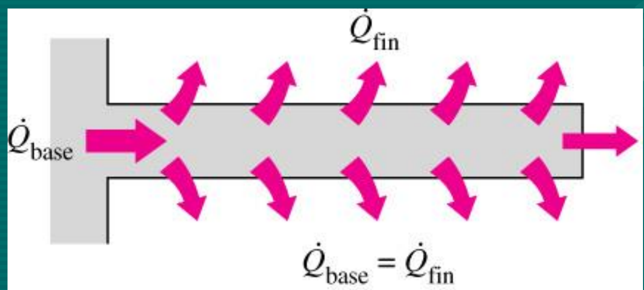
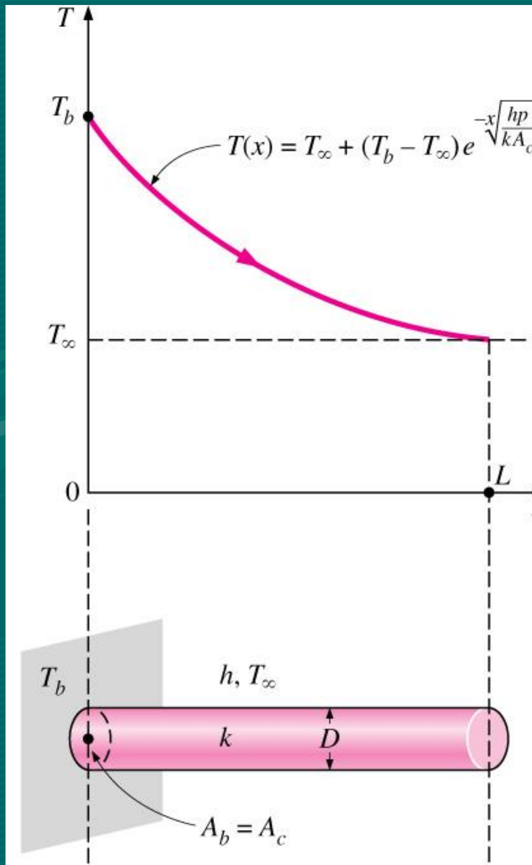


FIGURE 3–38
Under steady conditions, heat transfer from the exposed surfaces of the fin is equal to heat conduction to the fin at the base.



($p = \pi D$, $A_c = \pi D^2/4$ for a cylindrical fin)

FIGURE 3–37
A long circular fin of uniform cross section and the variation of temperature along it.

$$\dot{Q}_{\text{fin}} = \int_{A_{\text{fin}}} h[T(x) - T_\infty]\, dA_{\text{fin}} = \int_{A_{\text{fin}}} h\theta(x)\, dA_{\text{fin}}$$

Negligible Heat Loss from the Fin Tip (Insulated fin tip, $\dot{Q}_{\text{fin tip}} = 0$)

The fin tip can be assumed to be insulated, and the condition at the fin tip can be expressed as

Boundary condition at fin tip: $$\left.\frac{d\theta}{dx}\right|_{x=L} = 0$$

Adiabatic fin tip: $$\frac{T(x) - T_\infty}{T_b - T_\infty} = \frac{\cosh a(L - x)}{\cosh aL}$$

The rate of heat transfer from the fin can be determined again from Fourier's law of heat conduction:

Adiabatic fin tip: $$\dot{Q}_{\text{insulated tip}} = -kA_c \left.\frac{dT}{dx}\right|_{x=0} = \sqrt{hpkA_c}\,(T_b - T_\infty)\tanh aL$$

The heat transfer relations for the very long fin and the fin with negligible heat loss at the tip differ by the factor tanh *aL*, which approaches 1 as *L* becomes very large.

Convection (or Combined Convection and Radiation) from Fin Tip

A practical way of accounting for the heat loss from the fin tip is to replace the *fin length L* in the relation for the *insulated tip* case by a **corrected length** defined as

Corrected fin length: $$L_c = L + \frac{A_c}{p}$$

t: the thickness of the rectangular fins
D: the diameter of the cylindrical fins.

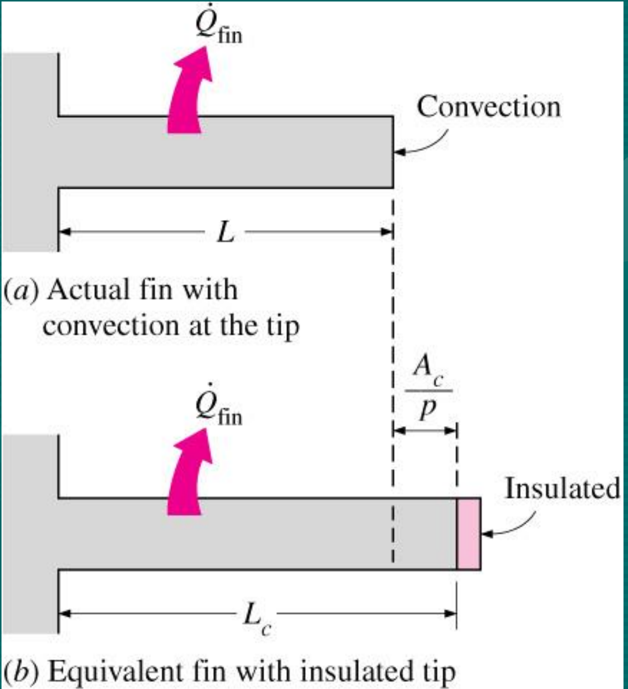


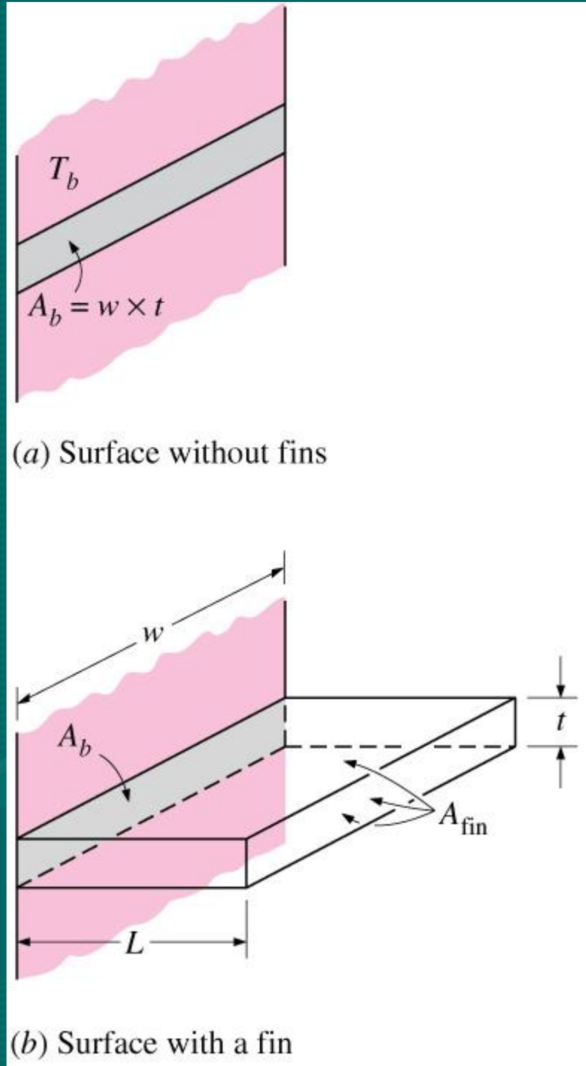
FIGURE 3–39
Corrected fin length L_c is defined such that heat transfer from a fin of length L_c with insulated tip is equal to heat transfer from the actual fin of length L with convection at the fin tip.

Fin Efficiency

In the limiting case of *zero thermal resistance* or *infinite thermal conductivity*, $(k \to \infty)$ the temperature of the fin will be uniform at the base value of T_b.

The heat transfer from the fin will be *maximum* in this case and can be expressed as

$$\dot{Q}_{\text{fin, max}} = hA_{\text{fin}}\,(T_b - T_\infty)$$



$$A_{\text{fin}} = 2 \times w \times L + w \times t$$
$$\cong 2 \times w \times L$$

FIGURE 3–40
Fins enhance heat transfer from a surface by enhancing surface area.

Fin efficiency can be defined as:

$$\eta_{\text{fin}} = \frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{fin, max}}} = \frac{\text{Actual heat transfer rate from the fin}}{\substack{\text{Ideal heat transfer rate from the fin} \\ \text{if the entire fin were at base temperature}}}$$

or

$$\dot{Q}_{\text{fin}} = \eta_{\text{fin}} \dot{Q}_{\text{fin, max}} = \eta_{\text{fin}} h A_{\text{fin}} (T_b - T_\infty)$$

For the cases of constant cross section of *very long fins* and *fins with insulated tips*, the fin efficiency can be expressed as

$$\eta_{\text{long fin}} = \frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{fin, max}}} = \frac{\sqrt{hpkA_c}\,(T_b - T_\infty)}{hA_{\text{fin}}\,(T_b - T_\infty)} = \frac{1}{L}\sqrt{\frac{kA_c}{hp}} = \frac{1}{aL}$$

and

$$\eta_{\text{insulated tip}} = \frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{fin, max}}} = \frac{\sqrt{hpkA_c}\,(T_b - T_\infty)\tanh aL}{hA_{\text{fin}}\,(T_b - T_\infty)} = \frac{\tanh aL}{aL}$$

since $A_{\text{fin}} = pL$ for fins with constant cross section.

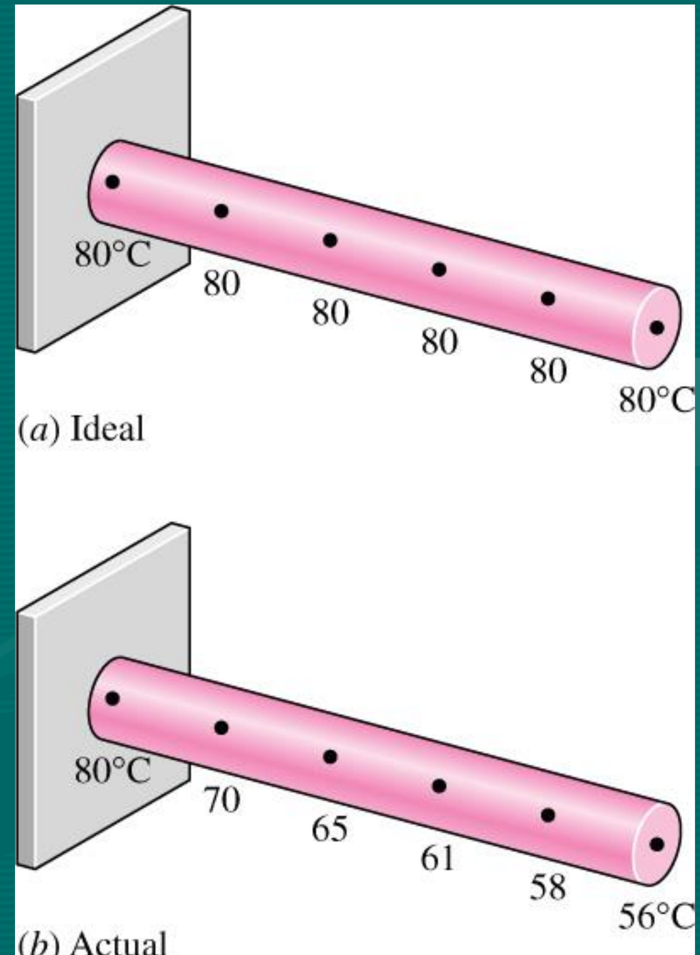


FIGURE 3–41
Ideal and actual temperature distribution along a fin.

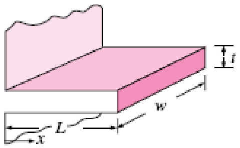
TABLE 3–3

Efficiency and surface areas of common fin configurations

Straight rectangular fins

$m = \sqrt{2h/kt}$
$L_c = L + t/2$
$A_{fin} = 2wL_c$

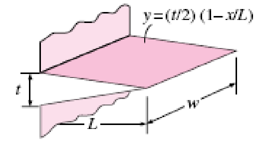
$$\eta_{fin} = \frac{\tanh mL_c}{mL_c}$$



Straight triangular fins

$m = \sqrt{2h/kt}$
$A_{fin} = 2w\sqrt{L^2 + (t/2)^2}$

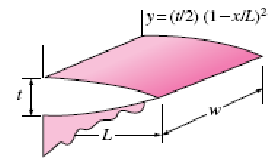
$$\eta_{fin} = \frac{1}{mL}\frac{I_1(2mL)}{I_0(2mL)}$$



Straight parabolic fins

$m = \sqrt{2h/kt}$
$A_{fin} = wL[C_1 + (L/t)\ln(t/L + C_1)]$
$C_1 = \sqrt{1 + (t/L)^2}$

$$\eta_{fin} = \frac{2}{1 + \sqrt{(2mL)^2 + 1}}$$

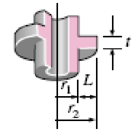


Circular fins of rectangular profile

$m = \sqrt{2h/kt}$
$r_{2c} = r_2 + t/2$
$A_{fin} = 2\pi(r_{2c}^2 - r_1^2)$

$$\eta_{fin} = C_2\frac{K_1(mr_1)I_1(mr_{2c}) - I_1(mr_1)K_1(mr_{2c})}{I_0(mr_1)K_1(mr_{2c}) + K_0(mr_1)I_1(mr_{2c})}$$

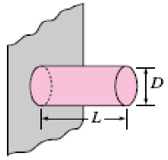
$$C_2 = \frac{2r_1/m}{r_{2c}^2 - r_1^2}$$



Pin fins of rectangular profile

$m = \sqrt{4h/kD}$
$L_c = L + D/4$
$A_{fin} = \pi DL_c$

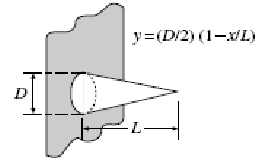
$$\eta_{fin} = \frac{\tanh mL_c}{mL_c}$$



Pin fins of triangular profile

$m = \sqrt{4h/kD}$
$A_{fin} = \dfrac{\pi D}{2}\sqrt{L^2 + (D/2)^2}$

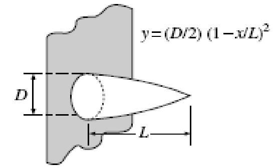
$$\eta_{fin} = \frac{2}{mL}\frac{I_2(2mL)}{I_1(2mL)}$$



Pin fins of parabolic profile

$m = \sqrt{4h/kD}$
$A_{fin} = \dfrac{\pi L^3}{8D}\left[C_3C_4 - \dfrac{L}{2D}ln(2DC_4/L + C_3)\right]$
$C_3 = 1 + 2(D/L)^2$
$C_4 = \sqrt{1 + (D/L)^2}$

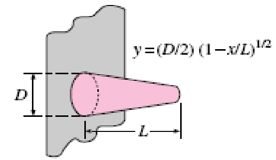
$$\eta_{fin} = \frac{2}{1 + \sqrt{(2mL/3)^2 + 1}}$$



Pin fins of parabolic profile (blunt tip)

$m = \sqrt{4h/kD}$
$A_{fin} = \dfrac{\pi D^4}{96L^2}\left\{[16(L/D)^2 + 1]^{3/2} - 1\right\}$

$$\eta_{fin} = \frac{3}{2mL}\frac{I_1(4mL/3)}{I_0(4mL/3)}$$



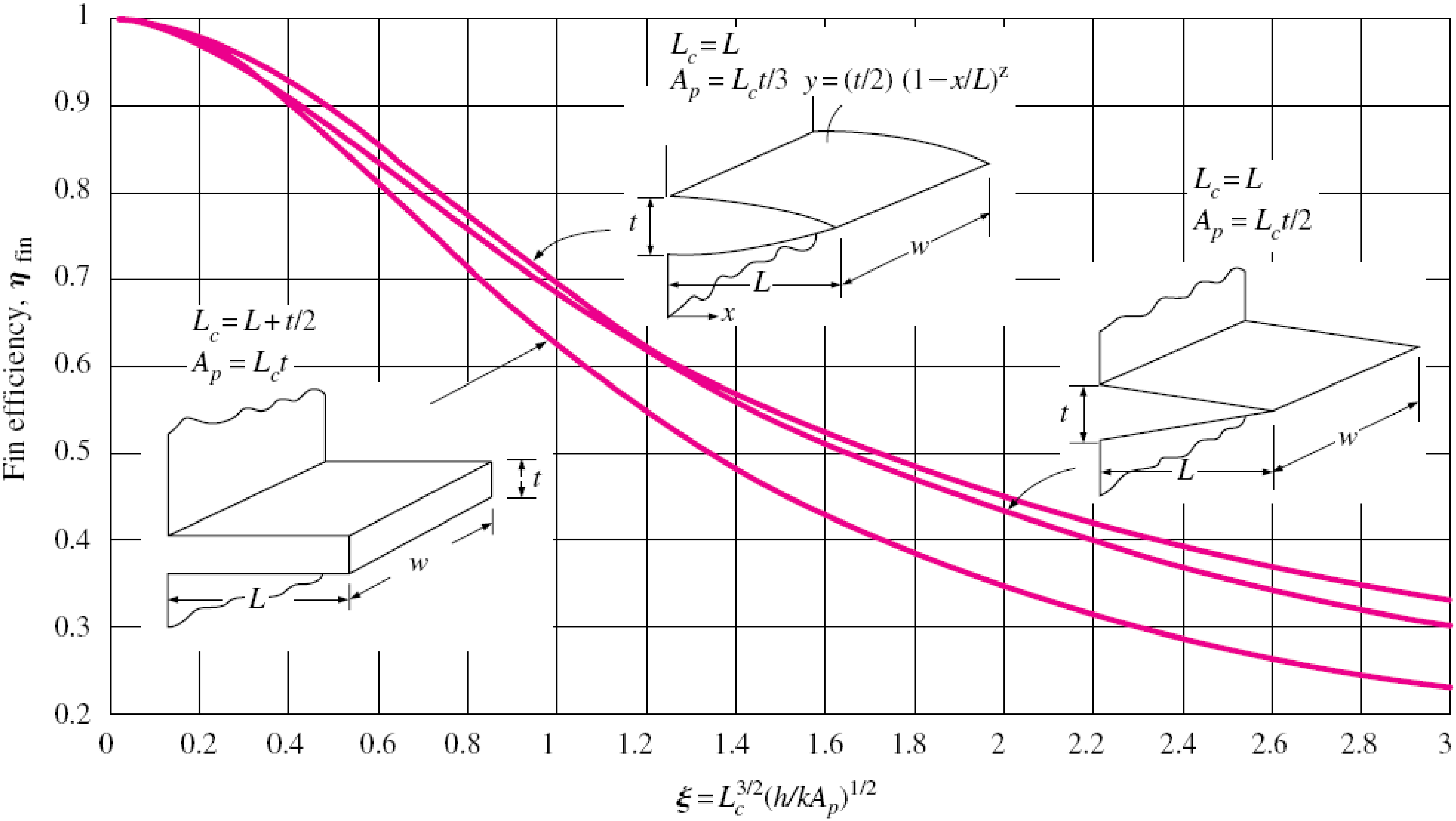


FIGURE 3–42

Efficiency of straight fins of rectangular, triangular, and parabolic profiles.

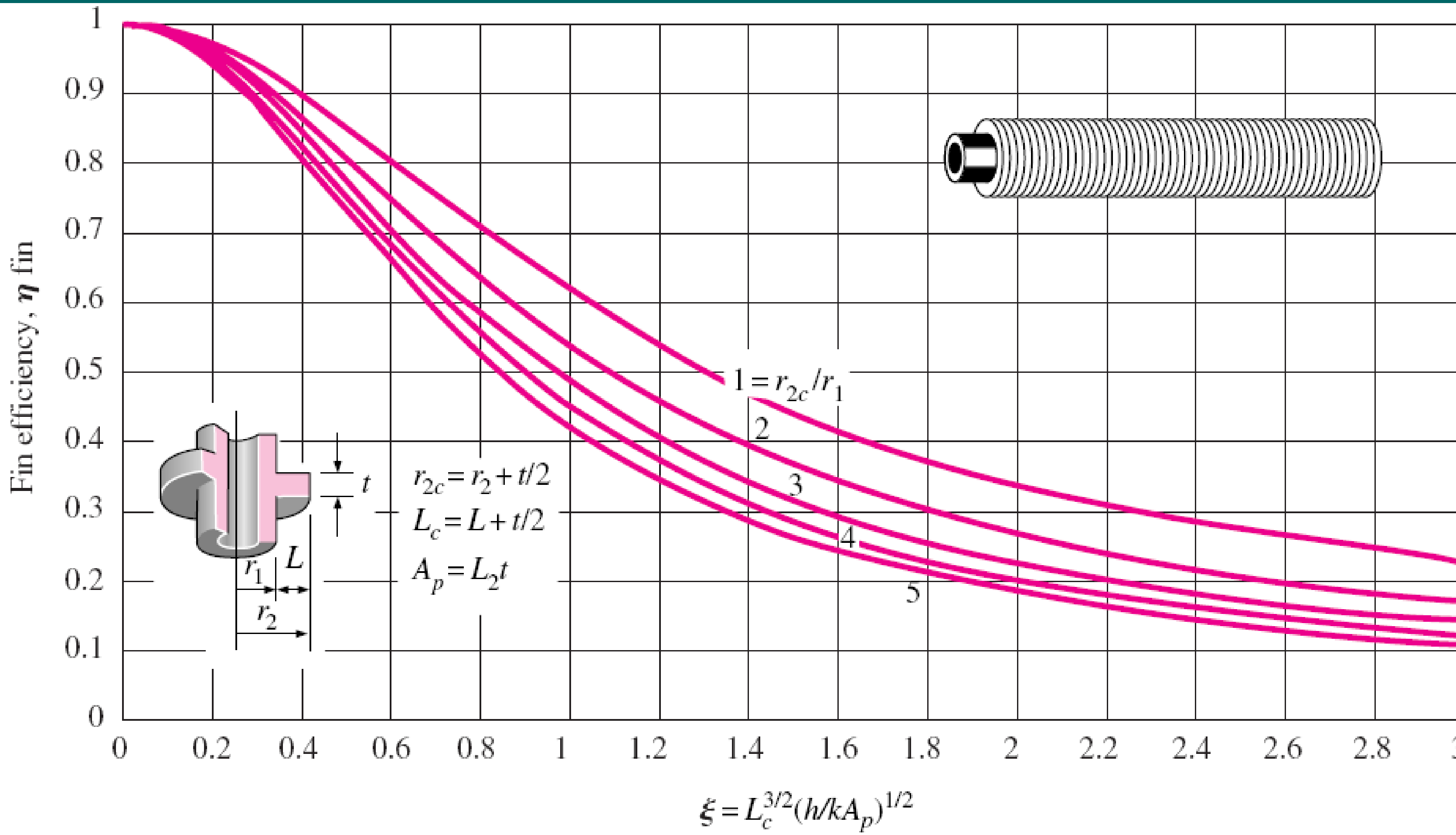


FIGURE 3–43

Efficiency of annular fins of constant thickness t.

- Fins with triangular and parabolic profiles contain less material and are more efficient than the ones with rectangular profiles, and thus are more suitable for applications requiring minimum weight such as space applications.

- An important consideration in the design of finned surfaces is the selection of the proper *fin length L*. Normally the *longer* the fin, the *larger* the heat transfer area and thus the *higher* the rate of heat transfer from the fin.

- The larger the fin, the bigger the mass, the higher the price, and the larger the fluid friction. Therefore, increasing the length of the fin beyond a certain value cannot be justified unless the added benefits outweigh the added cost.

- Fin lengths that cause the fin efficiency to drop below 60% percent usually cannot be justified economically and should be avoided. The efficiency of most fins used in practice is above 90%.

Fin Effectiveness

The performance of fins expressed in terms of the *fin effectiveness* ε_{fin} is defined

$$\varepsilon_{\text{fin}} = \frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{no fin}}} = \frac{\dot{Q}_{\text{fin}}}{hA_b\,(T_b - T_\infty)} = \frac{\text{Heat transfer rate from the fin of } \textit{base area } A_b}{\text{Heat transfer rate from the surface of } \textit{area } A_b}$$

A_b : the cross-sectional area of the fin at the base

$\dot{Q}_{\text{no fin}}$: the rate of heat transfer from this area if no fins are attached to the surface.

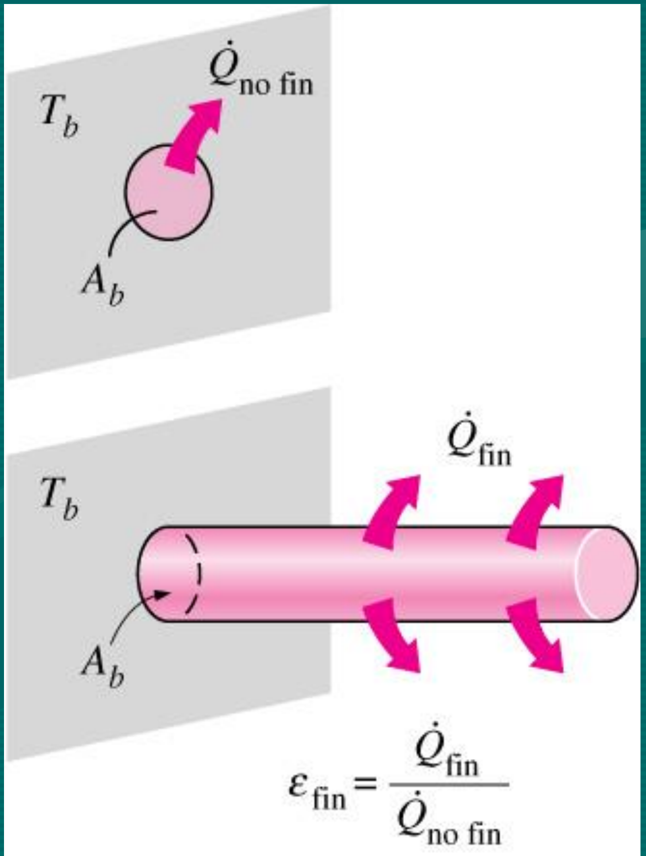


FIGURE 3–44
The effectiveness of a fin.

An effectiveness of ε_{fin}= 1 indicates that the addition of fins to the surface does not affect heat transfer at all.

An effectiveness of ε_{fin} < 1 indicates that the fin actually acts as *insulation*, slowing down the heat transfer from the surface.

An effectiveness of ε_{fin} > 1 indicates that fins are *enhancing* heat transfer from the surface, as they should.

Finned surfaces are designed on the basis of *maximizing* effectiveness for a specified cost or *minimizing* cost for a desired effectiveness.

The fin efficiency and fin effectiveness are related to each other by

$$\varepsilon_{\text{fin}} = \frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{no fin}}} = \frac{\dot{Q}_{\text{fin}}}{hA_b(T_b - T_\infty)} = \frac{\eta_{\text{fin}}\, hA_{\text{fin}}(T_b - T_\infty)}{hA_b(T_b - T_\infty)} = \frac{A_{\text{fin}}}{A_b}\,\eta_{\text{fin}}$$

The effectiveness of a sufficiently *long* fin of *uniform* cross section under steady conditions is determined to be

$$\varepsilon_{\text{long fin}} = \frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{no fin}}} = \frac{\sqrt{hpkA_c}\,(T_b - T_\infty)}{hA_b\,(T_b - T_\infty)} = \sqrt{\frac{kp}{hA_c}}$$

since $A_c = A_b$.

In the design and selection of the fins, the following should be taken into account:

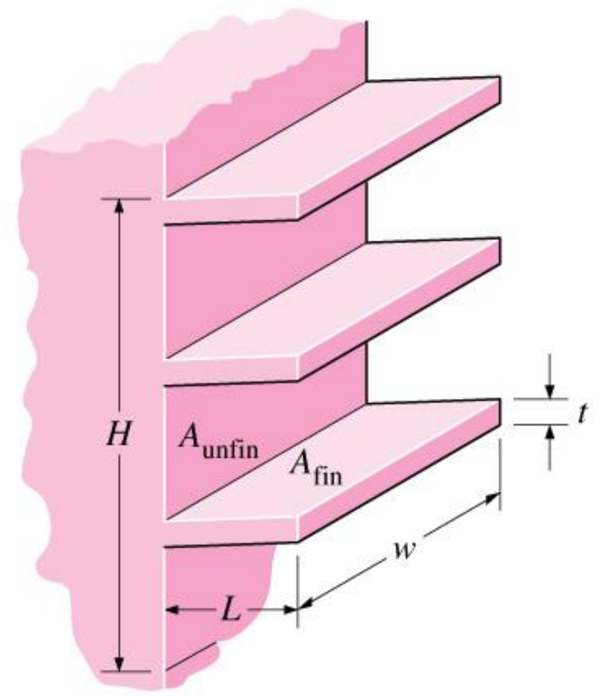
- The *thermal conductivity k* of the fin material should be as high as possible. Thus it is no coincidence that fins are made from metals, with copper, aluminum, and iron being the most common ones. Perhaps the most widely used fins are made of aluminum because of its low cost and weight and its resistance to corrosion.

- The ratio of the *perimeter* to the *cross-sectional area* of the fin p/A_c should be as high as possible. This criterion is satisfied by *thin* plate fins and *slender* pin fins.

- The use of fins is *most effective* in applications involving a *low convection heat transfer coefficient.*

The rate of heat transfer for a surface containing *n* fins can be expressed as

$$\begin{aligned}\dot{Q}_{\text{total, fin}} &= \dot{Q}_{\text{unfin}} + \dot{Q}_{\text{fin}} \\ &= hA_{\text{unfin}}\,(T_b - T_\infty) + \eta_{\text{fin}}\,hA_{\text{fin}}\,(T_b - T_\infty) \\ &= h(A_{\text{unfin}} + \eta_{\text{fin}}A_{\text{fin}})(T_b - T_\infty)\end{aligned}$$



$$A_{\text{no fin}} = w \times H$$
$$A_{\text{unfin}} = w \times H - 3 \times (t \times w)$$
$$A_{\text{fin}} = 2 \times L \times w + t \times w$$
$$\cong 2 \times L \times w \text{ (one fin)}$$

FIGURE 3–45
Various surface areas associated with a rectangular surface with three fins.

The **overall effectiveness** for a finned surface is defined as the ratio of the total heat transfer from the finned surface to the heat transfer from the same surface if there were no fins.

$$\varepsilon_{\text{fin, overall}} = \frac{\dot{Q}_{\text{total, fin}}}{\dot{Q}_{\text{total, no fin}}} = \frac{h(A_{\text{unfin}} + \eta_{\text{fin}}A_{\text{fin}})(T_b - T_\infty)}{hA_{\text{no fin}}\,(T_b - T_\infty)}$$

$A_{\text{no fin}}$: the area of the surface when there are no fins

A_{fin} : the total surface area of all the fins on the surface

A_{unfin} : the area of the unfinned portion of the surface

Proper Length of a Fin

To get a sense of the proper length of a fin, we compare heat transfer from a fin of finite length to heat transfer from an infinitely long fin under the same conditions. The ratio of these two heat transfers is

Heat transfer ratio:
$$\frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{long fin}}} = \frac{\sqrt{hpkA_c}\,(T_b - T_\infty)\tanh aL}{\sqrt{hpkA_c}\,(T_b - T_\infty)} = \tanh aL$$

Studies have shown that the error involved in one-dimensional fin analysis is negligible (less than about 1%) when

$$\frac{h\delta}{k} < 0.2$$

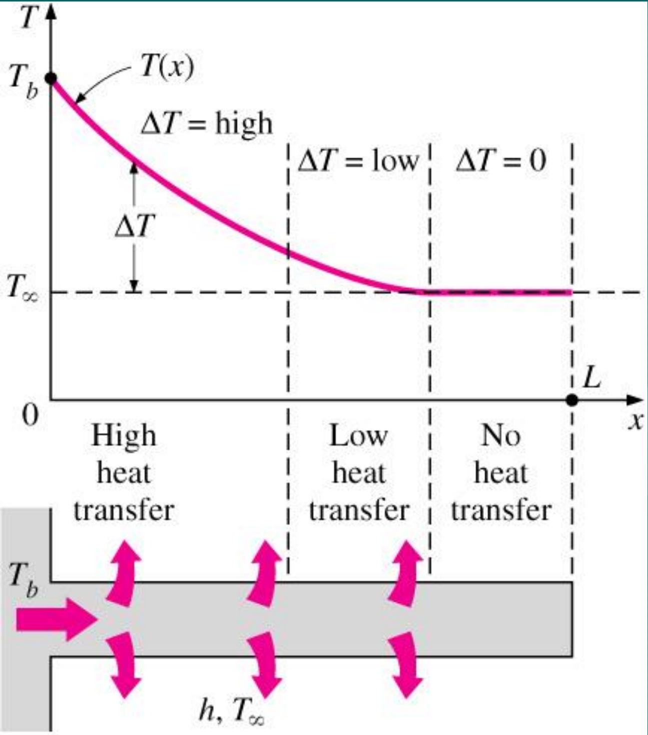


FIGURE 3–46
Because of the gradual temperature drop along the fin, the region near the fin tip makes little or no contribution to heat transfer.

The heat transfer performance of heat sinks is usually expressed in terms of their *thermal resistances R* in °C/W, which is defined as

$$\dot{Q}_{\text{fin}} = \frac{T_b - T_\infty}{R} = hA_{\text{fin}}\,\eta_{\text{fin}}\,(T_b - T_\infty)$$

EXAMPLE 3–11 Selecting a Heat Sink for a Transistor

A 60-W power transistor is to be cooled by attaching it to one of the commercially available heat sinks shown in Table 17–4. Select a heat sink that will allow the case temperature of the transistor not to exceed 90°C in the ambient air at 30°C.

SOLUTION A commercially available heat sink from Table 17–4 is to be selected to keep the case temperature of a transistor below 90°C.

Assumptions **1** Steady operating conditions exist. **2** The transistor case is isothermal at 90°C. **3** The contact resistance between the transistor and the heat sink is negligible.

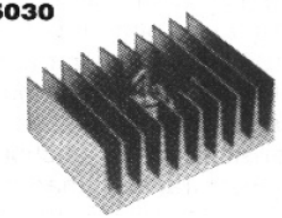
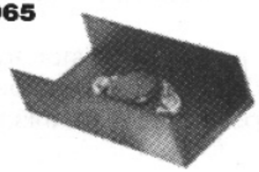
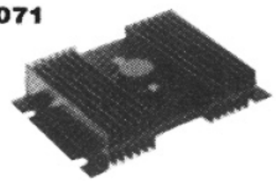
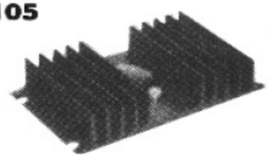
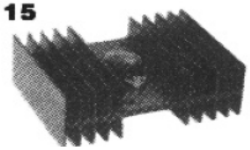
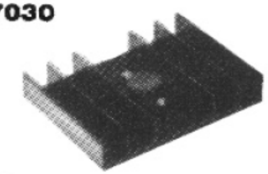
Analysis The rate of heat transfer from a 60-W transistor at full power is $\dot{Q} = 60$ W. The thermal resistance between the transistor attached to the heat sink and the ambient air for the specified temperature difference is determined to be

$$\dot{Q} = \frac{\Delta T}{R} \longrightarrow R = \frac{\Delta T}{\dot{Q}} = \frac{(90 - 30)^\circ\text{C}}{60\text{ W}} = 1.0^\circ\text{C/W}$$

Therefore, the thermal resistance of the heat sink should be below 1.0°C/W. An examination of Table 17–4 reveals that the HS 5030, whose thermal resistance is 0.9°C/W in the vertical position, is the only heat sink that will meet this requirement.

TABLE 3–6

Combined natural convection and radiation thermal resistance of various heat sinks used in the cooling of electronic devices between the heat sink and the surroundings. All fins are made of aluminum 6063T-5, are black anodized, and are 76 mm (3 in) long.

HS 5030		R = 0.9°C/W (vertical) R = 1.2°C/W (horizontal) Dimensions: 76 mm × 105 mm × 44 mm Surface area: 677 cm^2
HS 6065		R = 5°C/W Dimensions: 76 mm × 38 mm × 24 mm Surface area: 387 cm^2
HS 6071		R = 1.4°C/W (vertical) R = 1.8°C/W (horizontal) Dimensions: 76 mm × 92 mm × 26 mm Surface area: 968 cm^2
HS 6105		R = 1.8°C/W (vertical) R = 2.1°C/W (horizontal) Dimensions: 76 mm × 127 mm × 91 mm Surface area: 677 cm^2
HS 6115		R = 1.1°C/W (vertical) R = 1.3°C/W (horizontal) Dimensions: 76 mm × 102 mm × 25 mm Surface area: 929 cm^2
HS 7030		R = 2.9°C/W (vertical) R = 3.1°C/W (horizontal) Dimensions: 76 mm × 97 mm × 19 mm Surface area: 290 cm^2

EXAMPLE 3–12 Effect of Fins on Heat Transfer from Steam Pipes

Steam in a heating system flows through tubes whose outer diameter is $D_1 = 3$ cm and whose walls are maintained at a temperature of 120°C. Circular aluminum fins ($k = 180$ W/m · °C) of outer diameter $D_2 = 6$ cm and constant thickness $t = 2$ mm are attached to the tube, as shown in Fig. 17–48. The space between the fins is 3 mm, and thus there are 200 fins per meter length of the tube. Heat is transferred to the surrounding air at $T_\infty = 25$°C, with a combined heat transfer coefficient of $h = 60$ W/m^2 · °C. Determine the increase in heat transfer from the tube per meter of its length as a result of adding fins.

SOLUTION Circular aluminum fins are to be attached to the tubes of a heating system. The increase in heat transfer from the tubes per unit length as a result of adding fins is to be determined.

Assumptions **1** Steady operating conditions exist. **2** The heat transfer coefficient is uniform over the entire fin surfaces. **3** Thermal conductivity is constant. **4** Heat transfer by radiation is negligible.

Properties The thermal conductivity of the fins is given to be $k = 180$ W/m · °C.

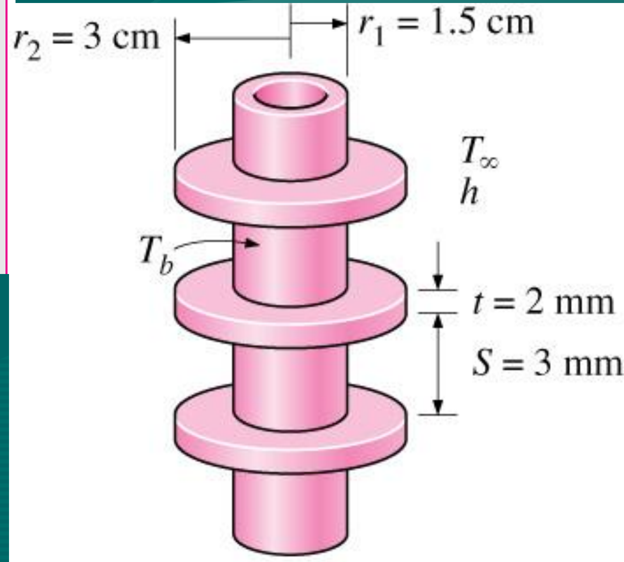


FIGURE 3–48
Schematic for Example 3–12.

Analysis In the case of no fins, heat transfer from the tube per meter of its length is determined from Newton's law of cooling to be

$$A_{\text{no fin}} = \pi D_1 L = \pi(0.03\text{ m})(1\text{ m}) = 0.0942\text{ m}^2$$
$$\begin{aligned}\dot{Q}_{\text{no fin}} &= hA_{\text{no fin}}(T_b - T_\infty)\\ &= (60\text{ W/m}^2\cdot{}^\circ\text{C})(0.0942\text{ m}^2)(120 - 25)^\circ\text{C}\\ &= 537\text{ W}\end{aligned}$$

The efficiency of the circular fins attached to a circular tube is plotted in Fig. 17–43. Noting that $L = \frac{1}{2}(D_2 - D_1) = \frac{1}{2}(0.06 - 0.03) = 0.015$ m in this case, we have

$$\left.\begin{aligned}\frac{r_2 + \frac{1}{2}t}{r_1} &= \frac{(0.03 + \frac{1}{2}\times 0.002)\text{ m}}{0.015\text{ m}} = 2.07\\ (L + \tfrac{1}{2}t)\sqrt{\frac{h}{kt}} &= (0.015 + \tfrac{1}{2}\times 0.002)\text{ m}\times\sqrt{\frac{60\text{ W/m}^2\cdot{}^\circ\text{C}}{(180\text{ W/m}\cdot{}^\circ\text{C})(0.002\text{ m})}} = 0.207\end{aligned}\right\}\ \eta_{\text{fin}} = 0.95$$

$$\begin{aligned}A_{\text{fin}} &= 2\pi(r_2^2 - r_1^2) + 2\pi r_2 t\\ &= 2\pi[(0.03\text{ m})^2 - (0.015\text{ m})^2] + 2\pi(0.03\text{ m})(0.002\text{ m})\\ &= 0.00462\text{ m}^2\end{aligned}$$

$$\begin{aligned}\dot{Q}_{\text{fin}} &= \eta_{\text{fin}}\dot{Q}_{\text{fin, max}} = \eta_{\text{fin}} hA_{\text{fin}}(T_b - T_\infty)\\ &= 0.95(60\text{ W/m}^2\cdot{}^\circ\text{C})(0.00462\text{ m}^2)(120 - 25)^\circ\text{C}\\ &= 25.0\text{ W}\end{aligned}$$

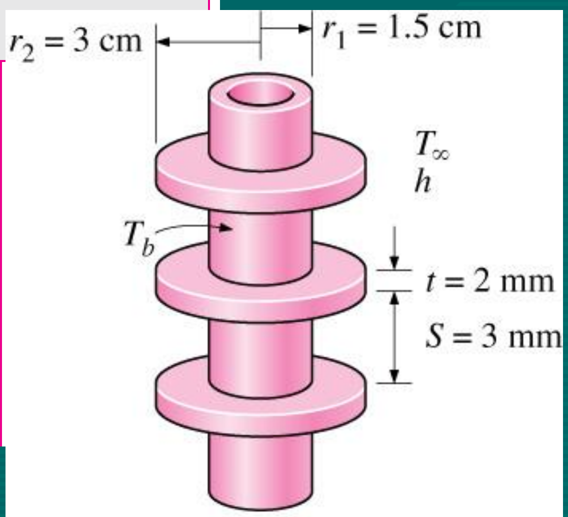


FIGURE 3–48
Schematic for Example 3–12.

Heat transfer from the unfinned portion of the tube is

$$A_{\text{unfin}} = \pi D_1 S = \pi(0.03\ \text{m})(0.003\ \text{m}) = 0.000283\ \text{m}^2$$

$$\begin{aligned}\dot{Q}_{\text{unfin}} &= hA_{\text{unfin}}(T_b - T_\infty) \\ &= (60\ \text{W/m}^2\cdot{}^\circ\text{C})(0.000283\ \text{m}^2)(120 - 25)^\circ\text{C} \\ &= 1.60\ \text{W}\end{aligned}$$

Noting that there are 200 fins and thus 200 interfin spacings per meter length of the tube, the total heat transfer from the finned tube becomes

$$\dot{Q}_{\text{total, fin}} = n(\dot{Q}_{\text{fin}} + \dot{Q}_{\text{unfin}}) = 200(25.0 + 1.6)\ \text{W} = 5320\ \text{W}$$

Therefore, the increase in heat transfer from the tube per meter of its length as a result of the addition of fins is

$$\dot{Q}_{\text{increase}} = \dot{Q}_{\text{total, fin}} - \dot{Q}_{\text{no fin}} = 5320 - 537 = \mathbf{4783\ W} \qquad \text{(per m tube length)}$$

Discussion The overall effectiveness of the finned tube is

$$\varepsilon_{\text{fin, overall}} = \frac{\dot{Q}_{\text{total, fin}}}{\dot{Q}_{\text{total, no fin}}} = \frac{5320\ \text{W}}{537\ \text{W}} = 9.9$$

That is, the rate of heat transfer from the steam tube increases by a factor of almost 10 as a result of adding fins. This explains the widespread use of finned surfaces.

HEAT TRANSFER IN COMMON CONFIGURATIONS

• We have dealt with 1-D simple geometries.

☞ The question: What happens if we have 2- or 3-D complicated geometries?

• The steady rate of heat transfer between two surfaces at *constant* temperatures T_1 and T_2 is expressed as

$$Q = Sk(T_1 - T_2)$$

S : the conduction shape factor **(which** has the dimension of *length)*

k : the thermal conductivity of the medium between the surfaces

☞ The conduction shape factor depends on the *geometry* of the system only.

A comparison of the following equations reveals that the conduction shape factor *S* is related to the thermal resistance *R* by $R = 1/kS$ or $S = 1/kR$.

$$\dot{Q}_{\text{cond, wall}} = \frac{T_1 - T_2}{R_{\text{wall}}} \qquad \text{(W)}$$

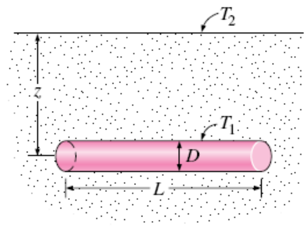
$$Q = Sk(T_1 - T_2)$$

TABLE 3–7

Conduction shape factors S for several configurations for use in $\dot{Q} = kS(T_1 - T_2)$ to determine the steady rate of heat transfer through a medium of thermal conductivity k between the surfaces at temperatures T_1 and T_2

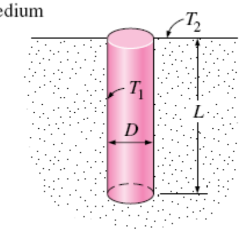
(1) Isothermal cylinder of length L buried in a semi-infinite medium ($L >> D$ and $z > 1.5D$)

$$S = \frac{2\pi L}{\ln(4z/D)}$$



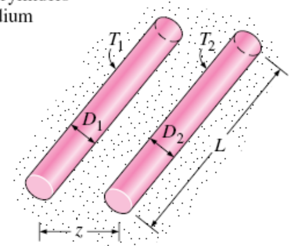
(2) Vertical isothermal cylinder of length L buried in a semi-infinite medium ($L >> D$)

$$S = \frac{2\pi L}{\ln(4L/D)}$$



(3) Two parallel isothermal cylinders placed in an infinite medium ($L >> D_1, D_2, z$)

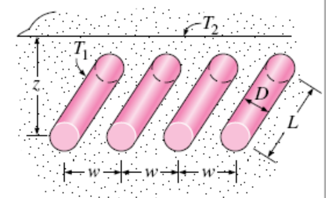
$$S = \frac{2\pi L}{\cosh^{-1}\left(\frac{4z^2 - D_1^2 - D_2^2}{2D_1D_2}\right)}$$



(4) A row of equally spaced parallel isothermal cylinders buried in a semi-infinite medium ($L >> D, z,$ and $w > 1.5D$)

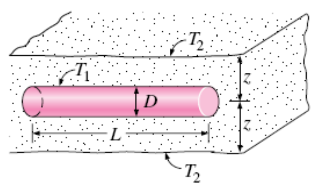
$$S = \frac{2\pi L}{\ln\left(\frac{2w}{\pi D}\sinh\frac{2\pi z}{w}\right)}$$

(per cylinder)



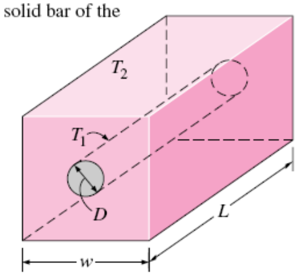
(5) Circular isothermal cylinder of length L in the midplane of an infinite wall ($z > 0.5D$)

$$S = \frac{2\pi L}{\ln(8z/\pi D)}$$



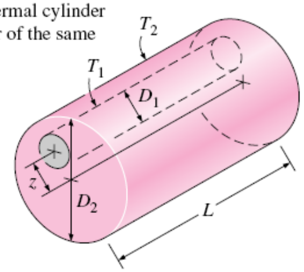
(6) Circular isothermal cylinder of length L at the center of a square solid bar of the same length

$$S = \frac{2\pi L}{\ln(1.08\,w/D)}$$



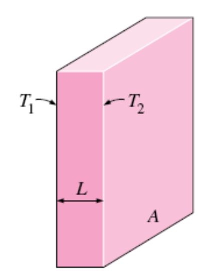
(7) Eccentric circular isothermal cylinder of length L in a cylinder of the same length ($L > D_2$)

$$S = \frac{2\pi L}{\cosh^{-1}\left(\frac{D_1^2 + D_2^2 - 4z^2}{2D_1D_2}\right)}$$



(8) Large plane wall

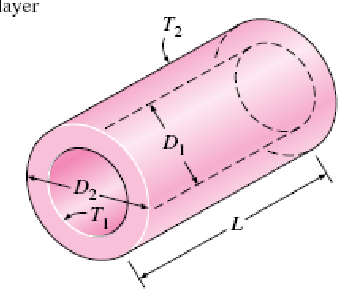
$$S = \frac{A}{L}$$



(continued)

(9) A long cylindrical layer

$$S = \frac{2\pi L}{\ln (D_2/D_1)}$$



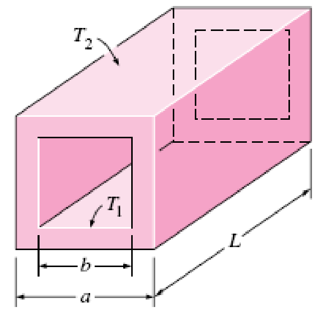
(10) A square flow passage

(*a*) For $a/b > 1.4$,

$$S = \frac{2\pi L}{0.93 \ln (0.948 a/b)}$$

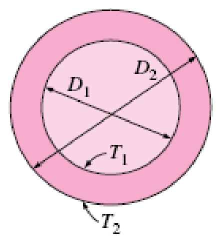
(*b*) For $a/b < 1.41$,

$$S = \frac{2\pi L}{0.785 \ln (a/b)}$$



(11) A spherical layer

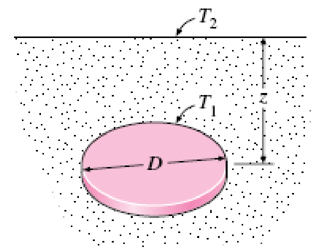
$$S = \frac{2\pi D_1 D_2}{D_2 - D_1}$$



(12) Disk buried parallel to the surface in a semi-infinite medium ($z >> D$)

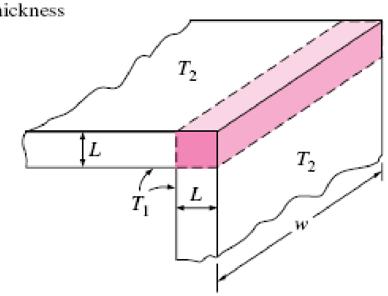
$S = 4D$

($S = 2D$ when $z = 0$)



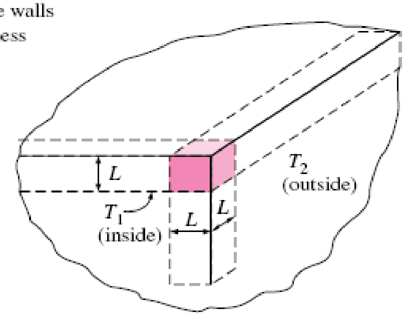
(13) The edge of two adjoining walls of equal thickness

$S = 0.54\,w$



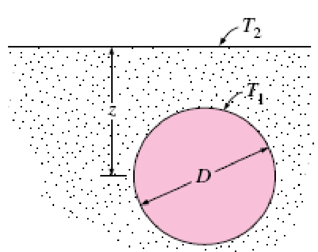
(14) Corner of three walls of equal thickness

$S = 0.15\,L$



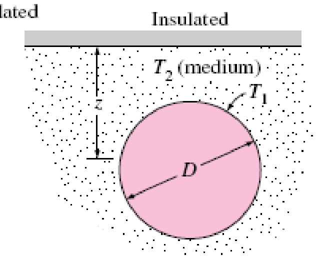
(15) Isothermal sphere buried in a semi-infinite medium

$$S = \frac{2\pi D}{1 - 0.25D/z}$$



(16) Isothermal sphere buried in a semi-infinite medium at T_2 whose surface is insulated

$$S = \frac{2\pi D}{1 + 0.25D/z}$$



EXAMPLE 3–13 Heat Loss from Buried Steam Pipes

A 30-m-long, 10-cm-diameter hot-water pipe of a district heating system is buried in the soil 50 cm below the ground surface, as shown in Fig. 17–49. The outer surface temperature of the pipe is 80°C. Taking the surface temperature of the earth to be 10°C and the thermal conductivity of the soil at that location to be 0.9 W/m · °C, determine the rate of heat loss from the pipe.

SOLUTION The hot-water pipe of a district heating system is buried in the soil. The rate of heat loss from the pipe is to be determined.

Assumptions **1** Steady operating conditions exist. **2** Heat transfer is two-dimensional (no change in the axial direction). **3** Thermal conductivity of the soil is constant.

Properties The thermal conductivity of the soil is given to be $k = 0.9$ W/m · °C.

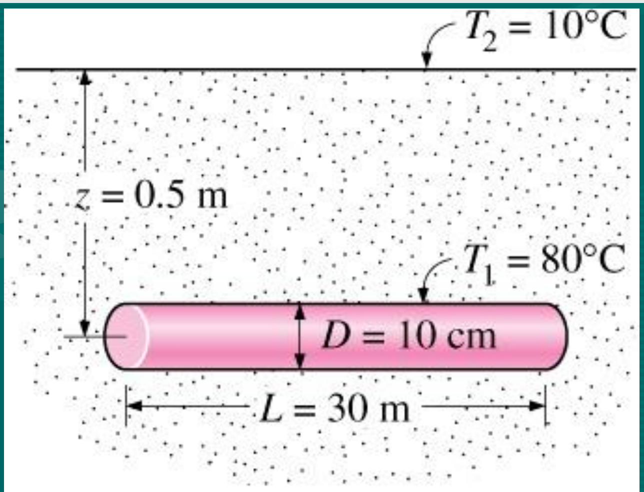


FIGURE 3–49
Schematic for Example 3–13.

Analysis The shape factor for this configuration is given in Table 17–5 to be

$$S = \frac{2\pi L}{\ln(4z/D)}$$

since $z > 1.5D$, where z is the distance of the pipe from the ground surface, and D is the diameter of the pipe. Substituting,

$$S = \frac{2\pi \times (30\ \text{m})}{\ln(4 \times 0.5/0.1)} = 62.9\ \text{m}$$

Then the steady rate of heat transfer from the pipe becomes

$$\dot{Q} = Sk(T_1 - T_2) = (62.9\ \text{m})(0.9\ \text{W/m}\cdot{}^\circ\text{C})(80 - 10)^\circ\text{C} = 3963\ \text{W}$$

Discussion Note that this heat is conducted from the pipe surface to the surface of the earth through the soil and then transferred to the atmosphere by convection and radiation.

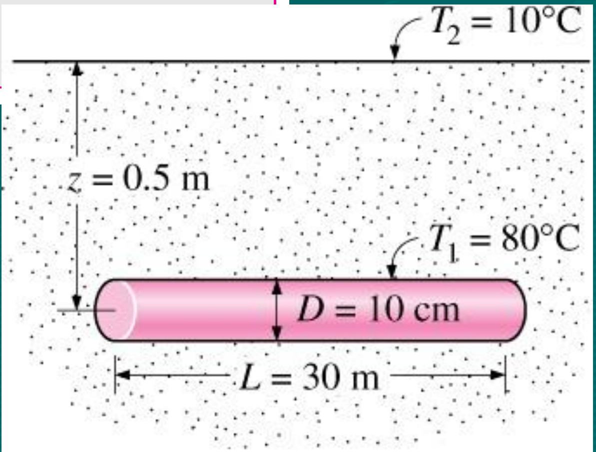


FIGURE 3–49
Schematic for Example 3–13.

Concluding Points:

- Steady and One-Dimensional Modeling of Heat Transfer through a Wall
- Conduction and Convection Resistances
- Analogy between Thermal and Electrical Resistances
- Radiation and Combined Heat Transfer Coefficients
- Overall Heat Transfer Coefficient
- Heat Transfer through a Plane and Multilayer Plane Walls
- Thermal Contact Resistance
- Generalized Thermal Resistance Networks
- Heat Conduction in Multilayered Cylinders and Spheres
- Critical Radius of Insulation for Cylindrical and Spherical Bodies
- Heat Transfer from Finned Surfaces
- Fin Efficiency, Fin Effectiveness and Overall Effectiveness
- Important Considerations in the Design and Selection of Fins
- Heat Transfer in Common Configurations and Conduction Shape Factors

HEAT AND MASS TRANSFER

Transient Heat Conduction

Outline

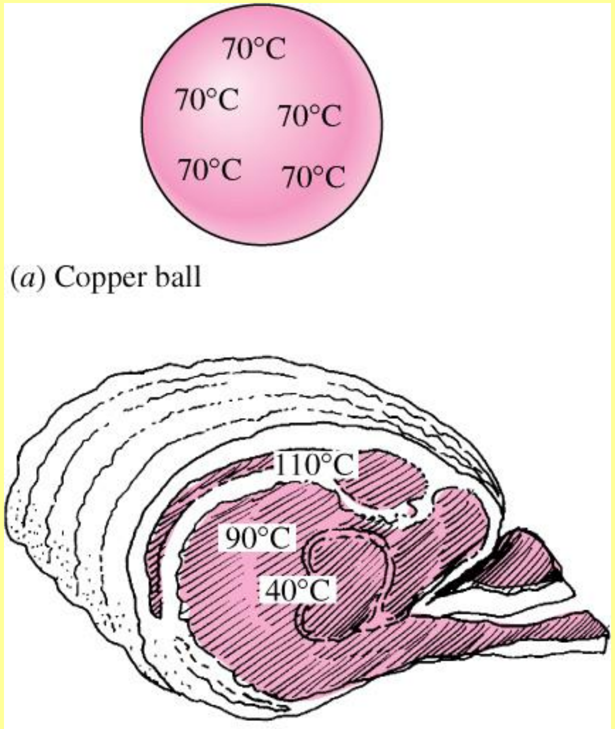
- Lumped system analysis
- Transient heat conduction in
  - large plane walls
  - long cylinders
  - spheres
- Transient heat conduction in semi-infinite solids
- Transient heat conduction in multidimensional systems

Objectives

- To assess when the spatial variation of temperature is negligible, and temperature varies nearly uniformly with time, making the simplified lumped system analysis applicable,
- To obtain analytical solutions for transient 1-D conduction problems in rectangular, cylindrical, and spherical geometries using the method of separation of variables, and understand why a one-term solution is usually a reasonable approximation,
- To solve the transient conduction problem in large mediums using the similarity variable, and predict the variation of temperature with time and distance from the exposed surface, and
- To construct solutions for multi-dimensional transient conduction problems using the product solution approach.

Lumped System Analysis

- In heat transfer analysis, some bodies are essentially isothermal and can be treated as a "lump" system.
- That is, $T = T(t)$



(*a*) Copper ball

(*b*) Roast beef

FIGURE 4–1
A small copper ball can be modeled as a lumped system, but a roast beef cannot.

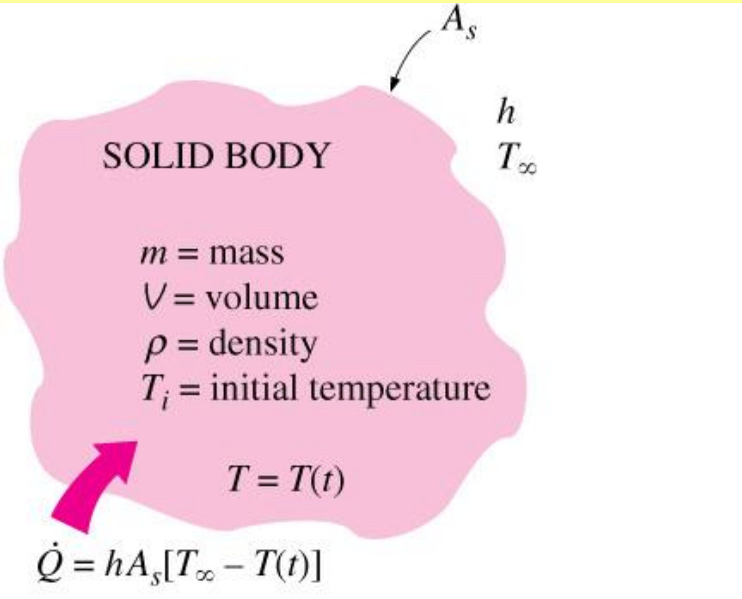


FIGURE 4–2
The geometry and parameters involved in the lumped system analysis.

During a differential time interval dt, the temperature of the body rises by a differential amount dT. An energy balance of the solid for the time interval dt can be expressed as

$$\begin{pmatrix}\text{Heat transfer into the body} \\ \text{during } dt\end{pmatrix} = \begin{pmatrix}\text{The increase in the} \\ \text{energy of the body} \\ \text{during } dt\end{pmatrix}$$

or

$$hA_s(T_\infty - T)\,dt = mc_p\,dT \qquad \text{(4–1)}$$

Noting that $m = \rho V$ and $dT = d(T - T_\infty)$ since T_∞ = constant, Eq. 4–1 can be rearranged as

$$\frac{d(T - T_\infty)}{T - T_\infty} = -\frac{hA_s}{\rho V c_p}\,dt \qquad \text{(4–2)}$$

Integrating from $t = 0$, at which $T = T_i$, to any time t, at which $T = T(t)$, gives

$$\ln \frac{T(t) - T_\infty}{T_i - T_\infty} = -\frac{hA_s}{\rho V c_p}\,t \qquad \text{(4–3)}$$

Taking the exponential of both sides and rearranging, we obtain

$$\frac{T(t) - T_\infty}{T_i - T_\infty} = e^{-bt} \qquad \text{(4–4)}$$

where

$$b = \frac{hA_s}{\rho V c_p} \qquad \text{(1/s)} \qquad \text{(4–5)}$$

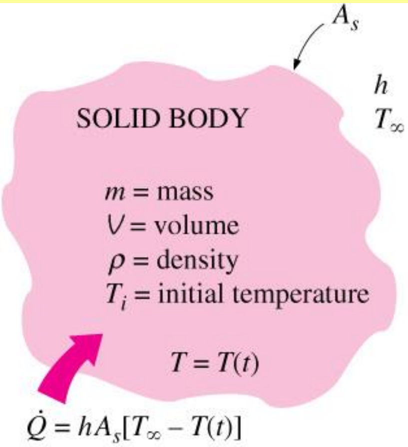


FIGURE 4–2
The geometry and parameters involved in the lumped system analysis.

b is a positive quantity (so-called the **time constant**).

Observations

- Equation 4–4 enables us to determine the temperature $T(t)$ of a body at time t, or alternatively, the time t required for the temperature to reach a specified value $T(t)$.
- The temperature of a body approaches the ambient temperature T exponentially.
- The temperature of the body changes rapidly at the beginning, but rather slowly later on.
- A large value of b indicates that the body approaches the ambient temperature in a short time.

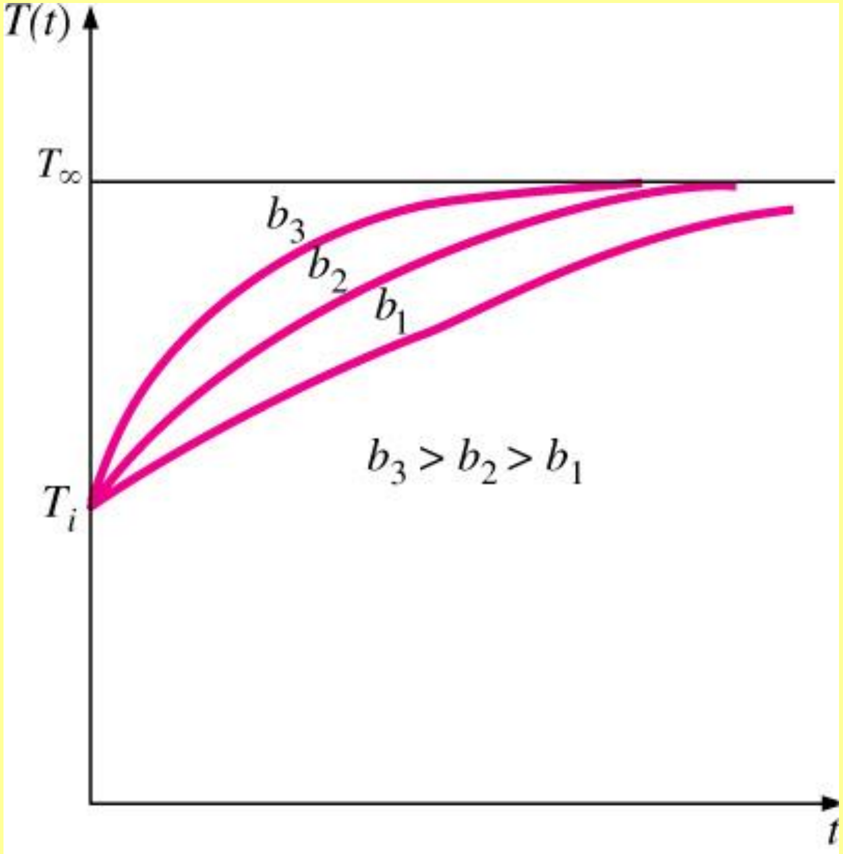


FIGURE 4–3

The temperature of a lumped system approaches the environment temperature as time gets larger.

Calculation of heat transfer

Once the temperature $T(t)$ at time t is available from Eq. 4–4, the *rate* of convection heat transfer between the body and its environment at that time can be determined from Newton's law of cooling as

$$\dot{Q}(t) = hA_s[T(t) - T_\infty] \qquad \text{(W)} \tag{4–6}$$

The *total amount* of heat transfer between the body and the surrounding medium over the time interval $t = 0$ to t is simply the change in the energy content of the body:

$$Q = mc_p[T(t) - T_i] \qquad \text{(kJ)} \tag{4–7}$$

The amount of heat transfer reaches its upper limit when the body reaches the surrounding temperature T_∞. Therefore, the *maximum* heat transfer between the body and its surroundings is (Fig. 4–4)

$$Q_{max} = mc_p(T_\infty - T_i) \qquad \text{(kJ)} \tag{4–8}$$

We could also obtain this equation by substituting the $T(t)$ relation from Eq. 4–4 into the $\dot{Q}(t)$ relation in Eq. 4–6 and integrating it from $t = 0$ to $t \to \infty$.

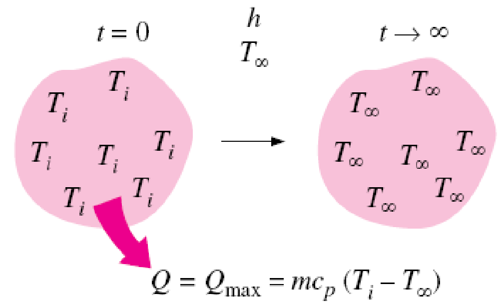


FIGURE 4–4
Heat transfer to or from a body reaches its maximum value when the body reaches the environment temperature.

$$L_c = \frac{V}{A_s}$$

and a **Biot number** Bi as

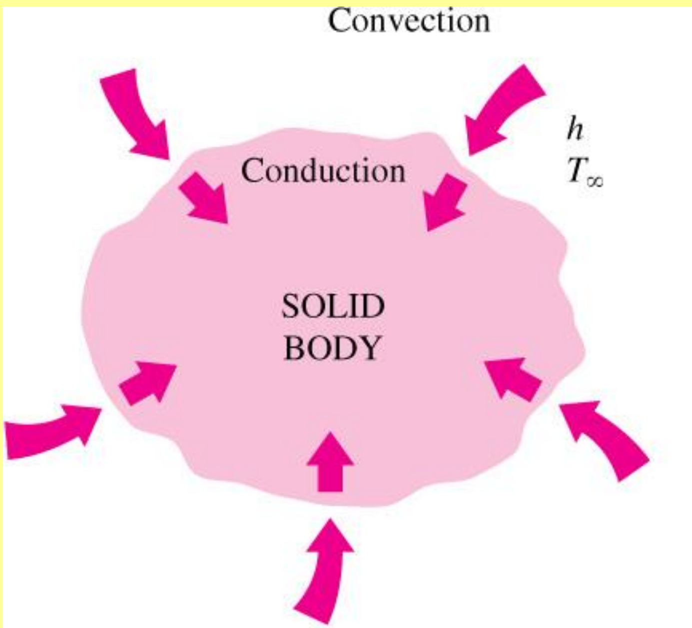
$$\mathrm{Bi} = \frac{hL_c}{k}$$

It can also be expressed as (Fig. 4–5)

$$\mathrm{Bi} = \frac{h}{k/L_c}\,\frac{\Delta T}{\Delta T} = \frac{\text{Convection at the surface of the body}}{\text{Conduction within the body}}$$

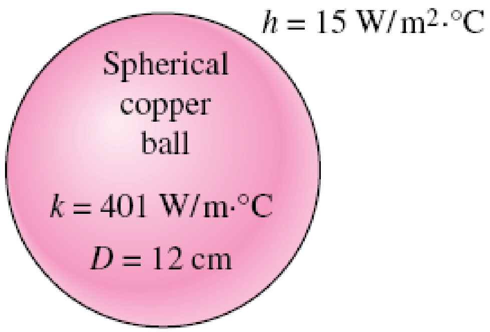
or

$$\mathrm{Bi} = \frac{L_c/k}{1/h} = \frac{\text{Conduction resistance within the body}}{\text{Convection resistance at the surface of the body}}$$



$$\mathrm{Bi} = \frac{\text{heat convection}}{\text{heat conduction}}$$

FIGURE 4–5
The Biot number can be viewed as the ratio of the convection at the surface to conduction within the body.



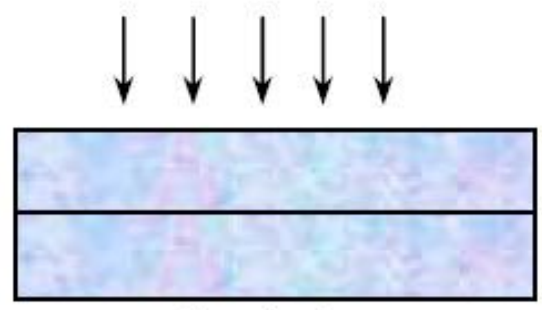
$$L_c = \frac{V}{A_s} = \frac{\frac{1}{6}\pi D^3}{\pi D^2} = \frac{1}{6}D = 0.02\ \mathrm{m}$$

$$\mathrm{Bi} = \frac{hL_c}{k} = \frac{15 \times 0.02}{401} = 0.00075 < 0.1$$

FIGURE 4–6
Small bodies with high thermal conductivities and low convection coefficients are most likely to satisfy the criterion for lumped system analysis.

➔ Lumped system analysis is *applicable* if **Bi < 0.1.**

4–110 Consider two 2-cm-thick large steel plates (k = 43 W/m · °C and $\alpha = 1.17 \times 10^{-5}$ m²/s) that were put on top of each other while wet and left outside during a cold winter night at −15°C. The next day, a worker needs one of the plates, but the plates are stuck together because the freezing of the water between the two plates has bonded them together. In an effort to melt the ice between the plates and separate them, the worker takes a large hair dryer and blows hot air at 50°C all over the exposed surface of the plate on the top. The convection heat transfer coefficient at the top surface is estimated to be 40 W/m² · °C. Determine how long the worker must keep blowing hot air before the two plates separate.



Analysis The characteristic length of the plates and the Biot number are

$$L_c = \frac{V}{A_s} = L = 0.02 \text{ m}$$

$$Bi = \frac{hL_c}{k} = \frac{(40 \text{ W/m}^2.°\text{C})(0.02 \text{ m})}{(43 \text{ W/m}.°\text{C})} = 0.019 < 0.1$$

Since $\text{Bi} < 0.1$, the lumped system analysis is applicable. Therefore,

$$b = \frac{hA_s}{\rho c_p V} = \frac{h}{\rho c_p L_c} = \frac{40 \text{ W/m}^2.°\text{C}}{(3.675\times10^6 \text{ J/m}^3.°\text{C})(0.02 \text{ m})} = 0.000544 \text{ s}^{-1}$$

$$\frac{T(t)-T_\infty}{T_i - T_\infty} = e^{-bt} \longrightarrow \frac{0-50}{-15-50} = e^{-(0.000544 \text{ s}^{-1})t} \longrightarrow t = \mathbf{482 \text{ s} = 8.0 \text{ min}}$$

where $\rho c_p = \dfrac{k}{\alpha} = \dfrac{43 \text{ W/m}.°\text{C}}{1.17\times10^{-5} \text{ m}^2/\text{s}} = 3.675\times10^6 \text{ J/m}^3.°\text{C}$

Transient Heat Conduction in Large Plane Walls, Long Cylinders, and Spheres

- In many transient heat transfer problems the Biot number is larger than 0.1, and lumped system can not be assumed.
- In these cases the temperature within the body changes appreciably from point to point as well as with time.
- It is constructive to first consider the variation of temperature with *time* and *position in* one-dimensional problems of rudimentary configurations such as a large plane wall, a long cylinder, and a sphere.

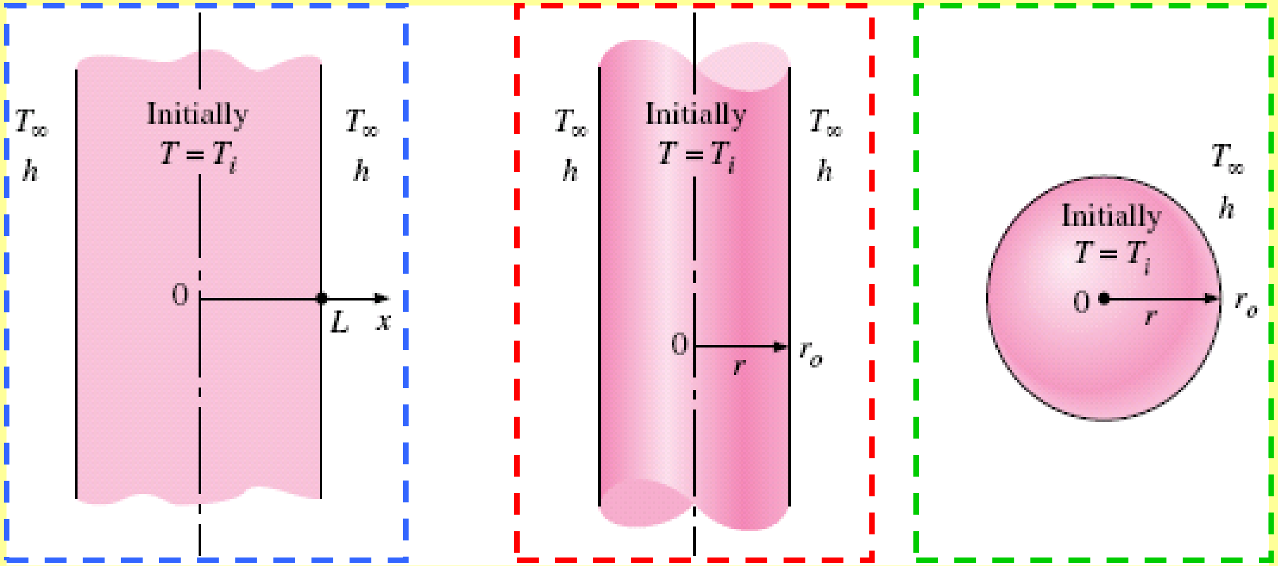
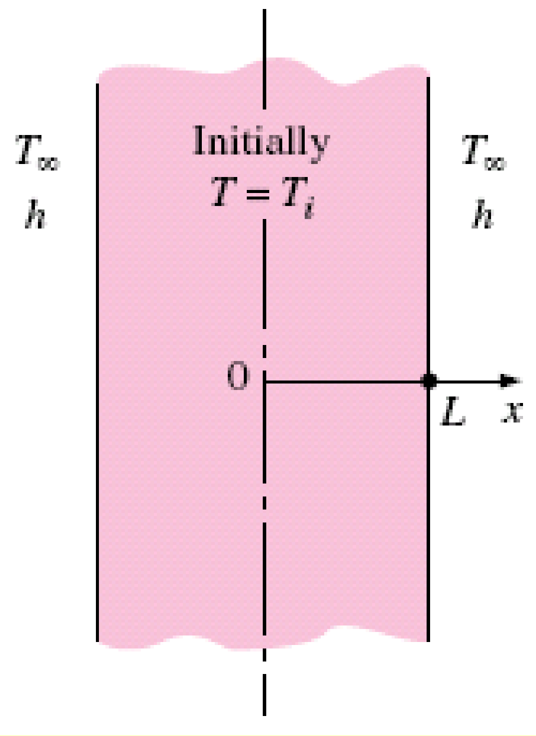


FIGURE 4–11
Schematic of the simple geometries in which heat transfer is one-dimensional.

A large Plane Wall

- A plane wall of thickness $2L$.
- Initially at a uniform temperature of T_i.
- At time $t=0$, the wall is immersed in a fluid at temperature T_∞.
- Constant heat transfer coefficient h.
- The height and the width of the wall are large relative to its thickness →one-dimensional approximation is valid.
- Constant thermophysical properties.
- No heat generation.
- There is thermal symmetry about the midplane passing through $x=0$.



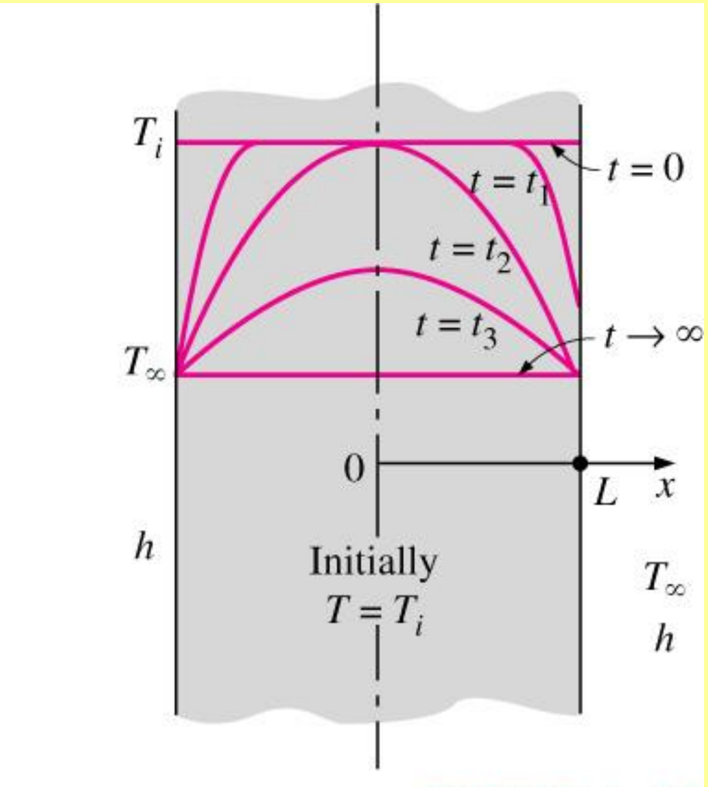


FIGURE 4–12
Transient temperature profiles in a plane wall exposed to convection from its surfaces for $T_i > T_\infty$.

The Heat Conduction Equation

- One-dimensional transient heat conduction equation problem $(0 \le x \le L)$:

Differential equation:

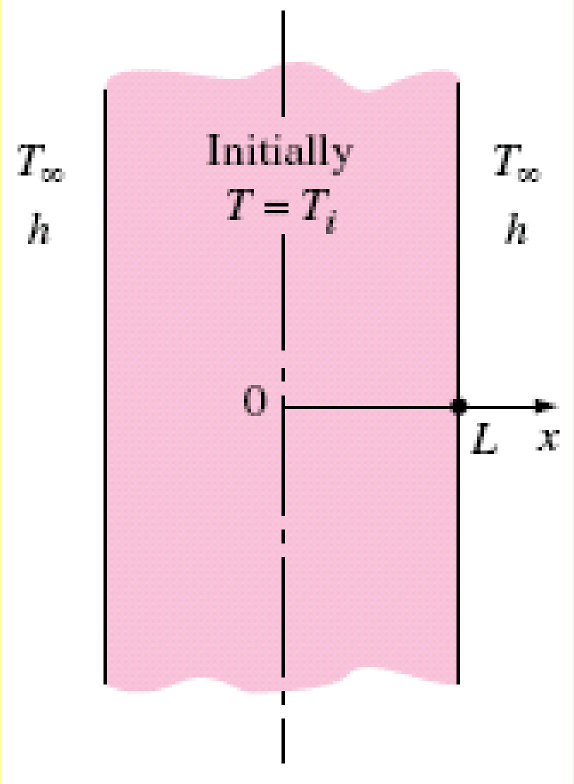
$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

Boundary conditions:

$$\begin{cases} \dfrac{\partial T(0,t)}{\partial x} = 0 \\ -k\dfrac{\partial T(L,t)}{\partial x} = h\left[T(L,t) - T_\infty\right] \end{cases}$$

Initial condition:

$$T(x,0) = T_i$$



The dimensionless time

Fourier number (Fo), $\tau = \alpha t/L^2$

Biot number $Bi = hL/k$

$$\tau = \frac{\alpha t}{L^2} = \frac{kL^2\,(1/L)}{\rho c_p L^3/t}\,\frac{\Delta T}{\Delta T} = \frac{\text{The rate at which heat is } \textit{conducted} \text{ across } L \text{ of a body of volume } L^3}{\text{The rate at which heat is } \textit{stored} \text{ in a body of volume } L^3}$$

Summary of the Solutions for One-Dimensional Transient Conduction

TABLE 4–1

Summary of the solutions for one-dimensional transient conduction in a plane wall of thickness $2L$, a cylinder of radius r_o and a sphere of radius r_o subjected to convention from all surfaces.*

Geometry	Solution	λ_n's are the roots of
Plane wall	$\theta = \sum_{n=1}^{\infty} \frac{4 \sin \lambda_n}{2\lambda_n + \sin(2\lambda_n)} e^{-\lambda_n^2 \tau} \cos(\lambda_n x / L)$	$\lambda_n \tan \lambda_n = \mathrm{Bi}$
Cylinder	$\theta = \sum_{n=1}^{\infty} \frac{2}{\lambda_n} \frac{J_1(\lambda_n)}{J_0^2(\lambda_n) + J_1^2(\lambda_n)} e^{-\lambda_n^2 \tau} J_0(\lambda_n r / r_o)$	$\lambda_n \frac{J_1(\lambda_n)}{J_0(\lambda_n)} = \mathrm{Bi}$
Sphere	$\theta = \sum_{n=1}^{\infty} \frac{4(\sin \lambda_n - \lambda_n \cos \lambda_n)}{2\lambda_n - \sin(2\lambda_n)} e^{-\lambda_n^2 \tau} \frac{\sin(\lambda_n x / L)}{\lambda_n x / L}$	$1 - \lambda_n \cot \lambda_n = \mathrm{Bi}$

*Here $\theta = (T - T_i)/(T_\infty - T_i)$ is the dimensionless temperature, $\mathrm{Bi} = hL/k$ or hr_o/k is the Biot number, $\mathrm{Fo} = \tau = \alpha t / L^2$ or $\alpha \tau / r_o^2$ is the Fourier number, and J_0 and J_1 are the Bessel functions of the first kind whose values are given in Table 4–3.

Approximate Analytical and Graphical Solutions

- For $\tau > 0.2$, keeping the first term and neglecting all the remaining terms in the series results in an error under 2%. Thus for $\tau > 0.2$ the **one-term approximation** can be used.

Plane wall: $$\theta_{\text{wall}} = \frac{T(x,t) - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau} \cos(\lambda_1 x/L), \quad \tau > 0.2$$

Cylinder: $$\theta_{\text{cyl}} = \frac{T(r,t) - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau} J_0(\lambda_1 r/r_o), \quad \tau > 0.2$$

Sphere: $$\theta_{\text{sph}} = \frac{T(r,t) - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau} \frac{\sin(\lambda_1 r/r_o)}{\lambda_1 r/r_o}, \quad \tau > 0.2$$

Center of plane wall $(x = 0)$: $$\theta_{0,\text{ wall}} = \frac{T_0 - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau}$$

Center of cylinder $(r = 0)$: $$\theta_{0,\text{ cyl}} = \frac{T_0 - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau}$$

Center of sphere $(r = 0)$: $$\theta_{0,\text{ sph}} = \frac{T_0 - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau}$$

TABLE 4–2

Coefficients used in the one-term approximate solution of transient one-dimensional heat conduction in plane walls, cylinders, and spheres (Bi = hL/k for a plane wall of thickness $2L$, and Bi = hr_o/k for a cylinder or sphere of radius r_o)

	Plane Wall		Cylinder		Sphere	
Bi	λ_1	A_1	λ_1	A_1	λ_1	A_1
0.01	0.0998	1.0017	0.1412	1.0025	0.1730	1.0030
0.02	0.1410	1.0033	0.1995	1.0050	0.2445	1.0060
0.04	0.1987	1.0066	0.2814	1.0099	0.3450	1.0120
0.06	0.2425	1.0098	0.3438	1.0148	0.4217	1.0179
0.08	0.2791	1.0130	0.3960	1.0197	0.4860	1.0239
0.1	0.3111	1.0161	0.4417	1.0246	0.5423	1.0298
0.2	0.4328	1.0311	0.6170	1.0483	0.7593	1.0592
0.3	0.5218	1.0450	0.7465	1.0712	0.9208	1.0880
0.4	0.5932	1.0580	0.8516	1.0931	1.0528	1.1164
0.5	0.6533	1.0701	0.9408	1.1143	1.1656	1.1441
0.6	0.7051	1.0814	1.0184	1.1345	1.2644	1.1713
0.7	0.7506	1.0918	1.0873	1.1539	1.3525	1.1978
0.8	0.7910	1.1016	1.1490	1.1724	1.4320	1.2236
0.9	0.8274	1.1107	1.2048	1.1902	1.5044	1.2488
1.0	0.8603	1.1191	1.2558	1.2071	1.5708	1.2732
2.0	1.0769	1.1785	1.5995	1.3384	2.0288	1.4793
3.0	1.1925	1.2102	1.7887	1.4191	2.2889	1.6227
4.0	1.2646	1.2287	1.9081	1.4698	2.4556	1.7202
5.0	1.3138	1.2403	1.9898	1.5029	2.5704	1.7870
6.0	1.3496	1.2479	2.0490	1.5253	2.6537	1.8338
7.0	1.3766	1.2532	2.0937	1.5411	2.7165	1.8673
8.0	1.3978	1.2570	2.1286	1.5526	2.7654	1.8920
9.0	1.4149	1.2598	2.1566	1.5611	2.8044	1.9106
10.0	1.4289	1.2620	2.1795	1.5677	2.8363	1.9249
20.0	1.4961	1.2699	2.2880	1.5919	2.9857	1.9781
30.0	1.5202	1.2717	2.3261	1.5973	3.0372	1.9898
40.0	1.5325	1.2723	2.3455	1.5993	3.0632	1.9942
50.0	1.5400	1.2727	2.3572	1.6002	3.0788	1.9962
100.0	1.5552	1.2731	2.3809	1.6015	3.1102	1.9990
∞	1.5708	1.2732	2.4048	1.6021	3.1416	2.0000

TABLE 4–3

The zeroth- and first-order Bessel functions of the first kind

η	$J_0(\eta)$	$J_1(\eta)$
0.0	1.0000	0.0000
0.1	0.9975	0.0499
0.2	0.9900	0.0995
0.3	0.9776	0.1483
0.4	0.9604	0.1960
0.5	0.9385	0.2423
0.6	0.9120	0.2867
0.7	0.8812	0.3290
0.8	0.8463	0.3688
0.9	0.8075	0.4059
1.0	0.7652	0.4400
1.1	0.7196	0.4709
1.2	0.6711	0.4983
1.3	0.6201	0.5220
1.4	0.5669	0.5419
1.5	0.5118	0.5579
1.6	0.4554	0.5699
1.7	0.3980	0.5778
1.8	0.3400	0.5815
1.9	0.2818	0.5812
2.0	0.2239	0.5767
2.1	0.1666	0.5683
2.2	0.1104	0.5560
2.3	0.0555	0.5399
2.4	0.0025	0.5202
2.6	−0.0968	−0.4708
2.8	−0.1850	−0.4097
3.0	−0.2601	−0.3391
3.2	−0.3202	−0.2613

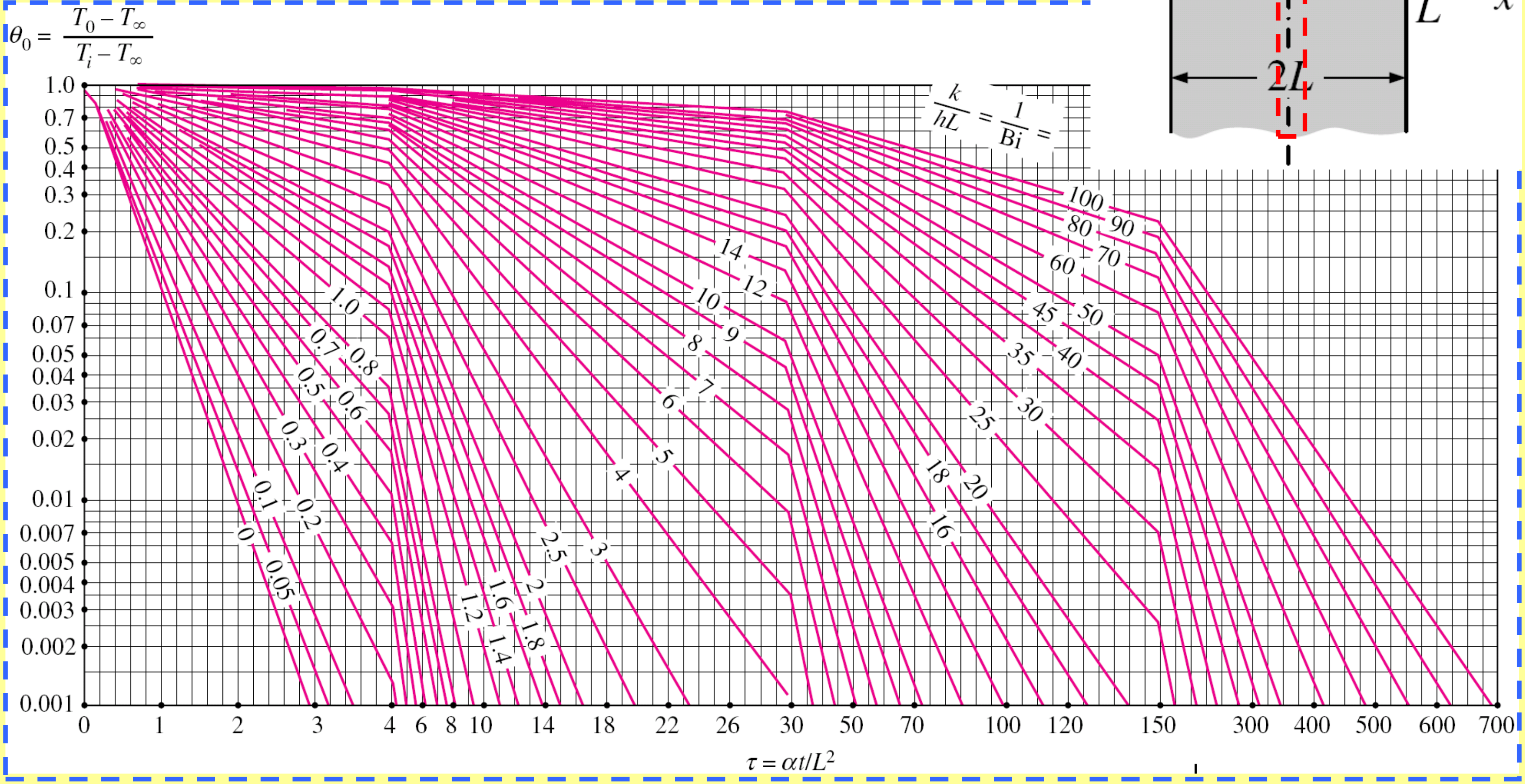
Heisler Charts

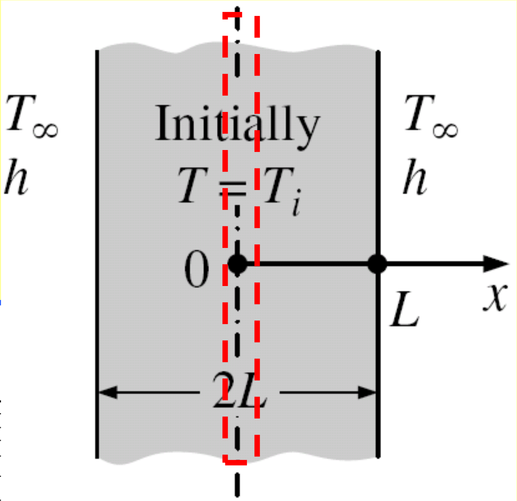
- The solution of the transient temperature for a large plane wall, long cylinder, and sphere are also presented in graphical form for $\tau > 0.2$, known as the *transient temperature charts* (also known as the Heisler Charts).

- There are *three* charts associated with each geometry:

  - the temperature T_0 at the *center* of the geometry at a given time t.
  - the temperature at *other locations* at the same time in terms of T_0.
  - the total amount of *heat transfer* up to the time t.

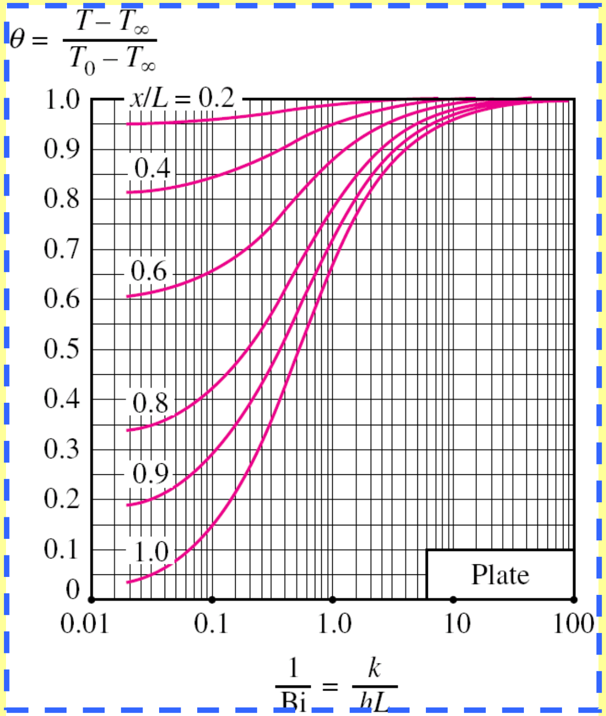
Heisler Charts – Plane Wall

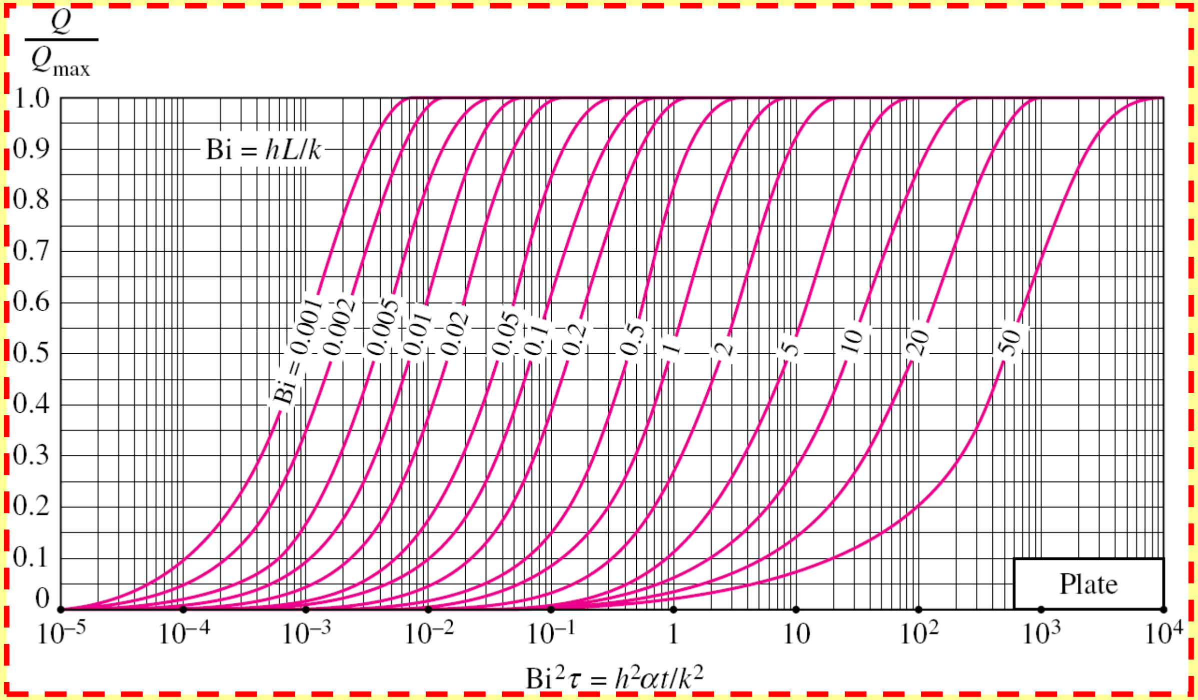




Midplane temperature

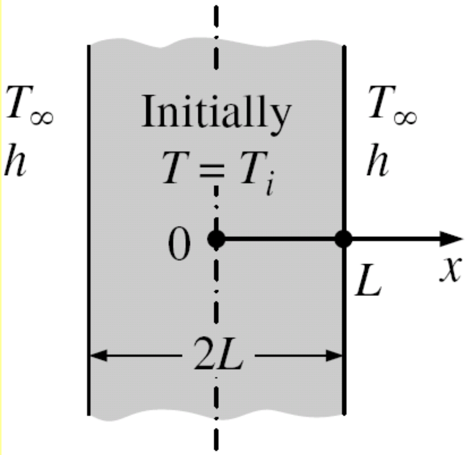
Heisler Charts – Plane Wall





Temperature distribution

Heat Transfer



Heat Transfer

- The *maximum* amount of heat that a body can gain (or lose) is

$$Q_{\max} = mc_p(T_\infty - T_i) = \rho V c_p(T_\infty - T_i)$$

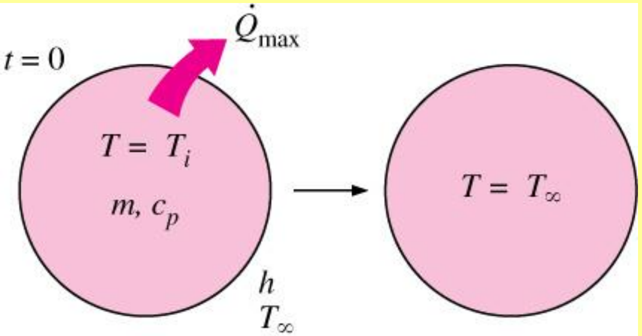
- The amount of heat transfer Q at a finite time t is can be expressed as

$$Q = \int_V \rho c_p[T(x,t) - T_i]dV \qquad \frac{Q}{Q_{\max}} = \frac{\int_V \rho c_p[T(x,t) - T_i]dV}{\rho c_\pi(T_\infty - T_i)V} = \frac{1}{V}\int_V (1 - \theta)dV$$

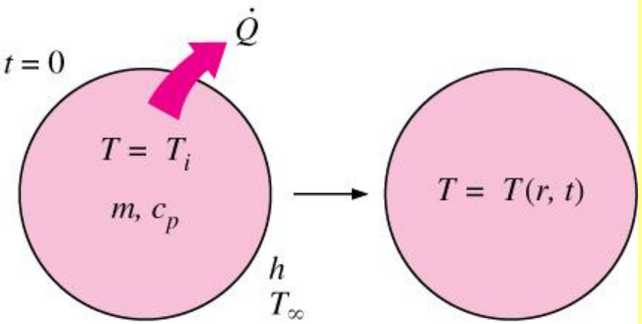
Plane wall: $$\left(\frac{Q}{Q_{\max}}\right)_{\text{wall}} = 1 - \theta_{0,\text{ wall}} \frac{\sin \lambda_1}{\lambda_1}$$

Cylinder: $$\left(\frac{Q}{Q_{\max}}\right)_{\text{cyl}} = 1 - 2\theta_{0,\text{ cyl}} \frac{J_1(\lambda_1)}{\lambda_1}$$

Sphere: $$\left(\frac{Q}{Q_{\max}}\right)_{\text{sph}} = 1 - 3\theta_{0,\text{ sph}} \frac{\sin \lambda_1 - \lambda_1 \cos \lambda_1}{\lambda_1^3}$$



(a) Maximum heat transfer ($t \to \infty$)



$$\left.\begin{array}{l} \mathrm{Bi} = \ldots \\ \dfrac{h^2\alpha t}{k^2} = \mathrm{Bi}^2\tau = \cdots \end{array}\right\} \quad \frac{Q}{Q_{max}} = \cdots$$

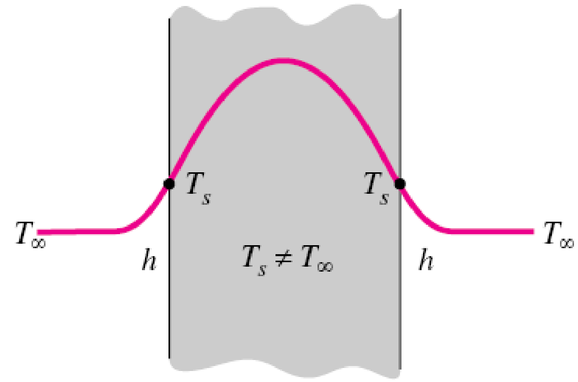
(Gröber chart)

(b) Actual heat transfer for time t

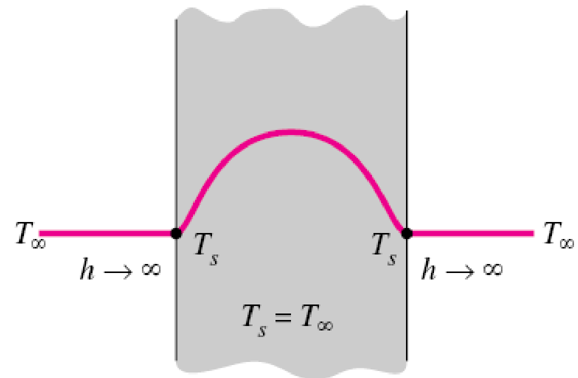
FIGURE 4–19
The fraction of total heat transfer Q/Q_{max} up to a specified time t is determined using the Gröber charts.

The Heisler Charts can only be used when:

- the body is initially at a *uniform* temperature,
- the **temperature** of the medium **surrounding the body** is *constant* and *uniform*.
- the **convection heat transfer coefficient** is *constant* and *uniform*, and there is no *heat generation* in the body.



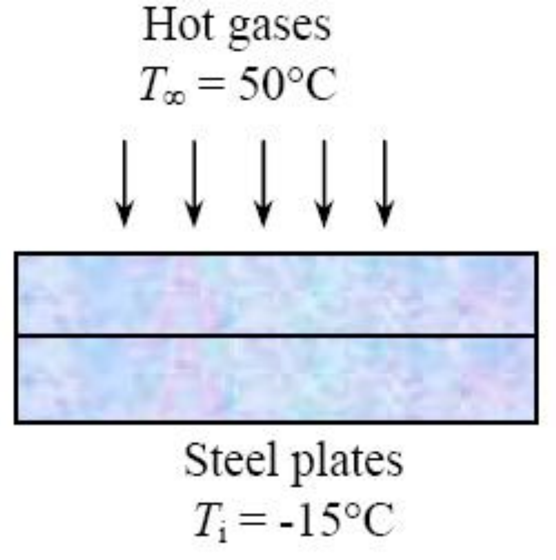
(a) Finite convection coefficient



(b) Infinite convection coefficient

FIGURE 4–18
The specified surface temperature corresponds to the case of convection to an environment at T_∞ with a convection coefficient h that is *infinite*.

4–110 Consider two 2-cm-thick large steel plates (k = 43 W/m · °C and $\alpha = 1.17 \times 10^{-5}$ m²/s) that were put on top of each other while wet and left outside during a cold winter night at −15°C. The next day, a worker needs one of the plates, but the plates are stuck together because the freezing of the water between the two plates has bonded them together. In an effort to melt the ice between the plates and separate them, the worker takes a large hair dryer and blows hot air at 50°C all over the exposed surface of the plate on the top. The convection heat transfer coefficient at the top surface is estimated to be 40 W/m² · °C. Determine how long the worker must keep blowing hot air before the two plates separate.



$$Bi = \frac{hL_c}{k} = \frac{(40\ \text{W/m}^2.°\text{C})(0.02\ \text{m})}{(43\ \text{W/m}.°\text{C})} = 0.019$$

$$\left.\begin{aligned} \frac{1}{Bi} &= \frac{1}{0.019} = 52.6 \\ \frac{T_0 - T_\infty}{T_i - T_\infty} &= \frac{0-50}{-15-50} = 0.769 \end{aligned}\right\} \tau = \frac{\alpha t}{r_o^2} = 15 > 0.2$$

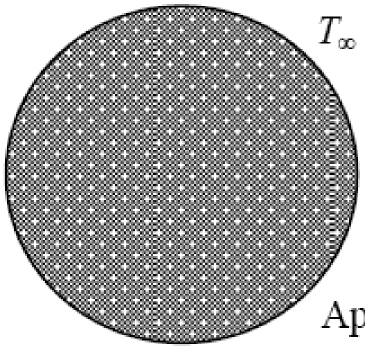
Then,

$$t = \frac{\tau r_o^2}{\alpha} = \frac{(15)(0.02\ \text{m})^2}{(1.17 \times 10^{-5}\ \text{m}^2/\text{s})} = \mathbf{513\ s}$$

4–53 A person puts a few apples into the freezer at $-15°C$ to cool them quickly for guests who are about to arrive. Initially, the apples are at a uniform temperature of 20°C, and the heat transfer coefficient on the surfaces is 8 $W/m^2 \cdot °C$. Treating the apples as 9-cm-diameter spheres and taking their properties to be $\rho = 840$ kg/m^3, $c_p = 3.81$ kJ/kg · °C, $k = 0.418$ W/m · °C, and $\alpha = 1.3 \times 10^{-7}$ m^2/s, determine the center and surface temperatures of the apples in 1 h. Also, determine the amount of heat transfer from each apple.

Analysis The Biot number is

$$Bi = \frac{hr_o}{k} = \frac{(8\ \text{W/m}^2.°\text{C})(0.045\ \text{m})}{(0.418\ \text{W/m}.°\text{C})} = 0.861$$



The constants λ_1 and A_1 corresponding to this Biot number are, from Table 4-2,

$$\lambda_1 = 1.476 \quad \text{and} \quad A_1 = 1.2390$$

The Fourier number is

$$\tau = \frac{\alpha t}{r_o^2} = \frac{(1.3\times10^{-7}\ \text{m}^2/\text{s})(1\ \text{h}\times3600\ \text{s/h})}{(0.045\ \text{m})^2} = 0.231 > 0.2$$

Then the temperature at the center of the apples becomes

$$\theta_{0,sph} = \frac{T_0 - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau} \longrightarrow \frac{T_0 - (-15)}{20 - (-15)} = (1.239)e^{-(1.476)^2(0.231)} = 0.749 \longrightarrow T_0 = \mathbf{11.2°C}$$

The temperature at the surface of the apples is

$$\theta(r_o,t)_{sph} = \frac{T(r_o,t)-T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2\tau} \frac{\sin(\lambda_1 r_o / r_o)}{\lambda_1 r_o / r_o} = (1.239)e^{-(1.476)^2(0.231)} \frac{\sin(1.476\,\text{rad})}{1.476} = 0.505$$

$$\frac{T(r_o,t)-(-15)}{20-(-15)} = 0.505 \longrightarrow T(r_o,t) = \mathbf{2.7°C}$$

The maximum possible heat transfer is

$$m = \rho V = \rho \frac{4}{3}\pi r_o^3 = (840\,\text{kg/m}^3)\left[\frac{4}{3}\pi(0.045\,\text{m})^{3.}\right] = 0.3206\,\text{kg}$$

$$Q_{\max} = mc_p(T_i - T_\infty) = (0.3206\,\text{kg})(3.81\,\text{kJ/kg.°C})[20-(-15)]°\text{C} = 42.75\,\text{kJ}$$

Then the actual amount of heat transfer becomes

$$\frac{Q}{Q_{\max}} = 1 - 3\theta_{o,sph} \frac{\sin(\lambda_1) - \lambda_1\cos(\lambda_1)}{\lambda_1^3} = 1 - 3(0.749)\frac{\sin(1.476\,\text{rad}) - (1.476)\cos(1.476\,\text{rad})}{(1.476)^3} = 0.402$$

$$Q = 0.402 Q_{\max} = (0.402)(42.75\,\text{kJ}) = \mathbf{17.2\,kJ}$$

Transient Heat Conduction in Semi-Infinite Solids

- A semi-infinite solid is an idealized body that has a *single plane surface* and extends to infinity in all directions.
- Assumptions:
  - constant thermophysical properties
  - no internal heat generation
  - uniform thermal conditions on its exposed surface
  - initially a uniform temperature of T_i throughout.
- Heat transfer in this case occurs only in the direction normal to the surface (the x direction) one-dimensional problem.

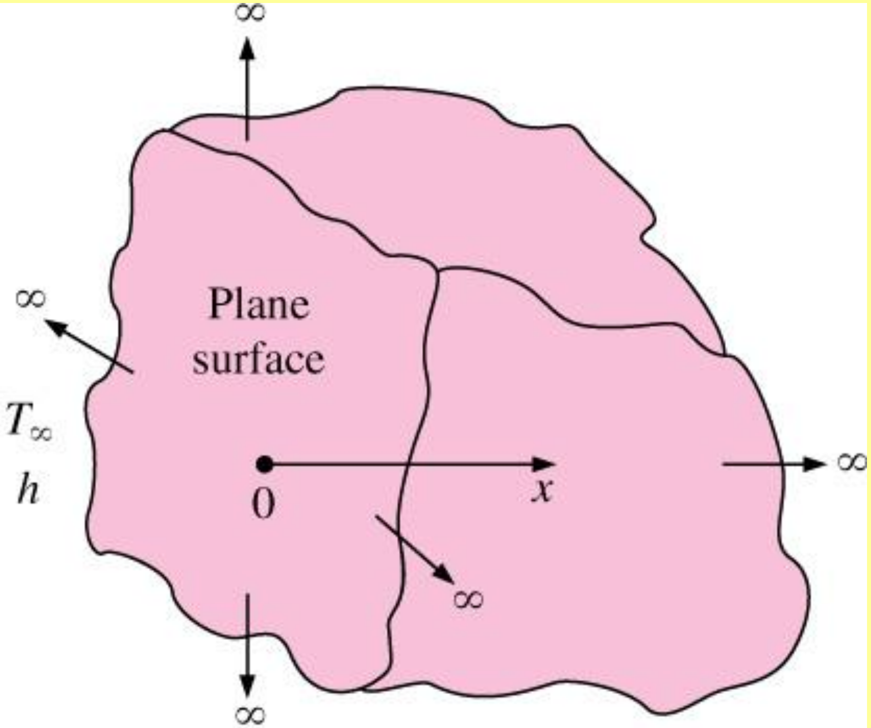


FIGURE 4–24
Schematic of a semi-infinite body.

Differential equation: $$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

Boundary conditions: $T(0, t) = T_s$ and $T(x \to \infty, t) = T_i$

Initial condition: $T(x, 0) = T_i$

Case 1: Specified Surface Temperature, T_s = constant (Fig. 4–27).

$$\frac{T(x, t) - T_i}{T_s - T_i} = \text{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right) \quad \text{and} \quad \dot{q}_s(t) = \frac{k(T_s - T_i)}{\sqrt{\pi \alpha t}}$$

Case 2: Specified Surface Heat Flux, $\dot{q}_s$ = constant.

$$T(x, t) - T_i = \frac{\dot{q}_s}{k}\left[\sqrt{\frac{4\alpha t}{\pi}}\exp\left(-\frac{x^2}{4\alpha t}\right) - x\,\text{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right)\right]$$

Case 3: Convection on the Surface, $\dot{q}_s(t) = h[T_\infty - T(0, t)]$.

$$\frac{T(x,t) - T_i}{T_\infty - T_i} = \text{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right) - \exp\left(\frac{hx}{k} + \frac{h^2 \alpha t}{k^2}\right)\text{erfc}\left(\frac{x}{2\sqrt{\alpha t}} + \frac{h\sqrt{\alpha t}}{k}\right)$$

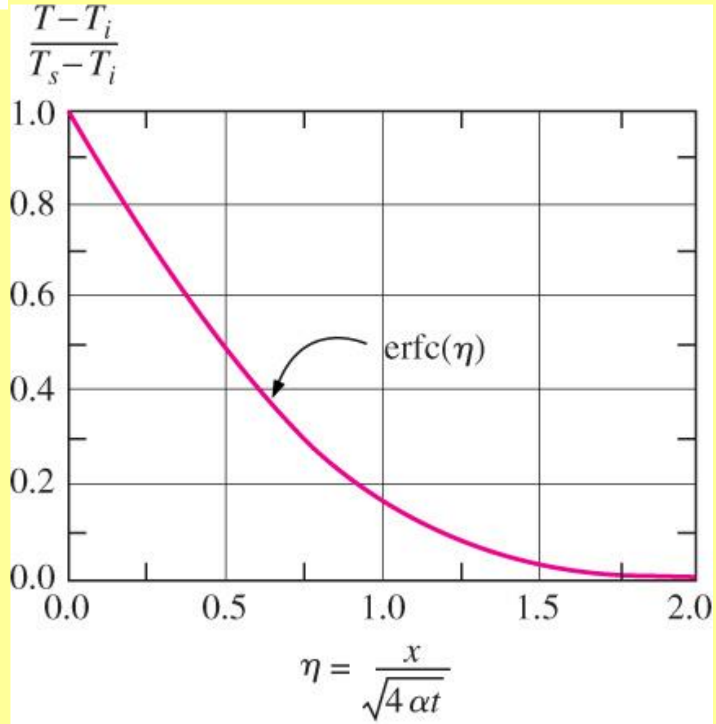


FIGURE 4–27

Dimensionless temperature distribution for transient conduction in a semi-infinite solid whose surface is maintained at a constant temperature T_s.

TABLE 4–4

The complementary error function

η	erfc (η)	η	erfc (η)	η	erfc (η)	η	erfc (η)	η	erfc (η)	η	erfc (η)
0.00	1.00000	0.38	0.5910	0.76	0.2825	1.14	0.1069	1.52	0.03159	1.90	0.00721
0.02	0.9774	0.40	0.5716	0.78	0.2700	1.16	0.10090	1.54	0.02941	1.92	0.00662
0.04	0.9549	0.42	0.5525	0.80	0.2579	1.18	0.09516	1.56	0.02737	1.94	0.00608
0.06	0.9324	0.44	0.5338	0.82	0.2462	1.20	0.08969	1.58	0.02545	1.96	0.00557
0.08	0.9099	0.46	0.5153	0.84	0.2349	1.22	0.08447	1.60	0.02365	1.98	0.00511
0.10	0.8875	0.48	0.4973	0.86	0.2239	1.24	0.07950	1.62	0.02196	2.00	0.00468
0.12	0.8652	0.50	0.4795	0.88	0.2133	1.26	0.07476	1.64	0.02038	2.10	0.00298
0.14	0.8431	0.52	0.4621	0.90	0.2031	1.28	0.07027	1.66	0.01890	2.20	0.00186
0.16	0.8210	0.54	0.4451	0.92	0.1932	1.30	0.06599	1.68	0.01751	2.30	0.00114
0.18	0.7991	0.56	0.4284	0.94	0.1837	1.32	0.06194	1.70	0.01612	2.40	0.00069
0.20	0.7773	0.58	0.4121	0.96	0.1746	1.34	0.05809	1.72	0.01500	2.50	0.00041
0.22	0.7557	0.60	0.3961	0.98	0.1658	1.36	0.05444	1.74	0.01387	2.60	0.00024
0.24	0.7343	0.62	0.3806	1.00	0.1573	1.38	0.05098	1.76	0.01281	2.70	0.00013
0.26	0.7131	0.64	0.3654	1.02	0.1492	1.40	0.04772	1.78	0.01183	2.80	0.00008
0.28	0.6921	0.66	0.3506	1.04	0.1413	1.42	0.04462	1.80	0.01091	2.90	0.00004
0.30	0.6714	0.68	0.3362	1.06	0.1339	1.44	0.04170	1.82	0.01006	3.00	0.00002
0.32	0.6509	0.70	0.3222	1.08	0.1267	1.46	0.03895	1.84	0.00926	3.20	0.00001
0.34	0.6306	0.72	0.3086	1.10	0.1198	1.48	0.03635	1.86	0.00853	3.40	0.00000
0.36	0.6107	0.74	0.2953	1.12	0.1132	1.50	0.03390	1.88	0.00784	3.60	0.00000

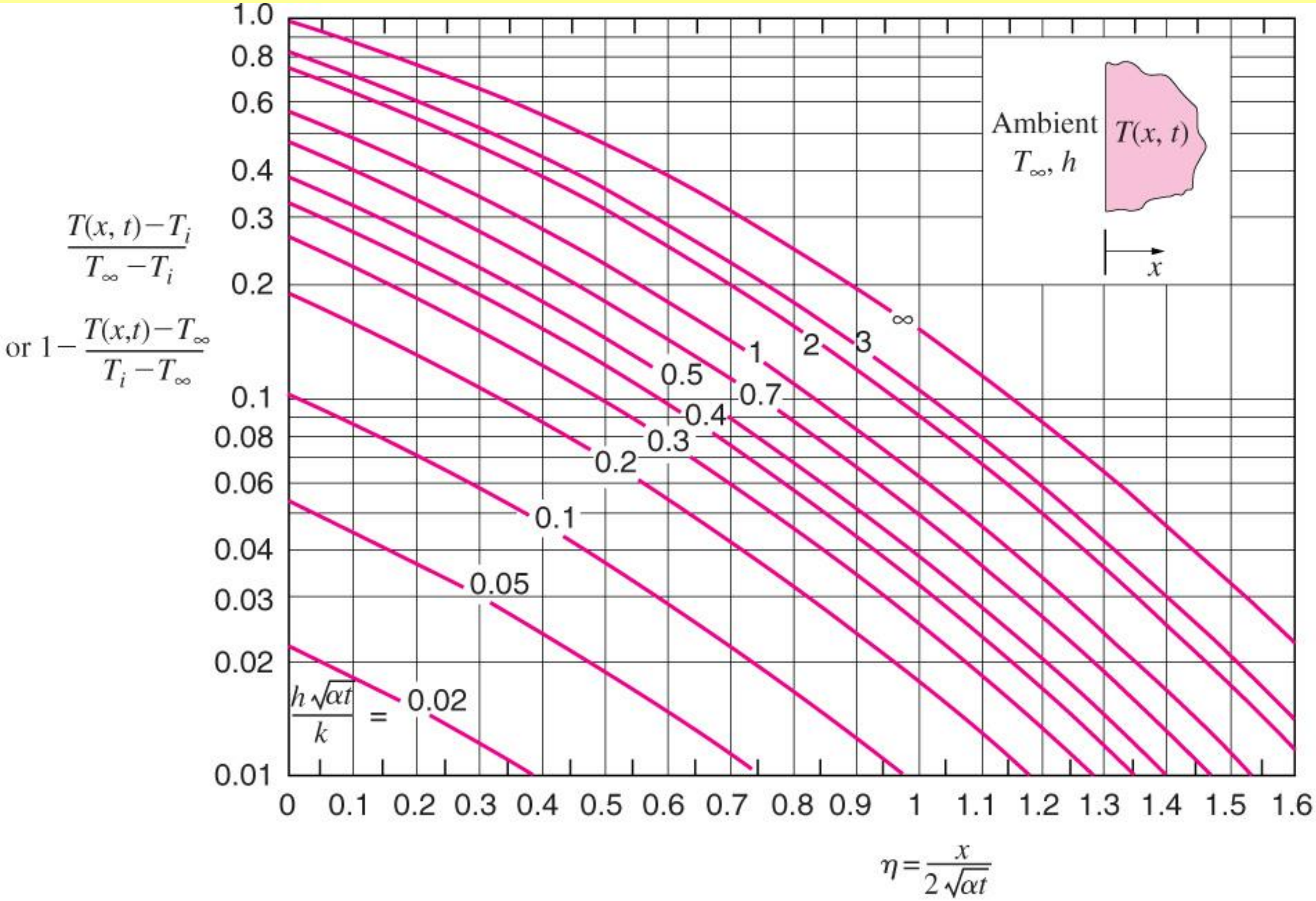
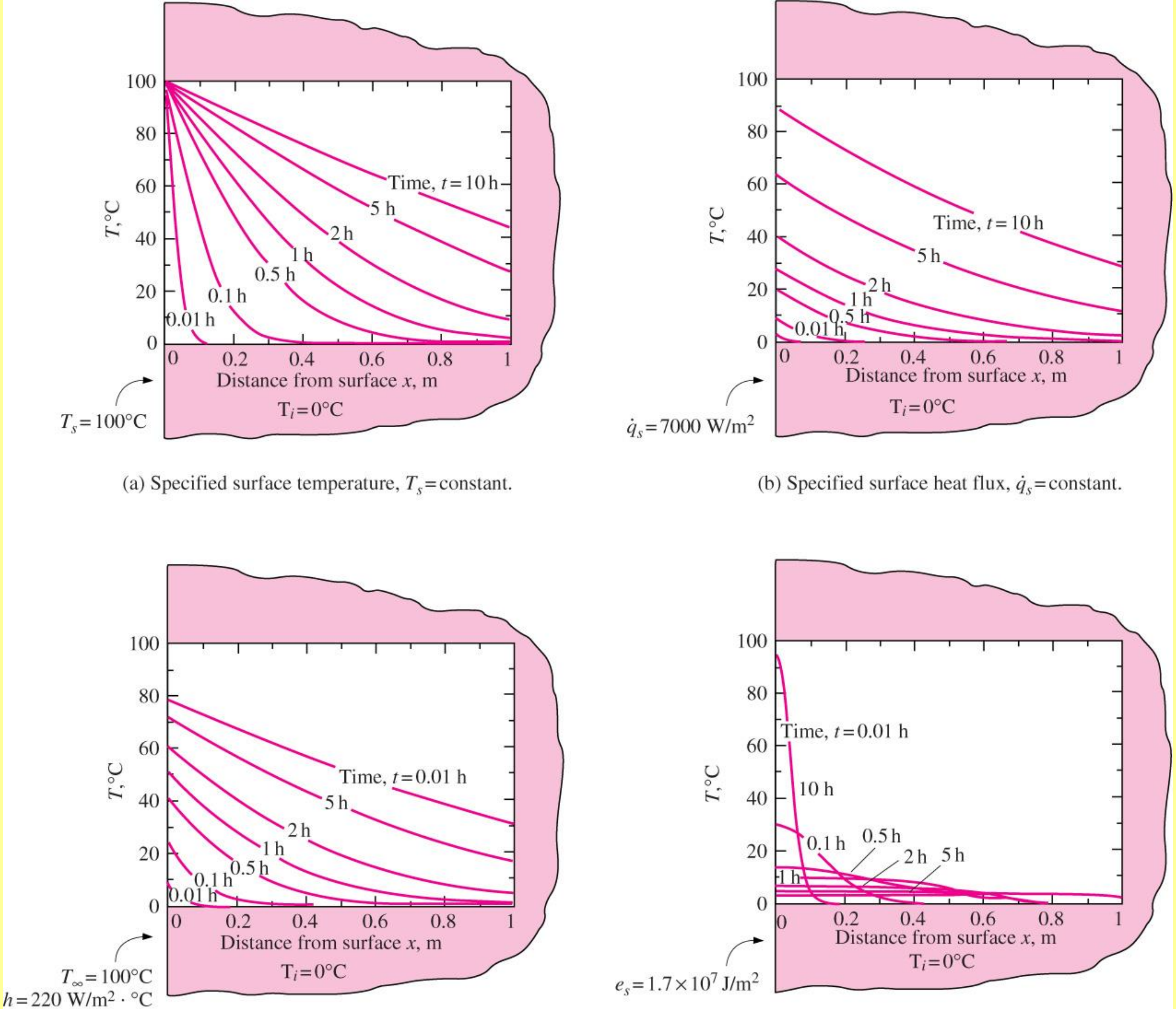


FIGURE 4–29

Variation of temperature with position and time in a semi-infinite solid initially at temperature T_i subjected to convection to an environment at T_∞ with a convection heat transfer coefficient of h (plotted using EES).



(a) Specified surface temperature, T_s = constant.

(b) Specified surface heat flux, $\dot{q}_s$ = constant.

(c) Convection at the surface

(d) Energy pulse at the surface, e_s = constant

FIGURE 4–28

Variations of temperature with position and time in a large cast iron block ($\alpha = 2.31 \times 10^{-5}$ m²/s, $k = 80.2$ W/m °C) initially at 0 °C under different thermal conditions on the surface.

Contact of Two Semi-Infinite Solids

$$\dot{q}_{s,A} = \dot{q}_{s,B} \rightarrow -\frac{k_A(T_s - T_{A,i})}{\sqrt{\pi \alpha_A t}} = \frac{k_B(T_s - T_{B,i})}{\sqrt{\pi \alpha_B t}} \rightarrow \frac{T_{A,i} - T_s}{T_s - T_{B,i}} = \sqrt{\frac{(k\rho c_p)_B}{(k\rho c_p)_A}}$$

$$T_s = \frac{\sqrt{(k\rho c_p)_A}\, T_{A,i} + \sqrt{(k\rho c_p)_B}\, T_{B,i}}{\sqrt{(k\rho c_p)_A} + \sqrt{(k\rho c_p)_B}}$$

Therefore, the interface temperature of two bodies brought into contact is dominated by the body with the larger $k\rho c_p$.

This also explains why a metal at room temperature feels colder than wood at the same temperature.

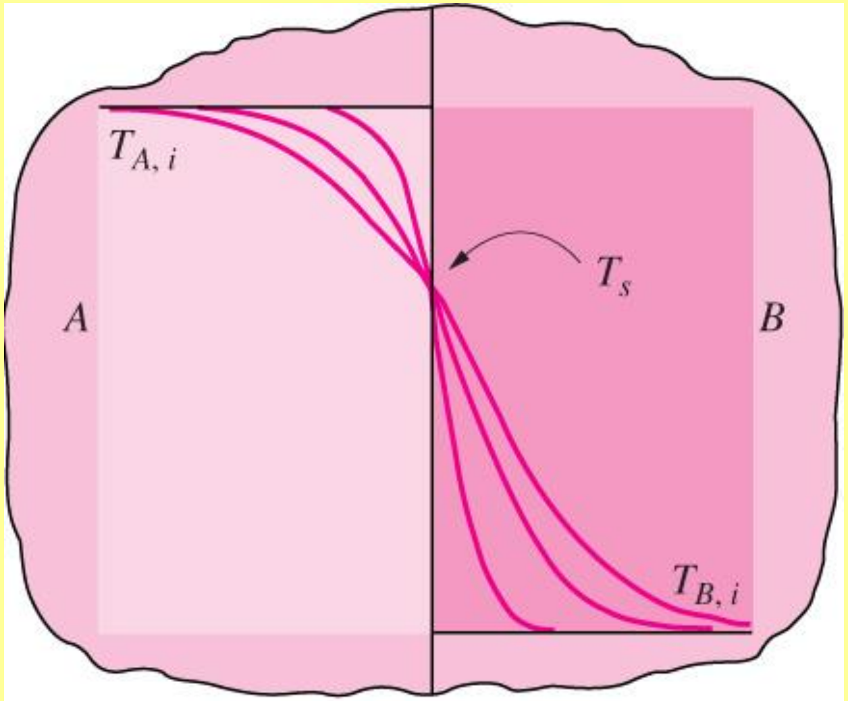


FIGURE 4–30
Contact of two semi-infinite solids of different initial temperatures.

4–111 Consider a curing kiln whose walls are made of 30-cm-thick concrete whose properties are $k = 0.9$ W/m · °C and $\alpha = 0.23 \times 10^{-5}$ m²/s. Initially, the kiln and its walls are in equilibrium with the surroundings at 6°C. Then all the doors are closed and the kiln is heated by steam so that the temperature of the inner surface of the walls is raised to 42°C and is maintained at that level for 2.5 h. The curing kiln is then opened and exposed to the atmospheric air after the stream flow is turned off. If the outer surfaces of the walls of the kiln were insulated, would it save any energy that day during the period the kiln was used for curing for 2.5 h only, or would it make no difference? Base your answer on calculations.

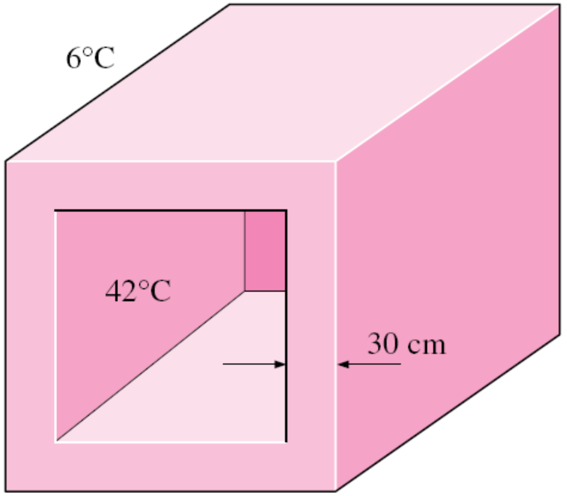
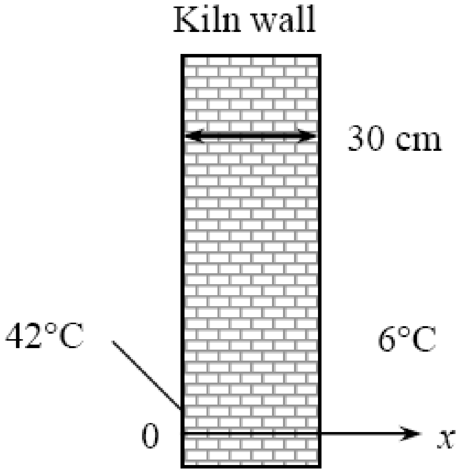


FIGURE P4–111

Analysis We determine the temperature at a depth of x = 0.3 m in 2.5 h using the analytical solution,

$$\frac{T(x,t)-T_i}{T_s-T_i} = erfc\left(\frac{x}{2\sqrt{\alpha t}}\right)$$



Substituting,

$$\frac{T(x,t)-6}{42-6} = erfc\left(\frac{0.3\text{ m}}{2\sqrt{(0.23\times10^{-5}\text{ m}^2/\text{s})(2.5\text{ h}\times3600\text{ s/h})}}\right)$$

$$= erfc(1.043) = 0.1402$$

$$T(x,t) = \mathbf{11.0\ °C}$$

which is greater than the initial temperature of 6°C. Therefore, heat will propagate through the 0.3 m thick wall in 2.5 h, and thus it may be desirable to insulate the outer surface of the wall to save energy.

Transient Heat Conduction in Multidimensional Systems

- Using a superposition approach called the **product solution**, the *one-dimensional* heat conduction solutions can also be used to construct solutions for some two-dimensional (and even three-dimensional) transient heat conduction problems.
- Provided that *all* surfaces of the solid are subjected to convection to the *same* fluid at temperature, the *same* heat transfer coefficient *h*, and the body involves no heat generation.

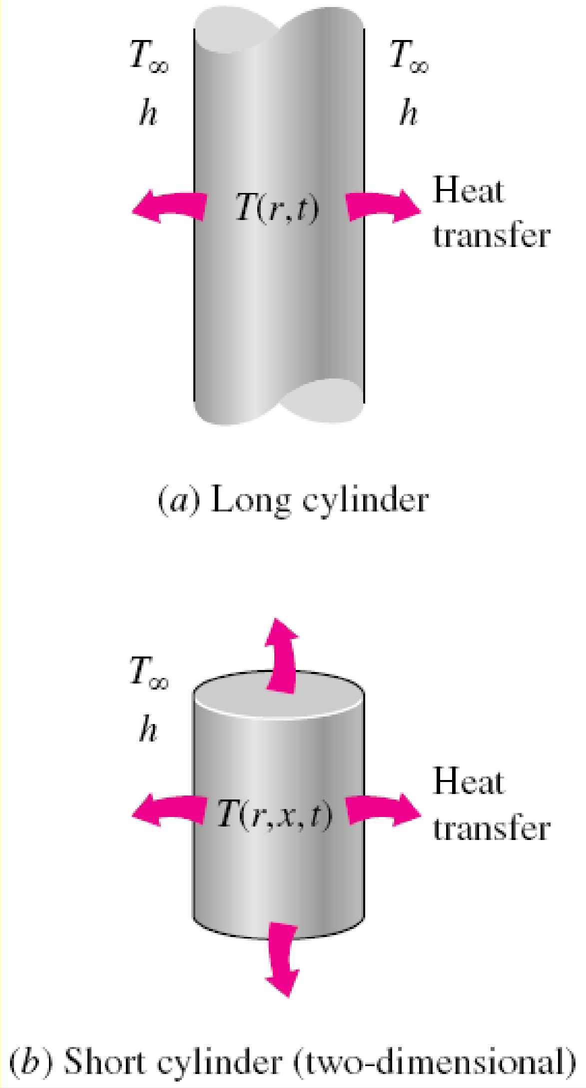


FIGURE 4–34
The temperature in a short cylinder exposed to convection from all surfaces varies in both the radial and axial directions, and thus heat is transferred in both directions.

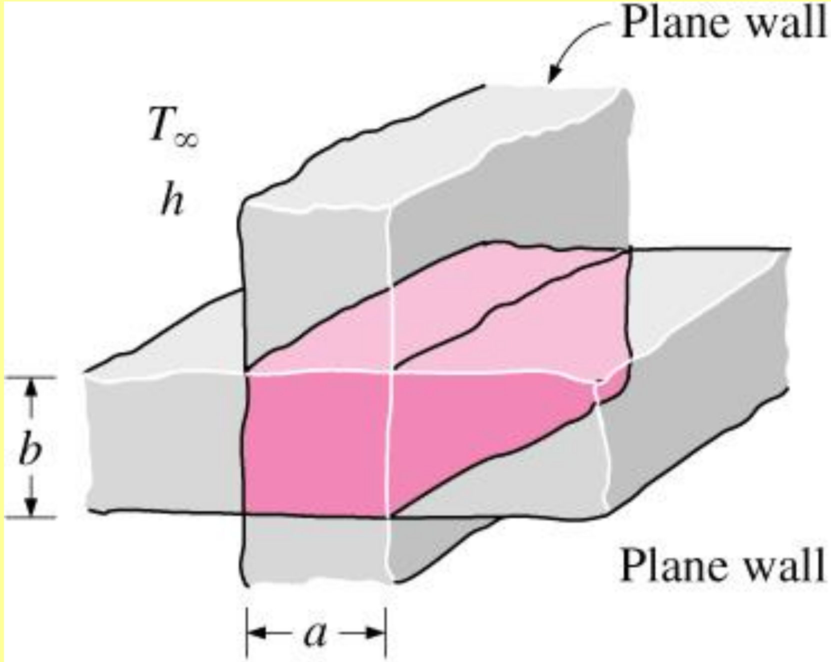


FIGURE 4–36

A long solid bar of rectangular profile $a \times b$ is the *intersection* of two plane walls of thicknesses a and b.

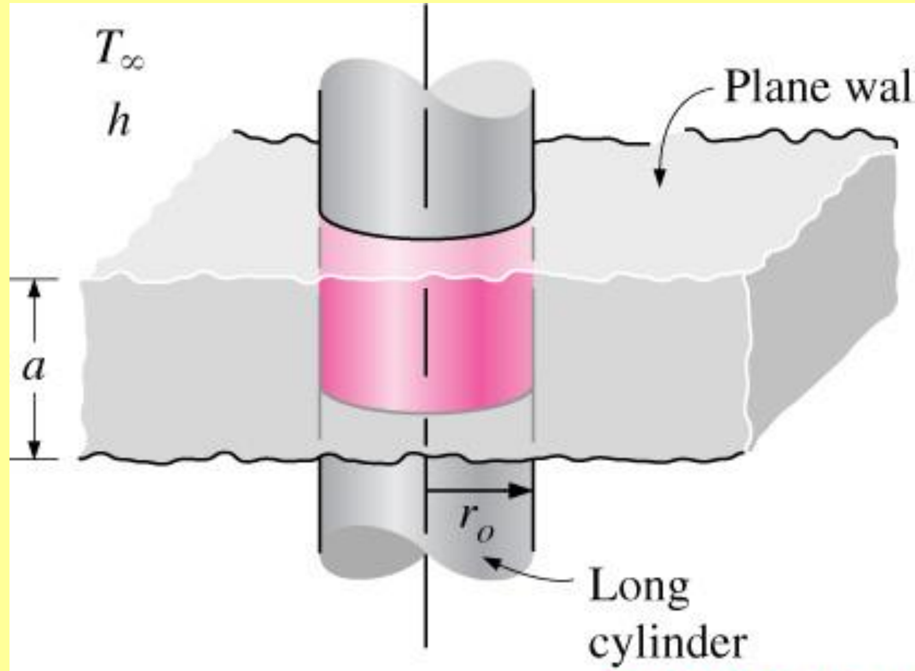
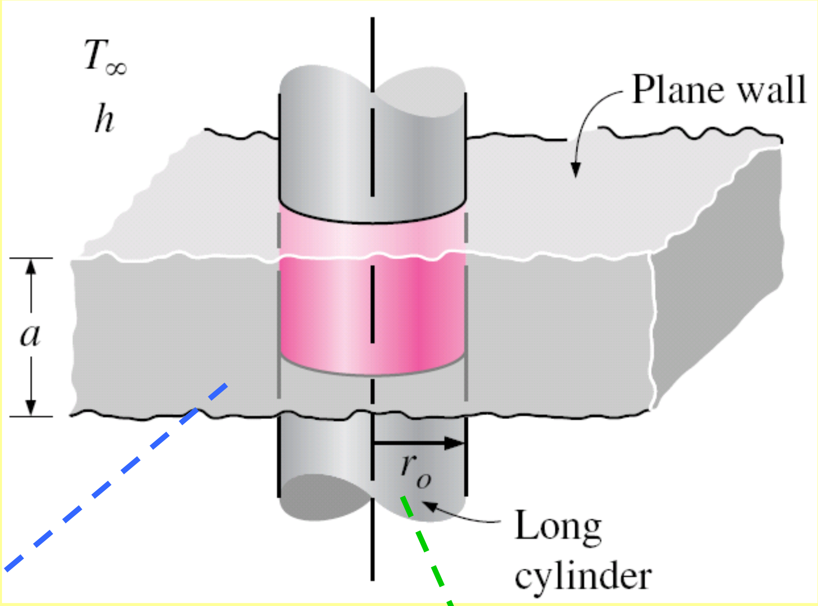


FIGURE 4–35

A short cylinder of radius r_o and height a is the *intersection* of a long cylinder of radius r_o and a plane wall of thickness a.

Example – short cylinder

- Height a and radius r_o.
- Initially uniform temperature T_i.
- No heat generation
- At time $t = 0$:
  - convection T_∞
  - heat transfer coefficient h
- The solution:



$$\left(\frac{T(r,x,t)-T_\infty}{T_i-T_\infty}\right)\Bigg|_{\substack{\text{Short}\\\text{Cylinder}}} = \left(\frac{T(x,t)-T_\infty}{T_i-T_\infty}\right)\Bigg|_{\substack{\text{plane}\\\text{wall}}} \text{X} \left(\frac{T(r,t)-T_\infty}{T_i-T_\infty}\right)\Bigg|_{\substack{\text{infinite}\\\text{cylinder}}} \quad \textbf{(4–50)}$$

- The solution can be generalized as follows: *the solution for a multidimensional geometry is the product of the solutions of the one-dimensional geometries whose intersection is the multidimensional body.*
- For convenience, the one-dimensional solutions are denoted by

$$\theta_{\text{wall}}(x, t) = \left(\frac{T(x, t) - T_\infty}{T_i - T_\infty}\right)_{\substack{\text{plane}\\ \text{wall}}}$$

$$\theta_{\text{cyl}}(r, t) = \left(\frac{T(r, t) - T_\infty}{T_i - T_\infty}\right)_{\substack{\text{infinite}\\ \text{cylinder}}}$$

$$\theta_{\text{semi-inf}}(x, t) = \left(\frac{T(x, t) - T_\infty}{T_i - T_\infty}\right)_{\substack{\text{semi-infinite}\\ \text{solid}}}$$

$$\left(\frac{T(x, y, t) - T_\infty}{T_i - T_\infty}\right)_{\substack{\text{rectangular}\\ \text{bar}}} = \theta_{\text{wall}}(x, t)\,\theta_{\text{wall}}(y, t)$$

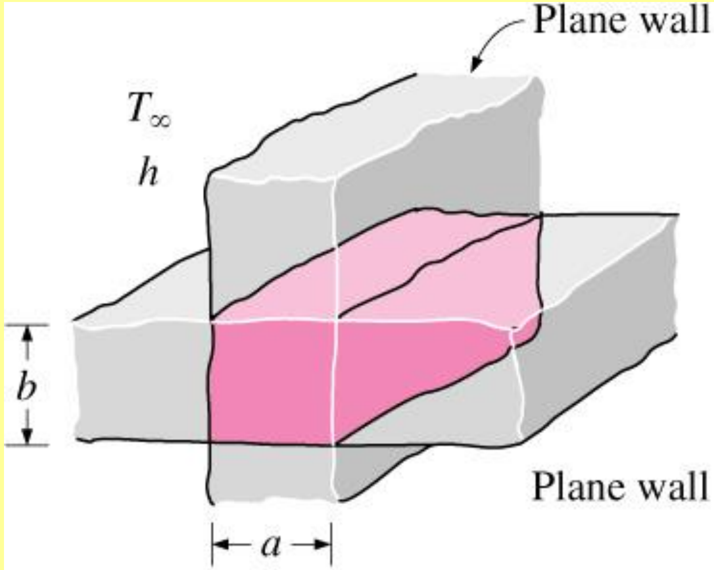
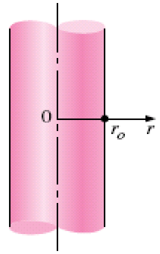
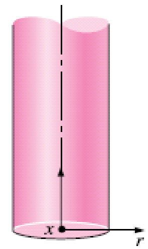
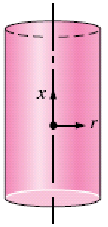
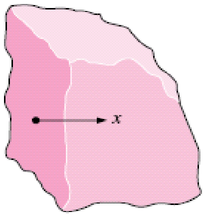
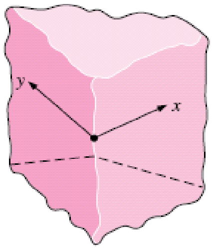
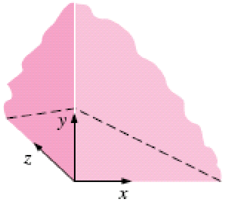
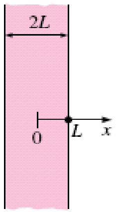
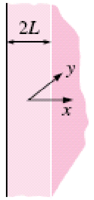
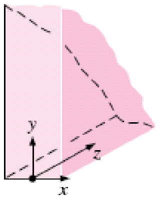
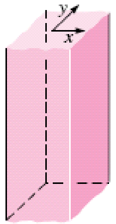
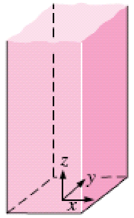
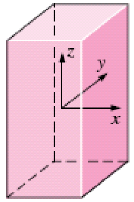


FIGURE 4–36

A long solid bar of rectangular profile $a \times b$ is the *intersection* of two plane walls of thicknesses a and b.

TABLE 4–5

Multidimensional solutions expressed as products of one-dimensional solutions for bodies that are initially at a uniform temperature T_i and exposed to convection from all surfaces to a medium at T_∞

 $\theta(r,t) = \theta_{cyl}(r,t)$ **Infinite cylinder**	 $\theta(x,r,t) = \theta_{cyl}(r,t)\,\theta_{semi\text{-}inf}(x,t)$ **Semi-infinite cylinder**	 $\theta(x,r,t) = \theta_{cyl}(r,t)\,\theta_{wall}(x,t)$ **Short cylinder**
 $\theta(x,t) = \theta_{semi\text{-}inf}(x,t)$ **Semi-infinite medium**	 $\theta(x,y,t) = \theta_{semi\text{-}inf}(x,t)\,\theta_{semi\text{-}inf}(y,t)$ **Quarter-infinite medium**	 $\theta(x,y,z,t) =$ $\theta_{semi\text{-}inf}(x,t)\,\theta_{semi\text{-}inf}(y,t)\,\theta_{semi\text{-}inf}(z,t)$ **Corner region of a large medium**
 $\theta(x,t) = \theta_{wall}(x,t)$ **Infinite plate (or plane wall)**	 $\theta(x,y,t) = \theta_{wall}(x,t)\,\theta_{semi\text{-}inf}(y,t)$ **Semi-infinite plate**	 $\theta(x,y,z,t) =$ $\theta_{wall}(x,t)\,\theta_{semi\text{-}inf}(y,t)\,\theta_{semi\text{-}inf}(z,t)$ **Quarter-infinite plate**
 $\theta(x,y,t) = \theta_{wall}(x,t)\,\theta_{wall}(y,t)$ **Infinite rectangular bar**	 $\theta(x,y,z,t) =$ $\theta_{wall}(x,t)\,\theta_{wall}(y,t)\,\theta_{semi\text{-}inf}(z,t)$ **Semi-infinite rectangular bar**	 $\theta(x,y,z,t) =$ $\theta_{wall}(x,t)\,\theta_{wall}(y,t)\,\theta_{wall}(z,t)$ **Rectangular parallelepiped**

Total Transient Heat Transfer

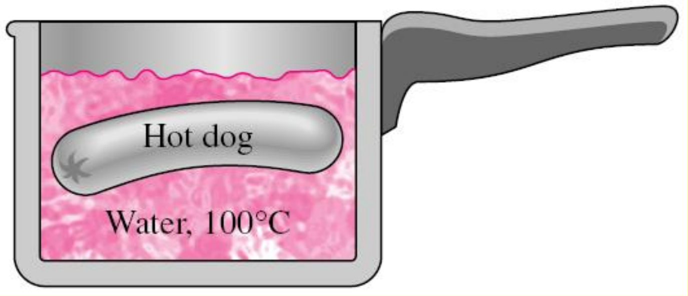
- The transient heat transfer for a two dimensional geometry formed by the intersection of two one-dimensional geometries 1 and 2 is:

$$\left(\frac{Q}{Q_{\max}}\right)_{total,\ 2D} = \left(\frac{Q}{Q_{\max}}\right)_1 + \left(\frac{Q}{Q_{\max}}\right)_2 \left[1 - \left(\frac{Q}{Q_{\max}}\right)_1\right]$$

- Transient heat transfer for a three-dimensional (intersection of three one-dimensional bodies 1, 2, and 3) is:

$$\left(\frac{Q}{Q_{\max}}\right)_{total,\ 3D} = \left(\frac{Q}{Q_{\max}}\right)_1 + \left(\frac{Q}{Q_{\max}}\right)_2 \left[1 - \left(\frac{Q}{Q_{\max}}\right)_1\right] + \left(\frac{Q}{Q_{\max}}\right)_3 \left[1 - \left(\frac{Q}{Q_{\max}}\right)_1\right]\left[1 - \left(\frac{Q}{Q_{\max}}\right)_2\right]$$

4–113 A hot dog can be considered to be a 12-cm-long cylinder whose diameter is 2 cm and whose properties are $\rho = 980$ kg/m³, $c_p = 3.9$ kJ/kg · °C, $k = 0.76$ W/m · °C, and $\alpha = 2 \times 10^{-7}$ m²/s. A hot dog initially at 5°C is dropped into boiling water at 100°C. The heat transfer coefficient at the surface of the hot dog is estimated to be 600 W/m² · °C. If the hot dog is considered cooked when its center temperature reaches 80°C, determine how long it will take to cook it in the boiling water.



$$Bi = \frac{hL}{k} = \frac{(600\ \text{W/m}^2.°\text{C})(0.06\ \text{m})}{(0.76\ \text{W/m.°C})} = 47.37 \longrightarrow \lambda_1 = 1.5380 \quad \text{and} \quad A_1 = 1.2726$$

$$Bi = \frac{hr_o}{k} = \frac{(600\ \text{W/m}^2.°\text{C})(0.01\ \text{m})}{(0.76\ \text{W/m.°C})} = 7.895 \longrightarrow \lambda_1 = 2.1249 \quad \text{and} \quad A_1 = 1.5514$$

$$\theta(0,0,t)_{block} = \theta(0,t)_{wall}\,\theta(0,t)_{cyl} = \left(A_1 e^{-\lambda_1^2 \tau}\right)\left(A_1 e^{-\lambda_1^2 \tau}\right)$$

$$\frac{80-100}{5-100} = \left\{(1.2726)\exp\left[-(1.5380)^2 \frac{(2\times 10^{-7})t}{(0.06)^2}\right]\right\}$$

$$\times\left\{(1.5514)\exp\left[-(2.1249)^2 \frac{(2\times 10^{-7})t}{(0.01)^2}\right]\right\} = 0.2105$$

which gives

t = **244 s = 4.1 min**

This hot dog can physically be formed by the intersection of an infinite plane wall of thickness 2*L* = 12 cm, and a long cylinder of radius *ro* = *D*/2 = 1 cm.

$$\tau_{cyl} = \frac{\alpha t}{r_o^2} = \frac{(2\times 10^{-7}\ \text{m}^2/\text{s})(244\ \text{s})}{(0.01\ \text{m})^2} = 0.49 > 0.2$$

Conclusions

- Lumped system analysis
- Transient heat conduction in
    - large plane walls
    - long cylinders
    - spheres
- Transient heat conduction in semi-infinite solids
- Transient heat conduction in multidimensional systems

HEAT AND MASS TRANSFER

Numerical Methods in Heat Transfer

Objectives

- To understand the limitations of analytical solutions of conduction problems, and the need for computation-intensive numerical methods,
- To express derivates as differences, and obtain finite difference formulations,
- To solve steady one- or two-dimensional conduction problems numerically using the finite difference method, and
- To solve transient one- or two-dimensional conduction problems using the finite difference method.

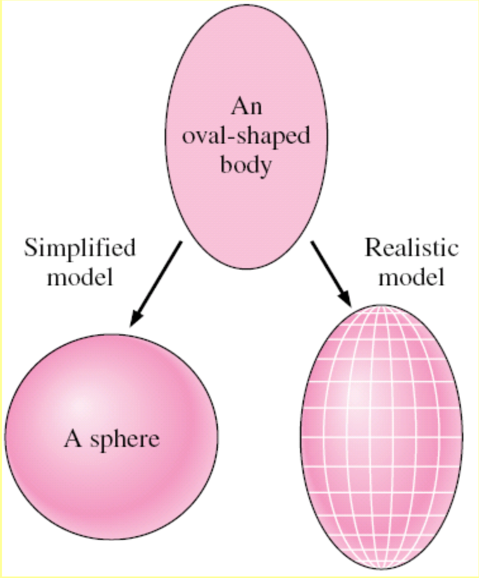
Why Numerical Methods?

➔Several ways for obtaining the numerical formulation of a heat conduction problem:
- the finite difference method,
- the finite element method,
- the boundary element method, and
- the energy balance (or control volume) method.

➔Each method has its own advantages and disadvantages!!! And each is used in practice.

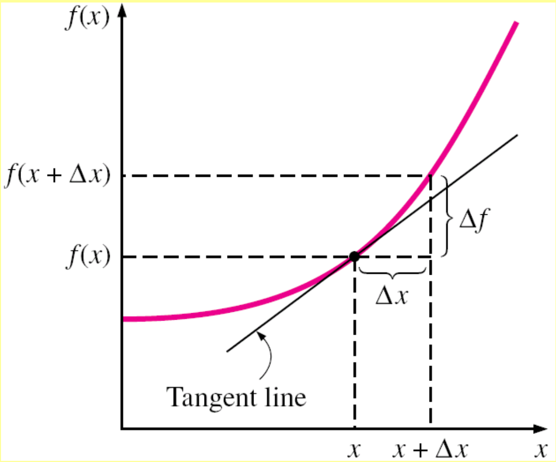
➔ **Key Reasons:**

1. **Limitations** — Analytical solution methods are limited to *highly simplified problems* in *simple geometries.*
2. **Better Modeling** — An "approximate" solution is usually more accurate than the "exact" solution of a crude mathematical model.
3. **Flexibility** — Engineering problems often require extensive *parametric studies.*
4. **Complications** — Analytical solutions for complex geometries and problems are not available.
5. **Human Nature**
   - Diminishing use of human brain power with expectation for powerful results.
   - Impressive presentation-style colorful output in graphical and tabular form.



Finite Difference Formulation of Differential Equations

- The numerical methods for solving differential equations are based on replacing the *differential equations* by *algebraic equations.*
- For **finite difference** method, this is done by replacing the *derivatives* by *differences.*
- A function f that depends on x.
- The **first derivative** of $f(x)$ at a point is equivalent to the *slope* of a line tangent to the curve at that point



$$\frac{df(x)}{dx} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \qquad \textbf{(5-5)}$$

- If we don't take the indicated limit, we will have the following *approximate* relation for the derivative:

$$\frac{df(x)}{dx} \cong \frac{f(x+\Delta x)-f(x)}{\Delta x} \qquad \textbf{(5-6)}$$

- The equation above can also be obtained by writing the *Taylor series expansion* of the function f about the point x,

$$f(x+\Delta x) = f(x) + \Delta x \frac{df(x)}{dx} + \frac{1}{2}\Delta x^2 \frac{d^2 f(x)}{dx^2} + \ldots \qquad \textbf{(5-7)}$$

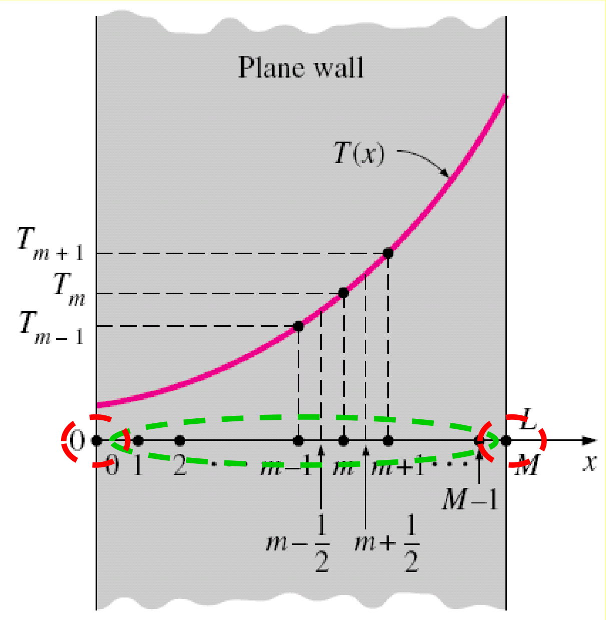
and neglecting all the terms except the first two.

- The first term neglected is proportional to Δx^2, and thus the *error* involved in each step is also proportional to Δx^2.
- However, the ***commutative error*** involved after M steps in the direction of length L is proportional to Δx since $M\Delta x^2=(L/\Delta x)\Delta x^2=L\Delta x$.

$$\boxed{\textbf{Error} = L\Delta x}$$

One-Dimensional Steady Heat Conduction

- Steady one-dimensional heat conduction in a plane wall of thickness L with heat generation.
- The wall is subdivided into M sections of equal thickness $\Delta x = L/M$.
- $M+1$ points 0, 1, 2, . . . , m-1, m, $m+1$, . . . , M called **nodes** or **nodal points**.
- The x-coordinate of any point m is $x_m = m\Delta x$.
- The temperature at that point is simply $T(x_m) = T_m$.



➔ Green: internal nodal points

➔ Red: boundary nodal points

- Using Eq. 5–6

$$\left.\frac{dT}{dx}\right|_{m-\frac{1}{2}} \cong \frac{T_m - T_{m-1}}{\Delta x} \quad ; \quad \left.\frac{dT}{dx}\right|_{m+\frac{1}{2}} \cong \frac{T_{m+1} - T_m}{\Delta x} \qquad \textbf{(5-8)}$$

- Noting that the second derivative is simply the derivative of the first derivative:

$$\left.\frac{d^2T}{dx^2}\right|_m \cong \frac{\left.\frac{dT}{dx}\right|_{m+\frac{1}{2}} - \left.\frac{dT}{dx}\right|_{m-\frac{1}{2}}}{\Delta x} = \frac{\frac{T_{m+1} - T_m}{\Delta x} - \frac{T_m - T_{m-1}}{\Delta x}}{\Delta x}$$

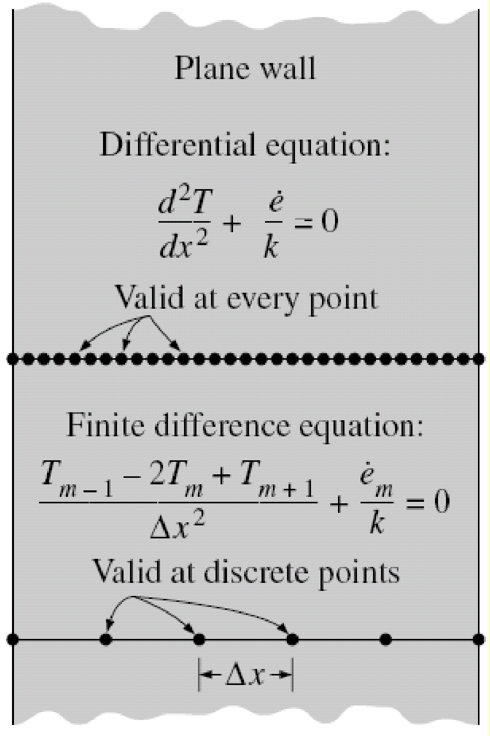
$$= \frac{T_{m-1} - 2T_m + T_{m+1}}{\Delta x^2} \qquad \textbf{(5-9)}$$

The governing equation for *steady one-dimensional* heat transfer in a plane wall with heat generation and constant thermal conductivity

$$\frac{d^2T}{dx^2} + \frac{\dot{e}}{k} = 0 \qquad \textbf{(5-10)}$$

$$\left.\frac{d^2T}{dx^2}\right|_m \cong \boxed{\frac{T_{m-1} - 2T_m + T_{m+1}}{\Delta x^2}} \qquad \textbf{(5-9)}$$

$$\frac{T_{m-1} - 2T_m + T_{m+1}}{\Delta x^2} + \frac{\dot{e}_m}{k} = 0, \quad m = 1, 2, 3, \ldots M - 1 \qquad \textbf{(5-11)}$$



-The equation is applicable to each of the *M*-1 interior nodes

→ *M*-1 equations for the determination of temperatures.

- The two additional equations needed to solve for the *M*+1 unknown nodal temperatures are obtained by applying the energy balance on the two elements at the boundaries.

Boundary Conditions

- A boundary node does not have a neighboring node on at least one side.
- We need to obtain the finite difference equations of boundary nodes separately in most cases (specified temperature boundary conditions is an exception).
- *Energy balance* on the volume elements of boundary nodes is applied.
- Boundary conditions frequently encountered are:

1. *specified temperature*,

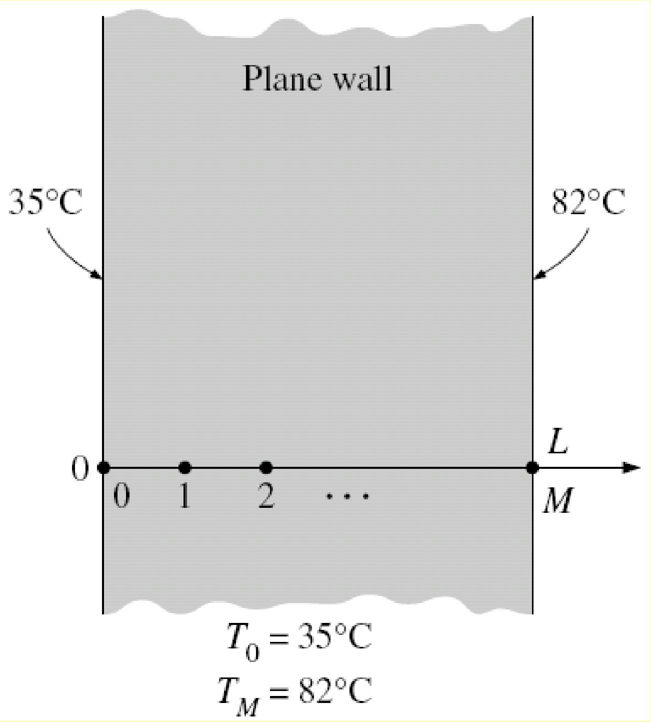
2. *specified heat flux*,

3. *convection*, and

4. *radiation* boundary conditions.

Boundary Conditions for Steady One-Dimensional Heat Conduction in a Plane Wall

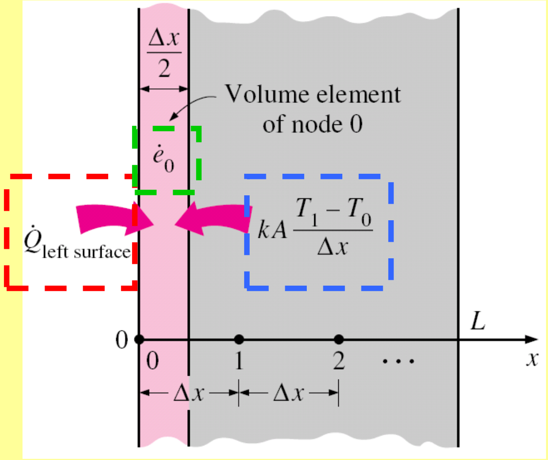
- Node number
  - at the left surface ($x=0$): is 0,
  - at the right surface at ($x=L$): is M
- The width of the volume element: $\Delta x/2$.



1. *Specified temperature boundary conditions*

- $T(0)=T_0$=Specified value
- $T(L)=T_M$=Specified value
- No need to write an energy balance unless the rate of heat transfer into or out of the medium is to be determined.

- An *energy balance* on the volume element at that boundary:



$$\sum_{All\ sides} \dot{Q} + \dot{E}_{gen,element} = 0 \quad \textbf{(5-20)}$$

- The finite difference formulation at the node $m=0$ can be expressed as:

$$\boxed{\dot{Q}_{left\ surface}} + \boxed{kA\frac{T_1 - T_0}{\Delta x}} + \boxed{\dot{e}_0\left(A\Delta x/2\right)} = 0 \qquad \textbf{(5-21)}$$

- The finite difference form of various boundary conditions can be obtained by replacing $\dot{Q}_{left\ surface}$ by a suitable expression.

2. Specified Heat Flux Boundary Condition

$$\boxed{\dot{q}_0 A} + kA\frac{T_1 - T_0}{\Delta x} + \dot{e}_0\left(A\Delta x/2\right) = 0 \qquad \textbf{(5-22)}$$

3. Convection Boundary Condition

$$\boxed{hA\left(T_\infty - T_0\right)} + kA\frac{T_1 - T_0}{\Delta x} + \dot{e}_0\left(A\Delta x/2\right) = 0 \qquad \textbf{(5-24)}$$

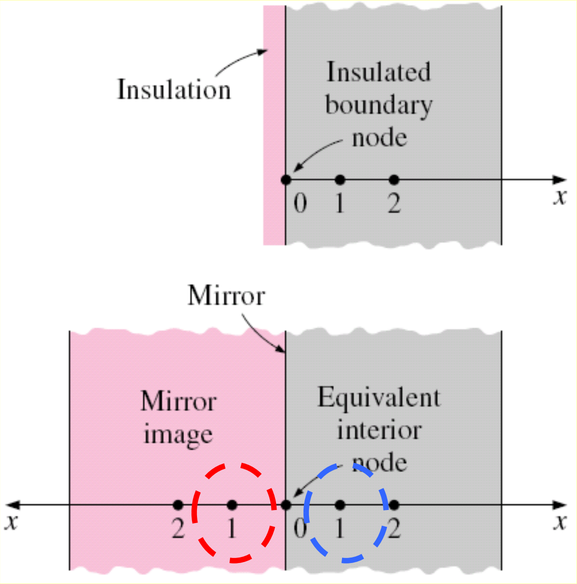
4. Radiation Boundary Condition

$$\boxed{\varepsilon\sigma A\left(T_{surr}^4 - T_0^4\right)} + kA\frac{T_1 - T_0}{\Delta x} + \dot{e}_0\left(A\Delta x/2\right) = 0 \qquad \textbf{(5-25)}$$

5. Combined Convection and Radiation

$$\boxed{hA(T_\infty - T_0) + \varepsilon\sigma A\left(T_{surr}^4 - T_0^4\right)} + kA\frac{T_1 - T_0}{\Delta x} + \dot{e}_0\left(A\Delta x/2\right) = 0 \qquad \textbf{(5-26)}$$

The Mirror Image Concept



- The finite difference formulation of a node on an insulated boundary can be treated as "zero" heat flux is Eq. 5–23.
- Another and more practical way is to treat the node on an insulated boundary as an interior node.
- By replacing the insulation on the boundary by a *mirror* and considering the reflection of the medium as its extension
- Using Eq. 5.11:

$$\frac{T_{m-1} - 2T_m + T_{m+1}}{\Delta x^2} + \frac{\dot{e}_m}{k} = 0$$

$$\rightarrow \frac{T_1 - 2T_0 + T_1}{\Delta x^2} + \frac{\dot{e}_m}{k} = 0 \quad \textbf{(5-30)}$$

Finite Differences Solution

- Usually a system of N algebraic equations in N unknown nodal temperatures that need to be solved simultaneously.
- There are numerous systematic approaches available which are broadly classified as
  - **direct** methods
    - Solve in a systematic manner following a series of well-defined steps
  - **iterative** methods
    - Start with an initial guess for the solution,
    - and iterate until solution converges
- The direct methods usually require a large amount of computer memory and computation time.
- The computer memory requirements for iterative methods are minimal.
- However, the convergence of iterative methods to the desired solution, however, may pose a problem.

Problem 1: **5–24** Consider a large uranium plate of thickness 5 cm and thermal conductivity $k = 28$ W/m · °C in which heat is generated uniformly at a constant rate of $\dot{e} = 6 \times 10^5$ W/m^3. One side of the plate is insulated while the other side is subjected to convection to an environment at 30°C with a heat transfer coefficient of $h = 60$ W/m^2 · °C. Considering six equally spaced nodes with a nodal spacing of 1 cm, (*a*) obtain the finite difference formulation of this problem and (*b*) determine the nodal temperatures under steady conditions by solving those equations.

Analysis The number of nodes is specified to be $M = 6$. Then the nodal spacing Δx becomes

$$\Delta x = \frac{L}{M-1} = \frac{0.05 \text{ m}}{6\text{-}1} = 0.01 \text{ m}$$

This problem involves 6 unknown nodal temperatures, and thus we need to have 6 equations to determine them uniquely. Node 0 is on insulated boundary, and thus we can treat it as an interior note by using the mirror image concept. Nodes 1, 2, 3, and 4 are interior nodes, and thus for them we can use the general finite difference relation expressed as

$$\frac{T_{m-1} - 2T_m + T_{m+1}}{\Delta x^2} + \frac{\dot{e}_m}{k} = 0 , \quad \text{for } m = 0, 1, 2, 3, \text{ and } 4$$

Finally, the finite difference equation for node 5 on the right surface subjected to convection is obtained by applying an energy balance on the half volume element about node 5 and taking the direction of all heat transfers to be towards the node under consideration:

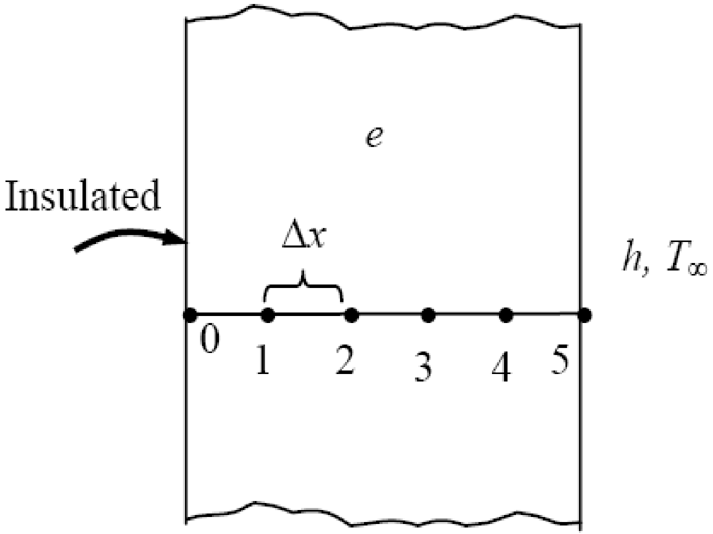
Node 0 (Left surface - insulated) : $\frac{T_1 - 2T_0 + T_1}{\Delta x^2} + \frac{\dot{e}}{k} = 0$

Node 1 (interior) : $\frac{T_0 - 2T_1 + T_2}{\Delta x^2} + \frac{\dot{e}}{k} = 0$

Node 2 (interior) : $\frac{T_1 - 2T_2 + T_3}{\Delta x^2} + \frac{\dot{e}}{k} = 0$

Node 3 (interior) : $\frac{T_2 - 2T_3 + T_4}{\Delta x^2} + \frac{\dot{e}}{k} = 0$

Node 4 (interior) : $\frac{T_3 - 2T_4 + T_5}{\Delta x^2} + \frac{\dot{e}}{k} = 0$



Node 5 (right surface - convection) : $h(T_\infty - T_5) + k\frac{T_4 - T_5}{\Delta x} + \dot{e}(\Delta x/2) = 0$

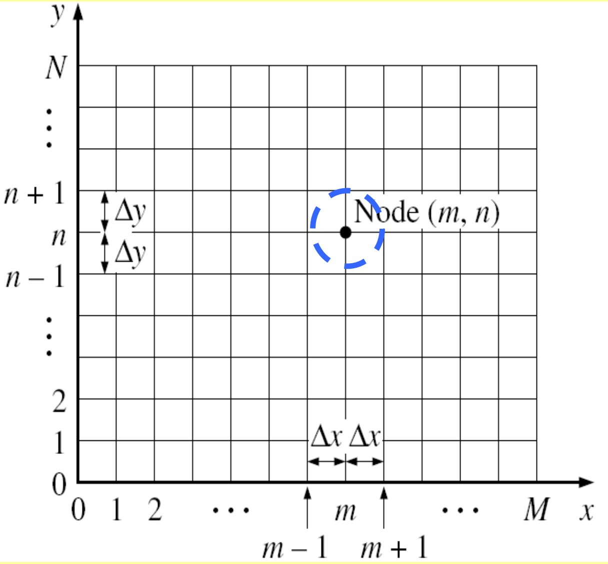
where $\Delta x = 0.01\,\text{m}$, $\dot{e} = 6\times10^5\ \text{W/m}^3$, $k = 28\ \text{W/m}\cdot°\text{C}$, $h = 60\ \text{W/m}^2\cdot°\text{C}$, and $T_\infty = 30°\text{C}$. This system of 6 equations with six unknown temperatures constitute the finite difference formulation of the problem.

(*b*) The 6 nodal temperatures under steady conditions are determined by solving the 6 equations above simultaneously with an equation solver to be

T_0 = **556.8°C**, T_1 = **555.7°C**, T_2 = **552.5°C**, T_3 = **547.1°C**, T_4 = **539.6°C**, and T_5 = **530.0°C**

Two-Dimensional Steady Heat Conduction

- The x-y plane of the region is divided into a rectangular mesh of nodal points spaced x and y.
- Numbering scheme: *double subscript notation* (m, n) where $m=0, 1, 2, \ldots, M$ is the node count in the x-direction and $n=0, 1, 2, \ldots, N$ is the node count in the y-direction.
- The coordinates of the node (m, n) are simply $x=mx$ and $y=ny$, and the temperature at the node (m, n) is denoted by $T_{m,n}$.
- A total of $(M+1)(N+1)$ nodes.

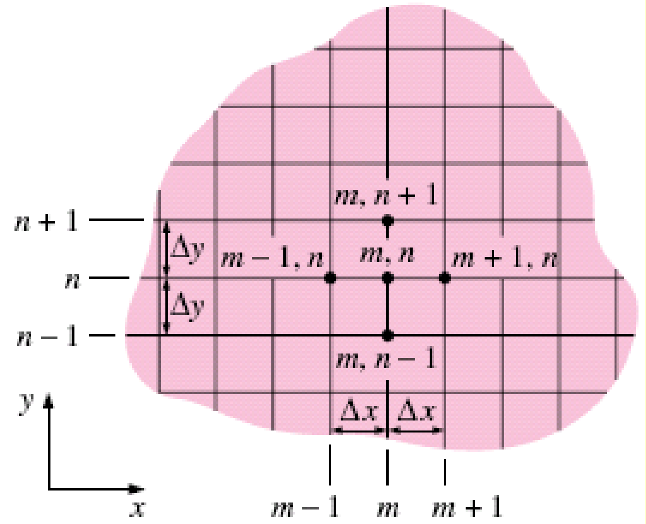


- The finite difference formulation given by Eq. 5-9 can easily be extended to two- or three-dimensional heat transfer problems by replacing each second derivative by a difference equation in that direction.
- For **steady two-dimensional heat conduction** with heat generation and constant thermal conductivity

$$\frac{T_{m+1,n} - 2T_{m,n} + T_{m-1,n}}{\Delta x^2} + \frac{T_{m,n+1} - 2T_{m,n} + T_{m,n-1}}{\Delta y^2}$$

$$+\frac{\dot{e}_{m,n}}{k} = 0 \qquad \textbf{(5-33)}$$

for $m=1, 2, 3, \ldots, M-1$ and $n=1, 2, 3, \ldots, N-1$.

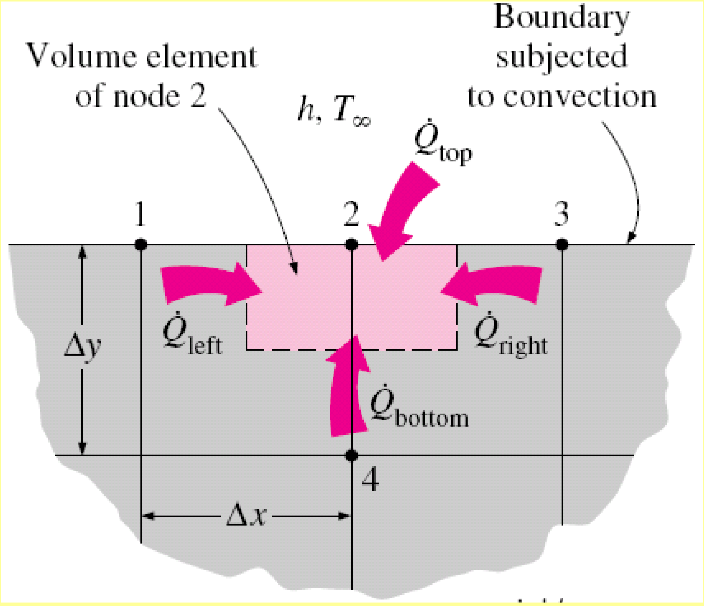


- For $\Delta x=\Delta y=l$, Eq. 5-33 reduces to

$$T_{m-1,n} + T_{m+1,n} + T_{m,n+1} + T_{m,n-1} - 4T_{m,n} + \frac{\dot{e}_{m,n} l^2}{k} = 0 \qquad \textbf{(5-34)}$$

Boundary Nodes

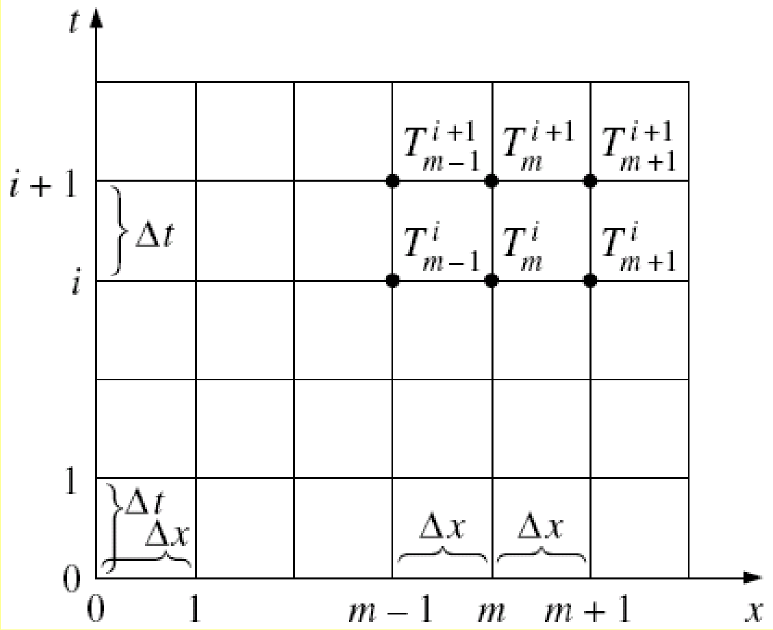
- The development of finite difference formulation of *boundary* nodes in two- (or three-) dimensional problems is similar to the development in the one-dimensional case discussed earlier.
- For heat transfer under *steady* conditions, the basic equation to keep in mind when writing an *energy balance* on a volume element is



$$\sum_{All\ sides} \dot{Q} + \dot{e}V_{element} = 0 \qquad \textbf{(5-36)}$$

Transient Heat Conduction

- The finite difference solution of transient problems requires *discretization in time* in addition to discretization in space.
- the unknown nodal temperatures are calculated repeatedly for each Δt until the solution at the desired time is obtained.
- In transient problems, the *superscript* ***i*** is used as the *index* or *counter* of time steps.
- $i=0$ corresponding to the specified initial condition.



- The energy balance on a volume element during a time interval Δt can be expressed as

$$\left(\begin{array}{c}\textbf{Heat transferred into}\\ \textbf{the volume element}\\ \textbf{from all of its surfaces}\\ \textbf{during } \boldsymbol{\Delta t}\end{array}\right) + \left(\begin{array}{c}\textbf{Heat generated}\\ \textbf{within the}\\ \textbf{volume element}\\ \textbf{during } \boldsymbol{\Delta t}\end{array}\right) = \left(\begin{array}{c}\textbf{The change in the}\\ \textbf{energy content of}\\ \textbf{the volume element}\\ \textbf{during } \boldsymbol{\Delta t}\end{array}\right)$$

- or

$$\Delta t \times \sum_{All\ sides} \dot{Q} + \Delta t \times \dot{E}_{gen,element} = \Delta E_{element} \qquad \textbf{(5-37)}$$

- Noting that $\Delta E_{\text{element}} = mc_p\Delta T = \rho V_{\text{element}} c_p \Delta T$, and dividing the earlier relation by Δt gives

$$\sum_{All\ sides} \dot{Q} + \dot{E}_{gen,element} = \frac{\Delta E_{element}}{\Delta t} = \rho V_{element} c_p \frac{\Delta T}{\Delta t} \qquad \textbf{(5-38)}$$

- or, for any node m in the medium and its volume element,

$$\sum_{All\ sides} \dot{Q} + \dot{E}_{gen,element} = \rho V_{element} c_p \frac{T_m^{i+1} - T_m^{i}}{\Delta t} \qquad \textbf{(5-39)}$$

Explicit and Implicit Method

- **Explicit method** — the known temperatures at the *previous* time step *i* is used for the terms on the left side of Eq. 5–39.

$$\sum_{All\ sides} \dot{Q}^{i} + \dot{E}^{i}_{gen,element} = \rho V_{element} c_p \frac{T_m^{i+1} - T_m^{i}}{\Delta t} \qquad \textbf{(5-40)}$$

- **Implicit method** — the *new* time step *i+1* is used for the terms on the left side of Eq. 5–39.

$$\sum_{All\ sides} \dot{Q}^{i+1} + \dot{E}^{i+1}_{gen,element} = \rho V_{element} c_p \frac{T_m^{i+1} - T_m^{i}}{\Delta t} \qquad \textbf{(5-41)}$$

Remarks:

- The *explicit method* is easy to implement but imposes a limit on the allowable time step to avoid instabilities in the solution.
- The *implicit method* requires the nodal temperatures to be solved simultaneously for each time step but imposes no limit on the magnitude of the time step.

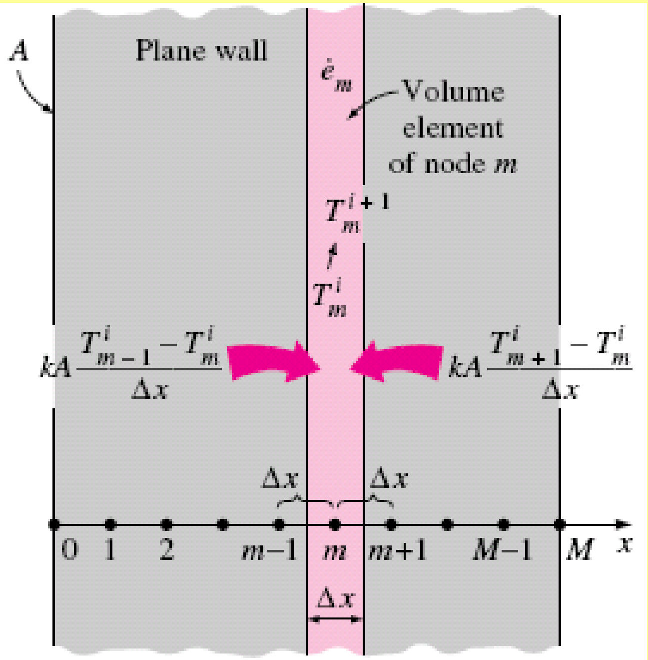
Transient Heat Conduction in a Plane Wall

- From Eq. 5–39 the interior node can be expressed on the basis of

$$kA\frac{T_{m-1}-T_m}{\Delta x}+kA\frac{T_{m+1}-T_m}{\Delta x}+\dot{e}_m A\Delta x=\rho A\Delta x c_p\frac{T_m^{i+1}-T_m^i}{\Delta t} \qquad \textbf{(5-42)}$$

- Canceling the surface area A and multiplying by $\Delta x/k$

$$T_{m-1}-2T_m+T_{m+1}+\frac{\dot{e}_m\Delta x^2}{k}=\frac{\Delta x^2}{\alpha\Delta t}\left(T_m^{i+1}-T_m^i\right) \qquad \textbf{(5-43)}$$



- Defining a dimensionless **mesh** **Fourier number** as

$$\tau = \frac{\alpha \Delta t}{\Delta x^2} \qquad \textbf{(5-44)}$$

- Eq. 5–43 reduces to

$$T_{m-1} - 2T_m + T_{m+1} + \frac{\dot{e}_m \Delta x^2}{k} = \frac{\left(T_m^{i+1} - T_m^i\right)}{\tau} \qquad \textbf{(5-45)}$$

- The *explicit* finite difference formulation

$$T_{m-1}^i - 2T_m^i + T_{m+1}^i + \frac{\dot{e}_m^i \Delta x^2}{k} = \frac{\left(T_m^{i+1} - T_m^i\right)}{\tau} \qquad \textbf{(5-46)}$$

- This equation can be solved *explicitly* for the new temperature (and thus the name *explicit* method)

$$T_m^{i+1} = \tau\left(T_{m-1}^i + T_{m+1}^i\right) + \left(1 - 2\tau\right)T_m^i + \tau \frac{\dot{e}_m^i \Delta x^2}{k} \qquad \textbf{(5-47)}$$

- The *implicit* finite difference formulation

$$T_{m-1}^{i+1} - 2T_{m}^{i+1} + T_{m+1}^{i+1} + \frac{\dot{e}_m^{i+1}\Delta x^2}{k} = \frac{\left(T_m^{i+1} - T_m^i\right)}{\tau} \qquad \textbf{(5-48)}$$

- which can be rearranged as

$$\tau T_{m-1}^{i+1} - \left(1 + 2\tau\right) T_{m}^{i+1} + \tau T_{m+1}^{i+1} + \tau \frac{\dot{e}_m^{i+1}\Delta x^2}{k} + T_m^i = 0 \qquad \textbf{(5-49)}$$

- The application of either the explicit or the implicit formulation to each of the *M*-1 interior nodes gives *M*-1 equations.
- The remaining two equations are obtained by applying the same method to the two boundary nodes unless, of course, the boundary temperatures are specified as constants (invariant with time).

Stability Criterion for Explicit Method

- The explicit method is easy to use, but it suffers from an undesirable feature: it is not unconditionally stable.
- The value of Δt must be maintained below a certain upper limit.
- It can be shown mathematically or by a physical argument based on the second law of thermodynamics that *the stability criterion is satisfied if the coefficients of all in the expressions* (*called the* **primary coefficients**) *are greater than or equal to zero for all* nodes m.
- All the terms involving for a particular node must be grouped together before this criterion is applied.

Explicit formulation:

$$T_0^{i+1} = a_0 T_0^i + \cdots$$

$$T_1^{i+1} = a_1 T_1^i + \cdots$$

$$\vdots$$

$$T_m^{i+1} = a_m T_m^i + \cdots$$

$$\vdots$$

$$T_M^{i+1} = a_M T_M^i + \cdots$$

Stability criterion:

$$a_m \geq 0, \quad m = 0, 1, 2, \ldots m, \ldots M$$

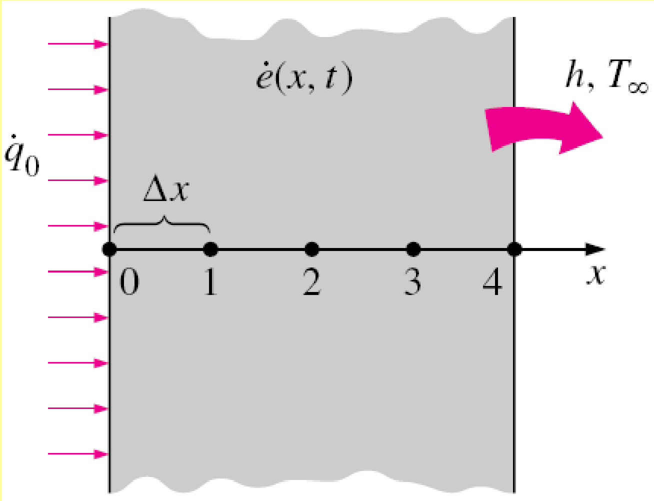
- Different equations for different nodes may result in different restrictions on the size of the time step Δt, and the criterion that is most restrictive should be used in the solution of the problem.
- In the case of transient one-dimensional heat conduction in a plane wall with specified surface temperatures, the explicit finite difference equations for all the nodes are obtained from Eq. 5–47. The coefficient of in the expression is $1-2\tau$.
- The stability criterion for all nodes in this case is $1-2\tau=0$ or

$$\tau = \frac{\alpha \Delta t}{\Delta x^2} \le \frac{1}{2} \quad \left(\begin{array}{c}\textbf{interior nodes, one-dimensional heat} \\ \textbf{transfer in rectangular coordinates}\end{array}\right) \qquad \textbf{(5-52)}$$

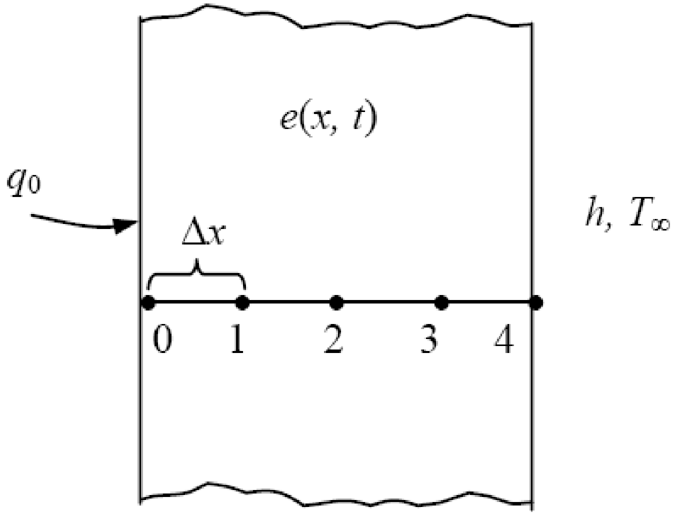
- The implicit method is *unconditionally stable*, and thus we can use any time step we please with that method

Problem 2:

5–76 Consider transient heat conduction in a plane wall with variable heat generation and constant thermal conductivity. The nodal network of the medium consists of nodes 0, 1, 2, 3, and 4 with a uniform nodal spacing of Δx. The wall is initially at a specified temperature. Using the energy balance approach, obtain the explicit finite difference formulation of the boundary nodes for the case of uniform heat flux $\dot{q}_0$ at the left boundary (node 0) and convection at the right boundary (node 4) with a convection coefficient of h and an ambient temperature of T_∞. Do not simplify.



Analysis Using the energy balance approach and taking the direction of all heat transfers to be towards the node under consideration, the *explicit* finite difference formulations become



Left boundary node:

$$kA\frac{T_1^i - T_0^i}{\Delta x} + \dot{q}_0 A + \dot{e}_0^i (A\Delta x/2) = \rho A \frac{\Delta x}{2} c_p \frac{T_0^{i+1} - T_0^i}{\Delta t}$$

Right boundary node:

$$kA\frac{T_3^i - T_4^i}{\Delta x} + hA(T_\infty^i - T_4^i) + \dot{e}_4^i (A\Delta x/2) = \rho A \frac{\Delta x}{2} c_p \frac{T_4^{i+1} - T_4^i}{\Delta t}$$

Two-Dimensional Transient Heat Conduction

- Heat may be generated in the medium at a rate $\dot{e}(x, y, t)$ which may vary with time and position.
- The thermal conductivity k of the medium is assumed to be constant.
- The transient finite difference formulation for a general interior node can be expressed on the basis of Eq. 5–39 as

$$k\Delta y \frac{T_{m-1,n} - T_{m,n}}{\Delta x} + k\Delta x \frac{T_{m,n+1} - T_{m,n}}{\Delta y} + k\Delta y \frac{T_{m+1,n} - T_{m,n}}{\Delta x}$$

$$+k\Delta x \frac{T_{m,n-1} - T_{m,n}}{\Delta y} + \dot{e}_{m,n}\Delta x \Delta y = \rho \Delta x \Delta y c_p \frac{T_{m,n}^{i+1} - T_{m,n}^{i}}{\Delta t} \qquad \textbf{(5-56)}$$

- Taking a square mesh ($\Delta x=\Delta y=l$) and dividing each term by k gives after simplifying

$$T_{m-1,n}+T_{m+1,n}+T_{m,n+1}+T_{m,n-1}-4T_{m,n}+\frac{\dot{e}_{m,n}l^2}{k}=\frac{T_{m,n}^{i+1}-T_{m,n}^{i}}{\tau} \quad \textbf{(5-57)}$$

- The *explicit* finite difference formulation

$$T^{i}_{m-1,n}+T^{i}_{m+1,n}+T^{i}_{m,n+1}+T^{i}_{m,n-1}-4T^{i}_{m,n}+\frac{\dot{e}^{i}_{m,n}l^2}{k}=\frac{T_{m,n}^{i+1}-T_{m,n}^{i}}{\tau} \quad \textbf{(5-59)}$$

- The *implicit* finite difference formulation

$$T_{m,n}^{i+1}=\tau\left(T^{i}_{m-1,n}+T^{i}_{m+1,n}+T^{i}_{m,n+1}+T^{i}_{m,n-1}\right)+\left(1-4\tau\right)T^{i}_{m,n}+\tau\frac{\dot{e}^{i}_{m,n}l^2}{k} \quad \textbf{(5-60)}$$

- The stability criterion that requires the coefficient of in the expression to be greater than or equal to zero for all nodes is equally valid for two or three-dimensional cases.
- In the case of transient two-dimensional heat transfer in rectangular coordinates, the coefficient of $T_m^{\,i}$ in the $T_m^{\,i+1}$ expression is $1\text{-}4\tau$.
- Thus the stability criterion for all interior nodes in this case is $1\text{-}4\tau>0$, or

$$\tau = \frac{\alpha \Delta t}{l^2} \le \frac{1}{4} \qquad \left(\begin{array}{c}\textbf{interior nodes, two-dimensional heat} \\ \textbf{transfer in rectangular coordinates}\end{array}\right) \qquad \textbf{(5-61)}$$

- The application of Eq. 5–60 to each of the (M-1)X(N-1) interior nodes gives (M-1)X(N-1) equations.
- The remaining equations are obtained by applying the method to the boundary nodes (unless the boundary temperatures are specified as being constant).

Conclusions

- Importance of numerical methods
- Finite difference formulation of differential equations
- One-dimensional steady heat conduction
- Two-dimensional steady heat conduction
- Transient heat conduction

HEAT AND MASS TRANSFER

Fundamentals of Convection

OUTLINE

- Physical Mechanism of Convection
- Nusselt Number
- Classification of Fluid Flows
- Velocity Boundary Layer
- Surface Shear Stress
- Thermal Boundary Layer
- Prandtl Number
- Laminar and Turbulent Flows
- Reynolds Number
- Solutions of Convection Equations
- Conclusions

PHYSICAL MECHANISM OF CONVECTION

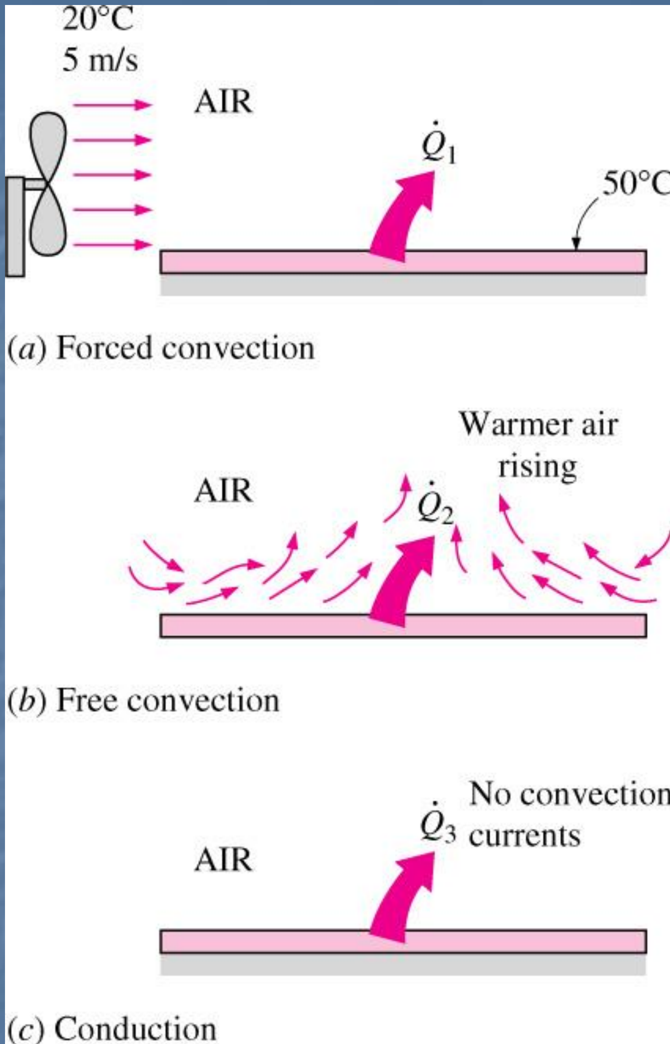


FIGURE 6–1
Heat transfer from a hot surface to the surrounding fluid by convection and conduction.

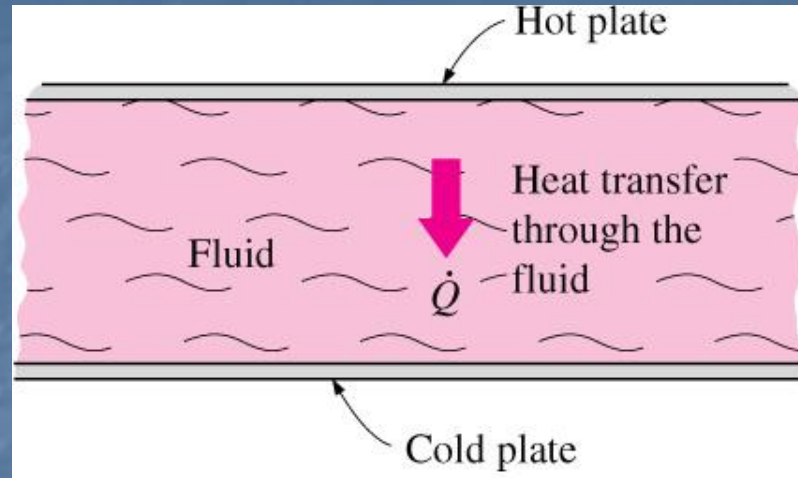


FIGURE 6–2
Heat transfer through a fluid sandwiched between two parallel plates.

The rate of convection heat transfer is observed to be proportional to the temperature difference and is conveniently expressed by **Newton's law of cooling** as

$$\dot{q}_{\text{conv}} = h(T_s - T_\infty) \qquad (\text{W/m}^2)$$

or

$$\dot{Q}_{\text{conv}} = hA_s(T_s - T_\infty) \qquad (\text{W})$$

The convection heat transfer strongly depends on the fluid properties *dynamic viscosity μ, thermal conductivity k, density ρ*, and *specific heat Cp*, as well as the *fluid velocity V*.

$$\dot{q}_{\text{conv}} = \dot{q}_{\text{cond}} = -k_{\text{fluid}} \left.\frac{\partial T}{\partial y}\right|_{y=0}$$

$(\partial T/\partial y)_{y=0}$: *the temperature gradient at the surface*

$$h = \frac{-k_{\text{fluid}}(\partial T/\partial y)_{y=0}}{T_s - T_\infty} \qquad (\text{W/m}^2 \cdot {}^\circ\text{C})$$

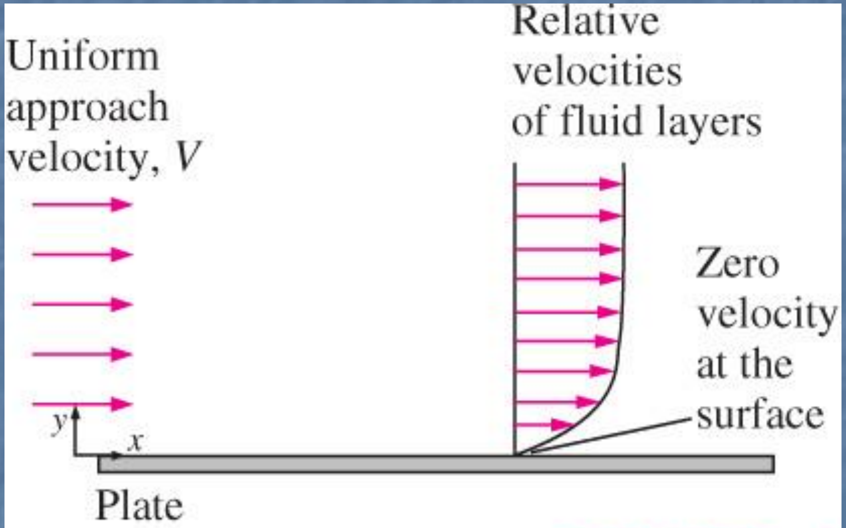


FIGURE 6–4

A fluid flowing over a stationary surface comes to a complete stop at the surface because of the no-slip condition.

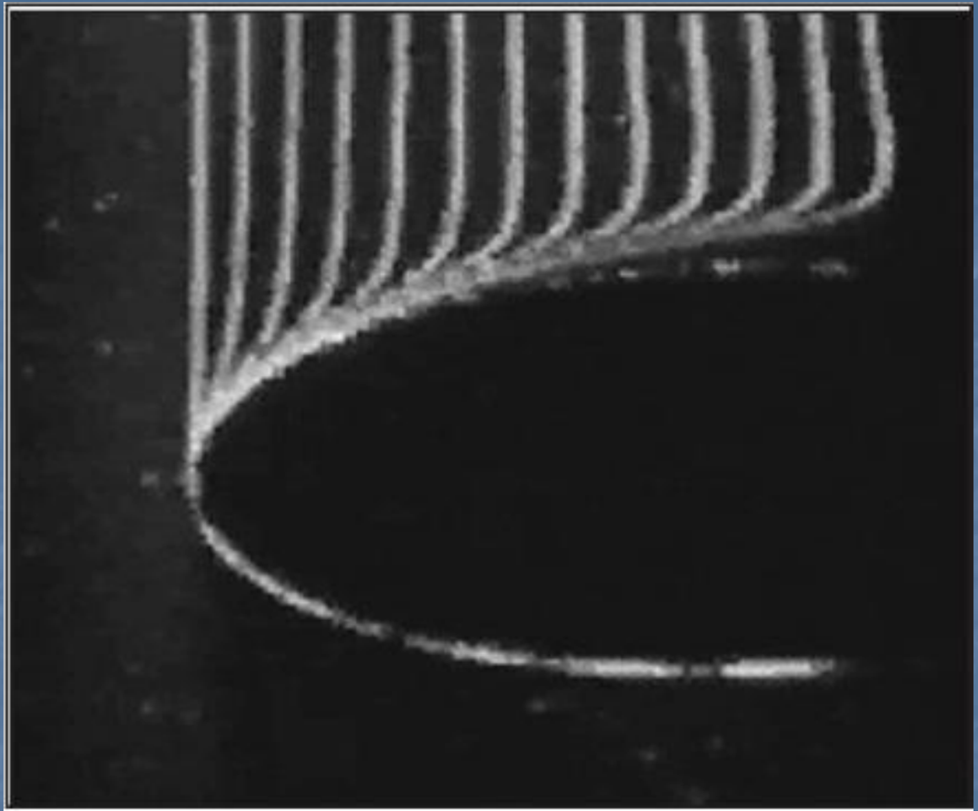


FIGURE 6–3

The development of a velocity profile due to the no-slip condition as a fluid flows over a blunt nose.

"Hunter Rouse: Laminar and Turbulent Flow Film." Copyright IIHR-Hydroscience & Engineering, The University of Iowa. Used by permission.

Nusselt Number (Nu)

The Nusselt number *(Nu)*:

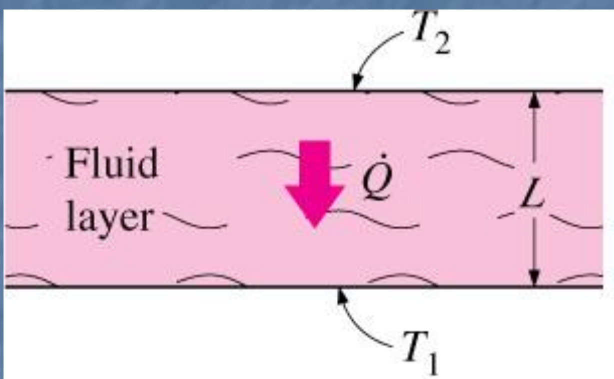
$$\mathrm{Nu} = \frac{hL_c}{k}$$

k : the thermal conductivity of the fluid
L_c : *the characteristic length.*

Heat flux (the rate of heat transfer per unit time per unit surface):

$$\dot{q}_{\text{conv}} = h\Delta T \quad \text{and} \quad \dot{q}_{\text{cond}} = k\frac{\Delta T}{L}$$

$$\frac{\dot{q}_{\text{conv}}}{\dot{q}_{\text{cond}}} = \frac{h\Delta T}{k\Delta T/L} = \frac{hL}{k} = \mathrm{Nu}$$



$$\Delta T = T_2 - T_1$$

FIGURE 6–5
Heat transfer through a fluid layer of thickness L and temperature difference ΔT.

A Nusselt number of Nu = 1 for a fluid layer represents heat transfer across the layer by pure conduction.

Classification of Fluid Flows

- Viscous versus inviscid regions of flow
- Internal versus external flow
- Compressible versus incompressible flow
- Laminar versus turbulent flow
- Natural (or unforced) versus forced flow
- Steady versus unsteady flow
- One-, two-, and three-dimensional flows

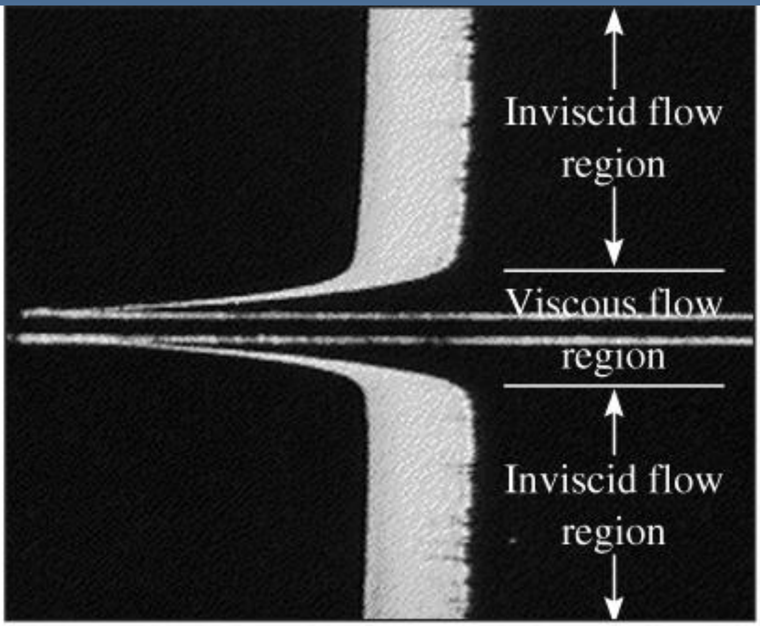


FIGURE 6–7

The flow of an originally uniform fluid stream over a flat plate, and the regions of viscous flow (next to the plate on both sides) and inviscid flow (away from the plate).

Fundamentals of Boundary Layers, National Committee from Fluid Mechanics Films, © Education Development Center.

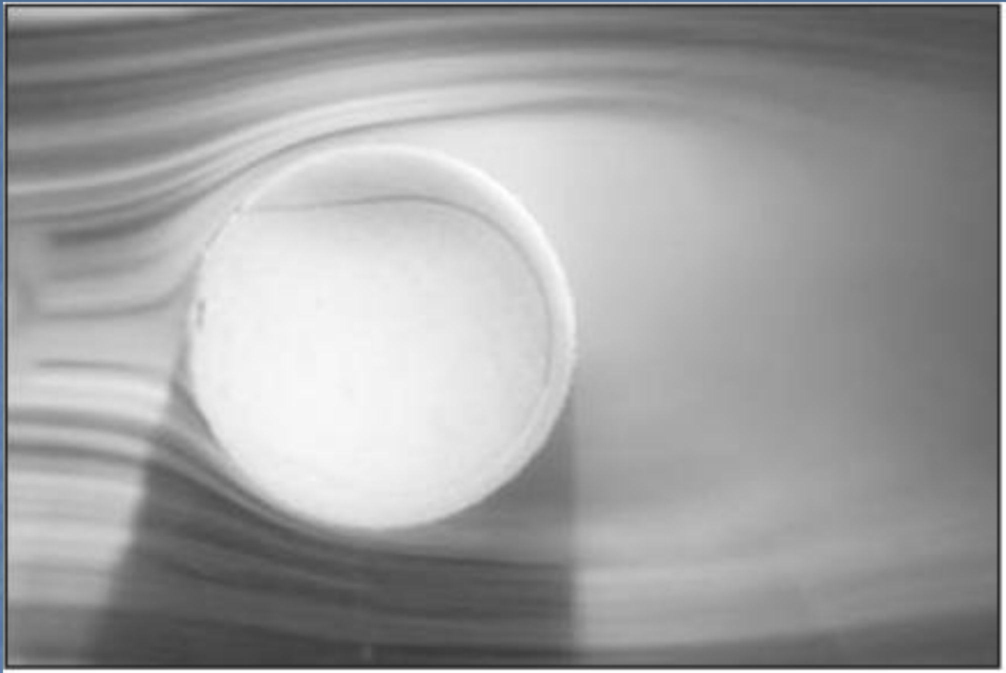


FIGURE 6–8

External flow over a tennis ball, and the turbulent wake region behind.

Courtesy NASA and Cislunar Aerospace, Inc.

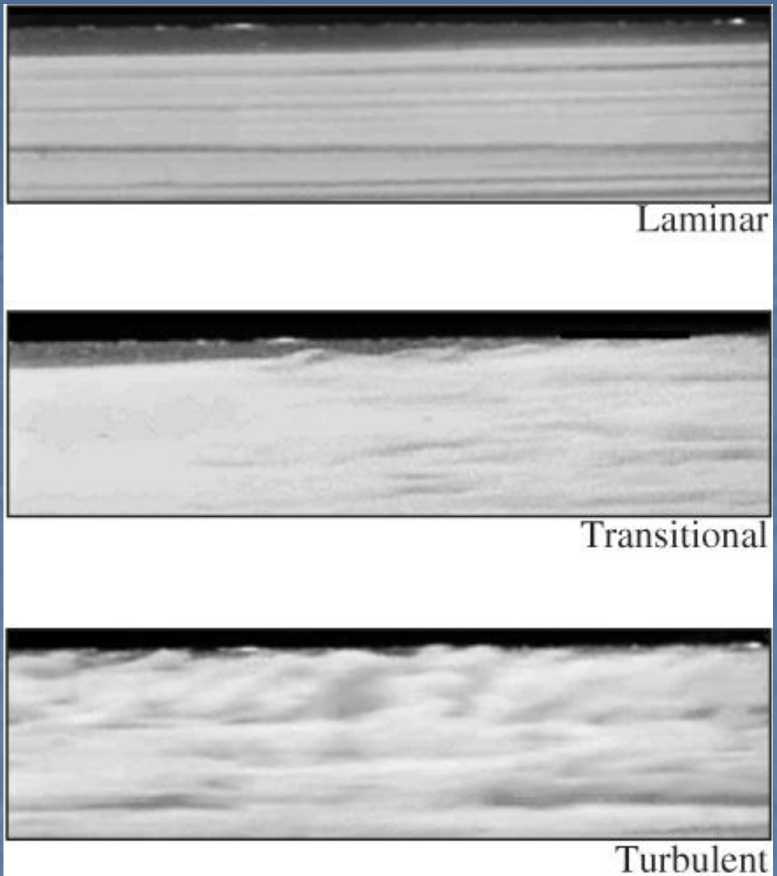


FIGURE 6–9
Laminar, transitional, and turbulent flows.
Courtesy ONERA, photograph by Werlé.

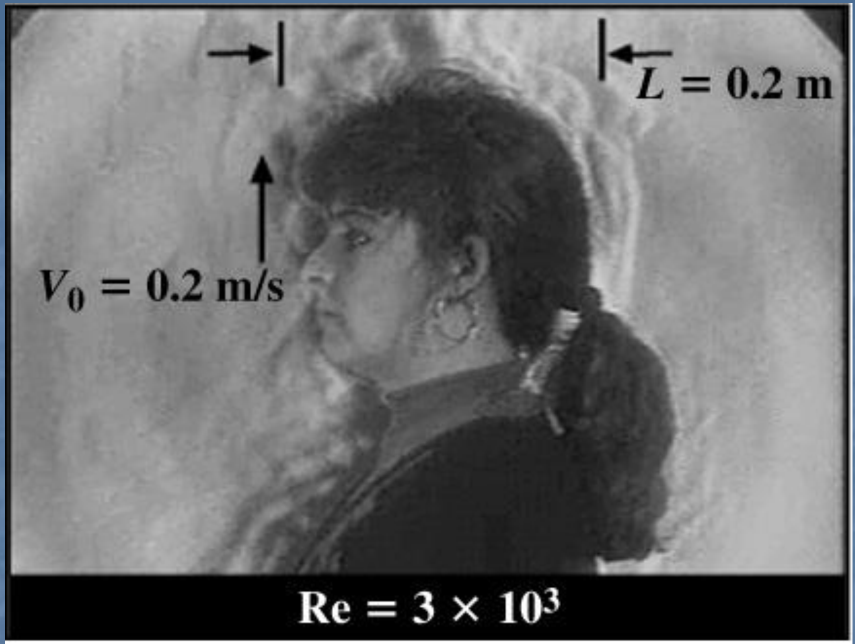
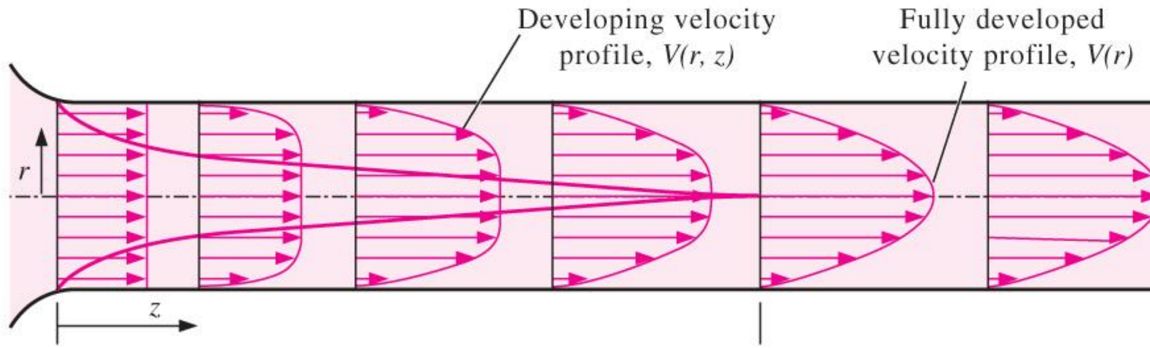


FIGURE 6–10
In this schlieren image of a girl, the rise of lighter, warmer air adjacent to her body indicates that humans and warm-blooded animals are surrounded by thermal plumes of rising warm air.
G. S. Settles, Gas Dynamics Lab, Penn State University. Used by permission.

Velocity Boundary Layer

FIGURE 6–11
The development of the velocity profile in a circular pipe. $V = V(r, z)$ and thus the flow is two-dimensional in the entrance region, and becomes one-dimensional downstream when the velocity profile fully develops and remains unchanged in the flow direction, $V = V(r)$.



The region of the flow above the plate bounded by δ is called the **velocity boundary layer**.

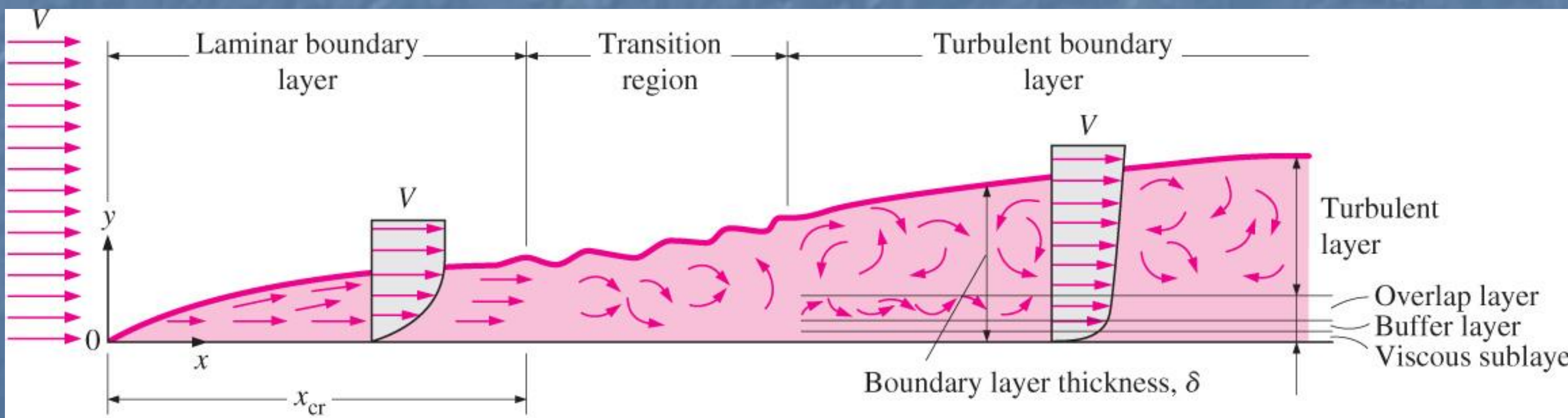


FIGURE 6–12
The development of the boundary layer for flow over a flat plate, and the different flow regimes.

Surface Shear Stress

- Friction force per unit area is called **shear stress**, and is denoted by τ.
- The shear stress for most fluids is proportional to the *velocity gradient.*
- The shear stress at the wall surface for these fluids is expressed as

$$\tau_s = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0} \qquad (\text{N/m}^2)$$

- The fluids that that obey the linear relationship above are called **Newtonian fluids.**
- The viscosity of a fluid is a measure of its *resistance to deformation.*
- The viscosities of liquids *decrease* with temperature, whereas the viscosities of gases *increase* with temperature.
- Surface shear stress and friction force:

$$\tau_s = C_f \frac{\rho V^2}{2} \qquad (\text{N/m}^2)$$

$$F_f = C_f A_s \frac{\rho V^2}{2} \qquad (\text{N})$$

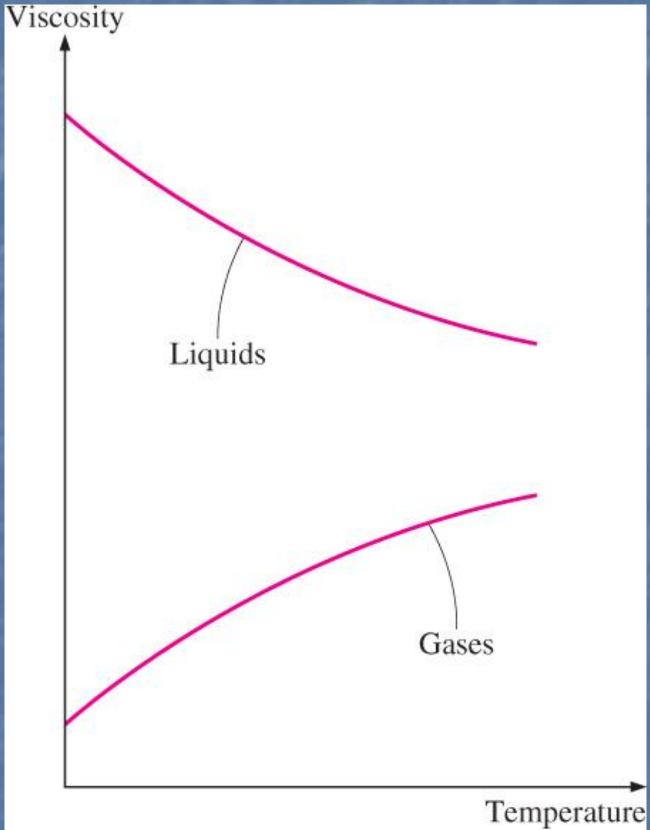


FIGURE 6–14

The viscosity of liquids decreases and the viscosity of gases increases with temperature.

THERMAL BOUNDARY LAYER

Thermal Boundary Layer: the flow region over the surface in which the temperature variation in the direction normal to the surface is significant.

The *thickness* of the thermal boundary layer δ_t at any location along the surface is defined as *the distance from the surface at which the temperature difference* $T - T_s$ *equals* $0.99\,(T - T_s)$.

The thickness of the thermal boundary layer increases in the flow direction, since the effects of heat transfer are felt at greater distances from the surface further downstream.

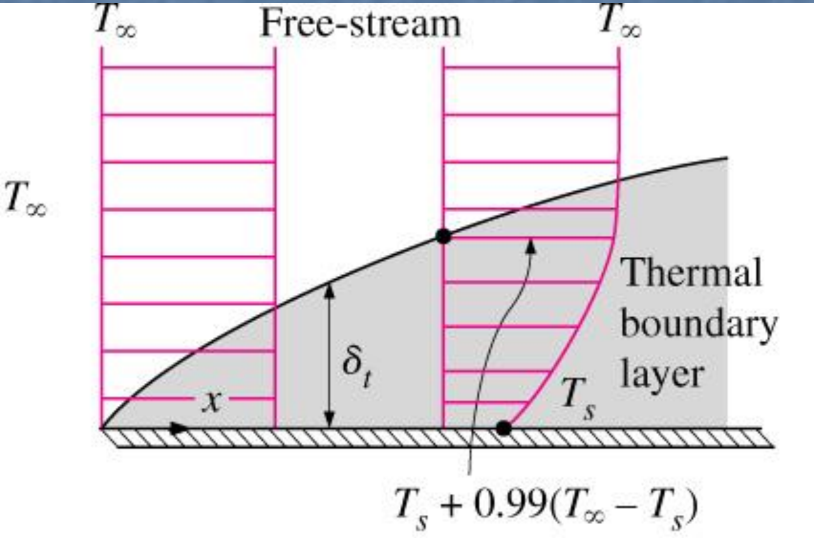


FIGURE 6–15
Thermal boundary layer on a flat plate (the fluid is hotter than the plate surface).

Prandtl Number (Pr)

The Prandtl numbers of fluids range from less than 0.01 for liquid metals to more than 100,000 for heavy oils (Table 6–2). The Prandtl number is in the order of 10 for water.

$$\mathrm{Pr} = \frac{\text{Molecular diffusivity of momentum}}{\text{Molecular diffusivity of heat}} = \frac{\nu}{\alpha} = \frac{\mu C_p}{k}$$

TABLE 6–2

Typical ranges of Prandtl numbers for common fluids

Fluid	Pr
Liquid metals	0.004–0.030
Gases	0.7–1.0
Water	1.7–13.7
Light organic fluids	5–50
Oils	50–100,000
Glycerin	2000–100,000

Laminar and Turbulent Flows

- **Laminar flow** — the flow is characterized by *smooth streamlines* and *highly-ordered motion.*
- **Turbulent flow** — the flow is characterized by *velocity fluctuations* and *highly-disordered motion.*
- The **transition** from **laminar** to **turbulent** flow does not occur suddenly.

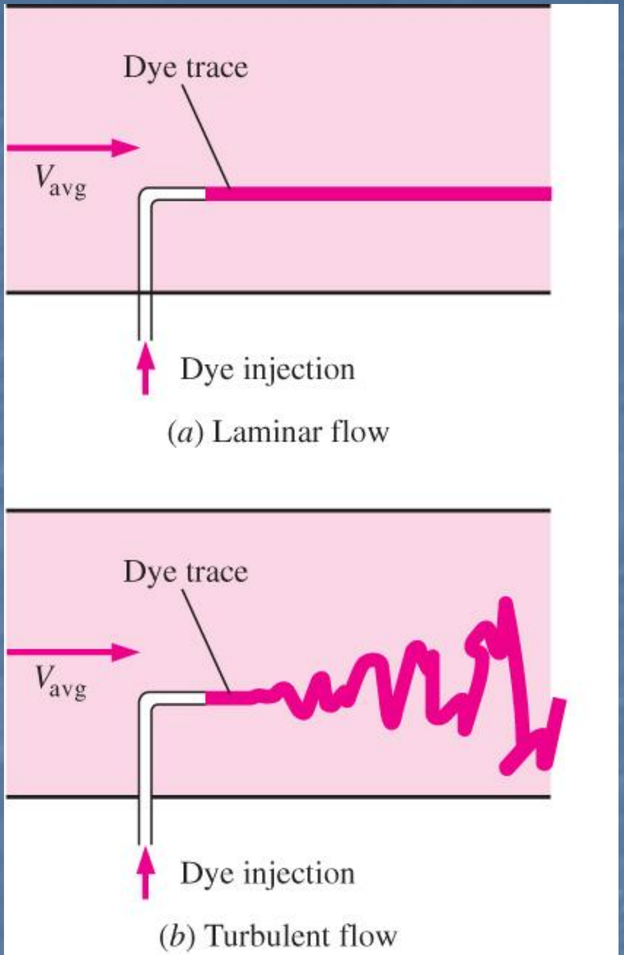


FIGURE 6–17
The behavior of colored fluid injected into the flow in laminar and turbulent flows in a pipe.

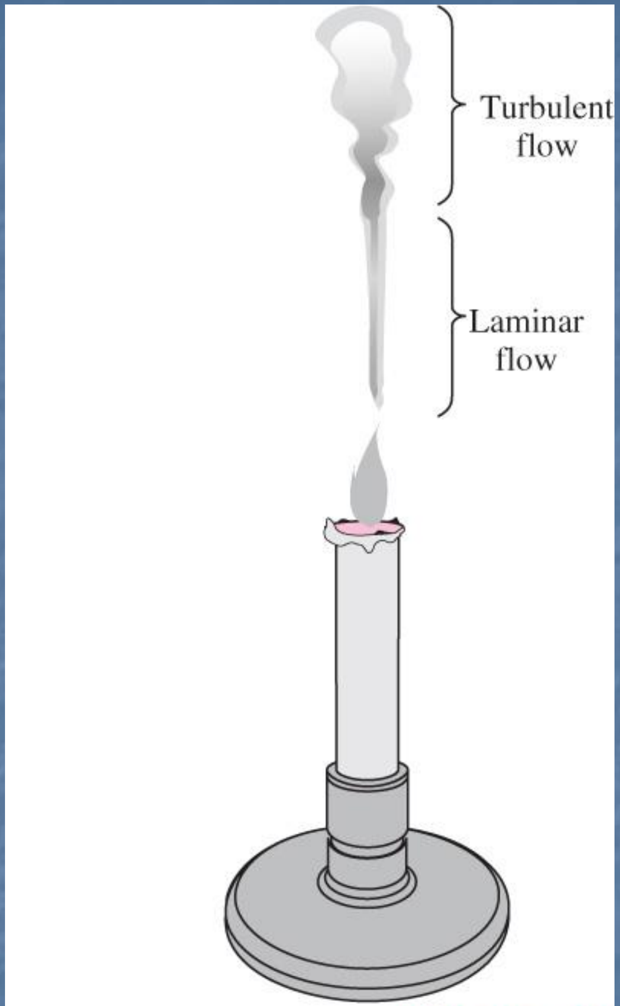
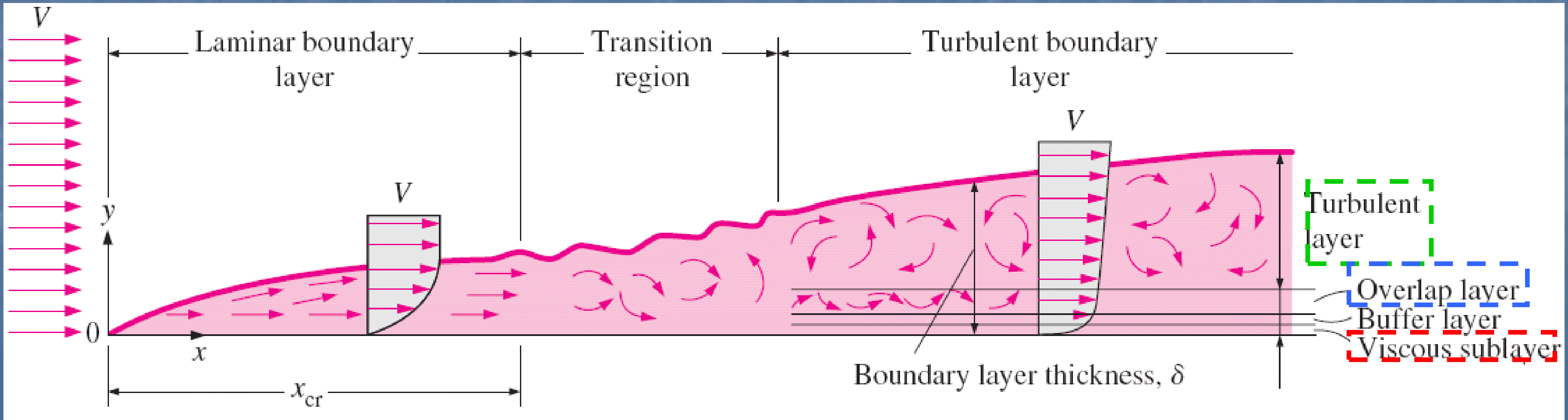


FIGURE 6–16
Laminar and turbulent flow regimes of candle smoke.

- The velocity profile in turbulent flow is much fuller than that in laminar flow, with a sharp drop near the surface.
- The turbulent boundary layer can be considered to consist of four regions:
  - Viscous sublayer
  - Buffer layer
  - Overlap layer
  - Turbulent layer
- The *intense mixing* in turbulent flow enhances heat and momentum transfer, which increases the friction force on the surface and the convection heat transfer rate.



Reynolds Number

- The transition from laminar to turbulent flow depends on the *surface geometry, surface roughness, flow velocity, surface temperature*, and *type of fluid.*
- The flow regime depends mainly on the ratio of the *inertia forces* to *viscous forces* in the fluid.
- This ratio is called the **Reynolds number**, which is expressed for external flow as

$$\text{Re} = \frac{\text{Inertia forces}}{\text{Viscous}} = \frac{VL_c}{\nu} = \frac{\rho VL_c}{\mu}$$

- At *large* Reynolds numbers (turbulent flow) the inertia forces are large relative to the viscous forces.
- At *small* or *moderate* Reynolds numbers (laminar flow), the viscous forces are large enough to suppress these fluctuations and to keep the fluid "inline."
- **Critical Reynolds number** — the Reynolds number at which the flow becomes turbulent.

Some important results from convection equations

The velocity boundary layer thickness $$\delta = \frac{4.91}{\sqrt{V/\nu x}} = \frac{4.91x}{\sqrt{\mathrm{Re}_x}}$$

The average local skin friction coefficient $$C_{f,x} = \frac{\tau_w}{\rho V^2/2} = 0.664\,\mathrm{Re}_x^{-1/2}$$

Local Nusselt number $$\mathrm{Nu}_x = \frac{h_x x}{k} = 0.332\,\mathrm{Pr}^{1/3}\mathrm{Re}_x^{1/2} \qquad \mathrm{Pr} > 0.6$$

The thermal boundary layer thickness $$\delta_t = \frac{\delta}{\mathrm{Pr}^{1/3}} = \frac{4.91x}{\mathrm{Pr}^{1/3}\sqrt{\mathrm{Re}_x}}$$

Reynold analogy $$C_{f,x}\frac{\mathrm{Re}_L}{2} = \mathrm{Nu}_x \qquad (\mathrm{Pr} = 1)$$

Modified Reynold analogy $$C_{f,x}\frac{\mathrm{Re}_L}{2} = \mathrm{Nu}_x\,\mathrm{Pr}^{\pm 1/3} \qquad \text{or} \qquad \frac{C_{f,x}}{2} = \frac{h_x}{\rho c_p V}\mathrm{Pr}^{2/3} \equiv j_H$$

CONCLUSIONS

- Physical Mechanism of Convection
- Nusselt Number
- Classification of Fluid Flows
- Velocity Boundary Layer
- Surface Shear Stress
- Thermal Boundary Layer
- Prandtl Number
- Laminar and Turbulent Flows
- Reynolds Number
- Solutions of Convection Equations

HEAT AND MASS TRANSFER

External Forced Convection

OBJECTIVES

- To distinguish between internal and external flow,
- To develop an intuitive understanding of friction drag and pressure drag, and evaluate the average drag and convection coefficients in external flow,
- To evaluate the drag and heat transfer associated with flow over a flat plate for both laminar and turbulent flow,
- To calculate the drag force exerted on cylinders during cross flow, and the average heat transfer coefficient, and
- To determine the pressure drop and the average heat transfer coefficient associated with flow across a tube bank for both in-line and staggered configurations.

Drag and Heat Transfer in External flow

- Fluid flow over solid bodies cause physical phenomena such as
  - *drag force*
    - automobiles
    - power lines
  - *lift force*
    - airplane wings
  - *cooling of metal or plastic sheets.*
- **Free-stream velocity** — the velocity of the fluid relative to an immersed solid body sufficiently far from the body.
- The fluid velocity ranges from zero at the surface (the no-slip condition) to the free-stream value away from the surface.

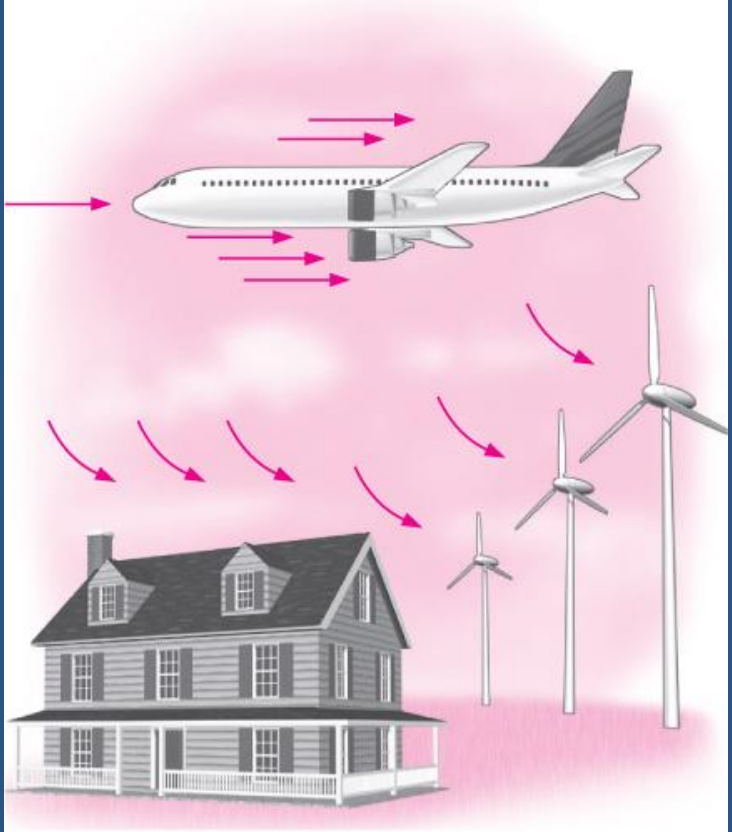


FIGURE 7–1
Flow over bodies is commonly encountered in practice.

Friction and Pressure Drag

- The force a flowing fluid exerts on a body in the flow direction is called **drag**.
- Drag is compose of:
  - pressure drag,
  - friction drag (skin friction drag).
- The drag force F_D depends on the
  - density ρ of the fluid,
  - the upstream velocity V, and
  - the size, shape, and orientation of the body.
- The dimensionless **drag coefficient** C_D is defined as

$$C_D = \frac{F_D}{\frac{1}{2}\rho V^2 A}$$

$$C_D = C_{D,\text{ friction}} + C_{D,\text{ pressure}}$$

For flat plate: $C_D = C_{D,\text{ friction}} = C_f$

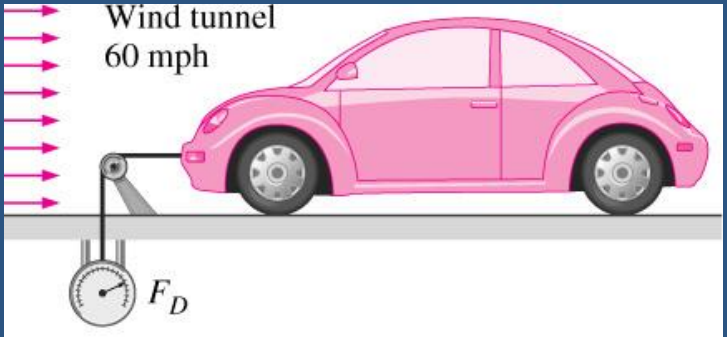


FIGURE 7–2
Schematic for measuring the drag force acting on a car in a wind tunnel.

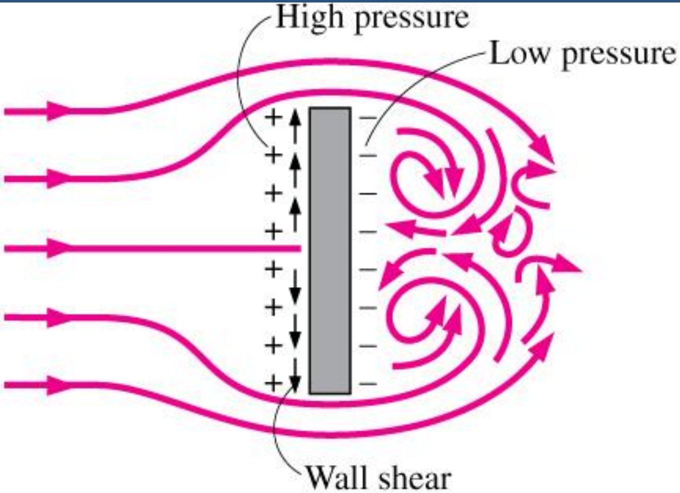


FIGURE 7–3
Drag force acting on a flat plate normal to the flow depends on the pressure only and is independent of the wall shear, which acts normal to the free-stream flow.

- At low Reynolds numbers, most drag is due to friction drag.
- The friction drag is also proportional to the surface area.
- The pressure drag is proportional to the frontal area and to the *difference* between the pressures acting on the front and back of the immersed body.
- The pressure drag is usually dominant for blunt bodies and negligible for streamlined bodies.
- When a fluid separates from a body, it forms a separated region between the body and the fluid stream.
- The larger the separated region, the larger the pressure drag.

FIGURE 7–5

Separation during flow over a tennis ball and the wake region.

Courtesy of NASA and Cislunar Aerospace, Inc.

Heat Transfer

Local and average Nusselt numbers:

$$\mathrm{Nu}_x = f_1(x^*, \mathrm{Re}_x, \mathrm{Pr}) \qquad \text{and} \qquad \mathrm{Nu} = f_2(\mathrm{Re}_L, \mathrm{Pr})$$

Average Nusselt number:

$$\mathrm{Nu} = C\,\mathrm{Re}_L^m \mathrm{Pr}^n$$

Film temperature:

$$T_f = \frac{T_s + T_\infty}{2}$$

Average friction coefficient:

$$C_D = \frac{1}{L}\int_0^L C_{D,x}\,dx$$

Average heat transfer coefficient:

$$h = \frac{1}{L}\int_0^L h_x\,dx$$

The heat transfer rate:

$$\dot{Q} = hA_s(T_s - T_\infty)$$

PARALLEL FLOW OVER FLAT PLATES

The Re number at a distance *x* from the leading edge of a flat plate is expressed as

$$\mathrm{Re}_x = \frac{\rho \mathcal{V} x}{\mu} = \frac{\mathcal{V} x}{\upsilon}$$

For flow over a flat plate, transition from laminar to turbulent usually occurs at

$$\mathrm{Re}_{cr} = \frac{\rho \mathcal{V} x_{cr}}{\mu} = 5 \times 10^5$$

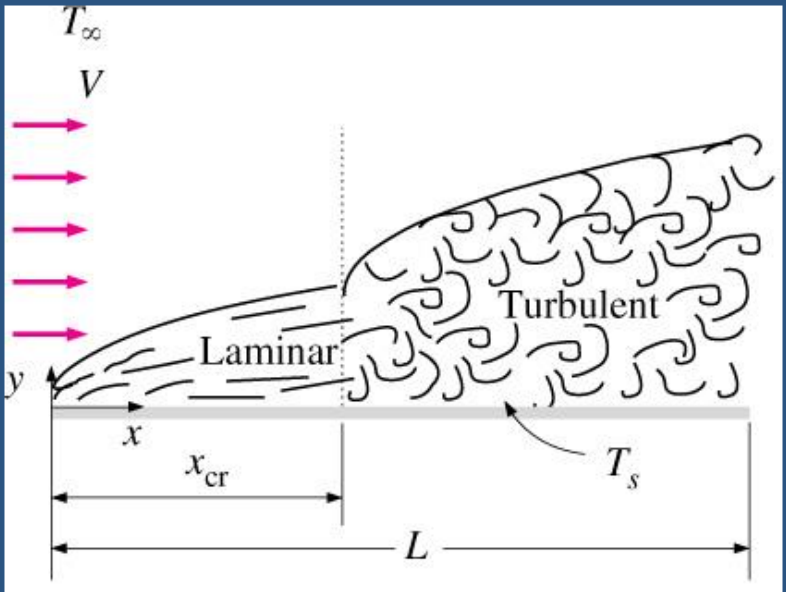


FIGURE 7–6
Laminar and turbulent regions of the boundary layer during flow over a flat plate.

The value of the critical Reynolds number for a flat plate may vary from 10^5 to 3×10^6, depending on the surface roughness and the turbulence level of the free stream

Friction Coefficient

Laminar: $\delta_{v,x} = \dfrac{4.91x}{\mathrm{Re}_x^{1/2}}$ and $C_{f,x} = \dfrac{0.664}{\mathrm{Re}_x^{1/2}}$, $\mathrm{Re}_x < 5 \times 10^5$

Turbulent: $\delta_{v,x} = \dfrac{0.38x}{\mathrm{Re}_x^{1/5}}$ and $C_{f,x} = \dfrac{0.059}{\mathrm{Re}_x^{1/5}}$, $5 \times 10^5 \leq \mathrm{Re}_x \leq 10^7$

Laminar: $C_f = \dfrac{1.33}{\mathrm{Re}_L^{1/2}}$ $\mathrm{Re}_L < 5 \times 10^5$

Turbulent: $C_f = \dfrac{0.074}{\mathrm{Re}_L^{1/5}}$ $5 \times 10^5 \leq \mathrm{Re}_L \leq 10^7$

$$C_f = \frac{1}{L}\left(\int_0^{x_{cr}} C_{f,x\ \mathrm{laminar}}\,dx + \int_{x_{cr}}^{L} C_{f,x,\ \mathrm{turbulent}}\,dx\right)$$

$$C_f = \frac{0.074}{\mathrm{Re}_L^{1/5}} - \frac{1742}{\mathrm{Re}_L} \qquad 5 \times 10^5 \leq \mathrm{Re}_L \leq 10^7$$

Rough surface, turbulent: $C_f = \left(1.89 - 1.62 \log \dfrac{\varepsilon}{L}\right)^{-2.5}$

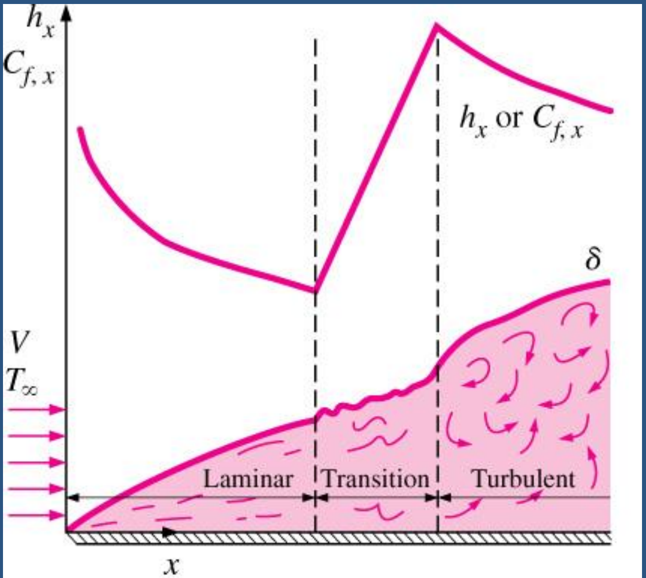


FIGURE 7–9
The variation of the local friction and heat transfer coefficients for flow over a flat plate.

Heat Transfer Coefficient

The local Nusselt number at a location *x* for laminar flow over a flat plate:

$$\textit{Laminar:} \qquad \mathrm{Nu}_x = \frac{h_x x}{k} = 0.332\ \mathrm{Re}_x^{0.5}\ \mathrm{Pr}^{1/3} \qquad \mathrm{Pr} > 0.60$$

The corresponding relation for turbulent flow:

$$\textit{Turbulent:} \qquad \mathrm{Nu}_x = \frac{h_x x}{k} = 0.0296\ \mathrm{Re}_x^{0.8}\ \mathrm{Pr}^{1/3} \qquad \begin{array}{l} 0.6 \le \mathrm{Pr} \le 60 \\ 5 \times 10^5 \le \mathrm{Re}_x \le 10^7 \end{array}$$

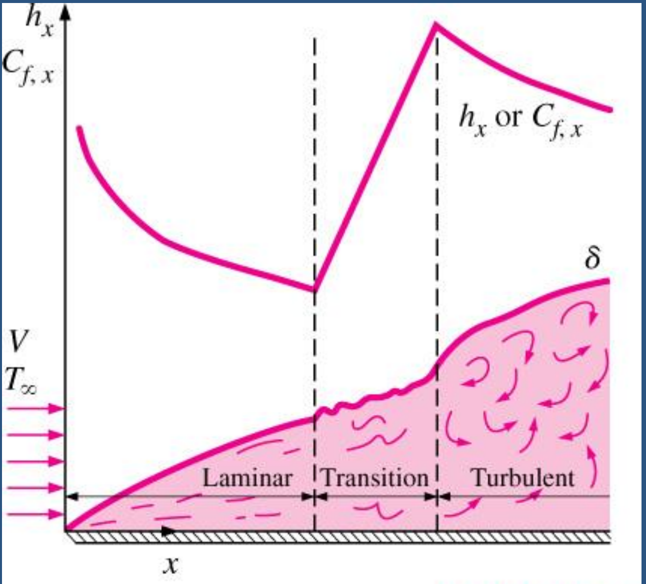


FIGURE 7–9
The variation of the local friction and heat transfer coefficients for flow over a flat plate.

The *average* Nusselt number over the entire plate:

$$\textit{Laminar:} \qquad \mathrm{Nu} = \frac{hL}{k} = 0.664\ \mathrm{Re}_L^{0.5}\ \mathrm{Pr}^{1/3} \qquad \mathrm{Re}_L < 5 \times 10^5$$

$$\textit{Turbulent:} \qquad \mathrm{Nu} = \frac{hL}{k} = 0.037\ \mathrm{Re}_L^{0.8}\ \mathrm{Pr}^{1/3} \qquad \begin{array}{l} 0.6 \le \mathrm{Pr} \le 60 \\ 5 \times 10^5 \le \mathrm{Re}_L \le 10^7 \end{array}$$

The *average* heat transfer coefficient over the entire plate:

$$h = \frac{1}{L}\left(\int_0^{x_{cr}} h_{x,\ \text{laminar}}\, dx + \int_{x_{cr}}^{L} h_{x,\ \text{trubulent}}\, dx \right)$$

The *average* Nusselt number over the *entire* plate:

$$\text{Nu} = \frac{hL}{k} = (0.037\,\text{Re}_L^{0.8} - 871)\text{Pr}^{1/3} \qquad \begin{array}{l} 0.6 \le \text{Pr} \le 60 \\ 5 \times 10^5 \le \text{Re}_L \le 10^7 \end{array}$$

Liquid metals (*e.g.,* mercury) have high thermal conductivities and are commonly used in applications that require high heat transfer rates.

$$\text{Nu}_x = 0.565(\text{Re}_x\,\text{Pr})^{1/2} \qquad \text{Pr} < 0.05$$

Churchill and Ozoe proposed the following relation which is applicable for *all* **Pr** *numbers:*

$$\text{Nu}_x = \frac{h_x x}{k} = \frac{0.3387\,\text{Pr}^{1/3}\,\text{Re}_x^{1/2}}{[1 + (0.0468/\text{Pr})^{2/3}]^{1/4}}$$

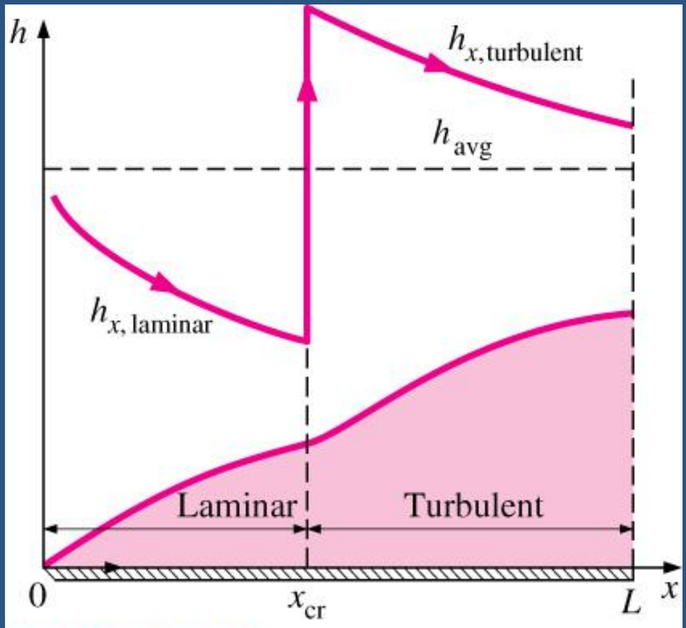


FIGURE 7–10
Graphical representation of the average heat transfer coefficient for a flat plate with combined laminar and turbulent flow.

Flat Plate with Unheated Starting Length

The local Nusselt numbers for both laminar and turbulent flows are:

Laminar: $$\text{Nu}_x = \frac{\text{Nu}_{x\,(\text{for } \xi=0)}}{[1-(\xi/x)^{3/4}]^{1/3}} = \frac{0.332\,\text{Re}_x^{0.5}\,\text{Pr}^{1/3}}{[1-(\xi/x)^{3/4}]^{1/3}}$$

Turbulent: $$\text{Nu}_x = \frac{\text{Nu}_{x\,(\text{for } \xi=0)}}{[1-(\xi/x)^{9/10}]^{1/9}} = \frac{0.0296\,\text{Re}_x^{0.8}\,\text{Pr}^{1/3}}{[1-(\xi/x)^{9/10}]^{1/9}}$$

for $x > \xi$

Laminar: $$h = \frac{2[1-(\xi/x)^{3/4}]}{1-\xi/L}\,h_{x=L}$$

Turbulent: $$h = \frac{5[1-(\xi/x)^{9/10}]}{4(1-\xi/L)}\,h_{x=L}$$

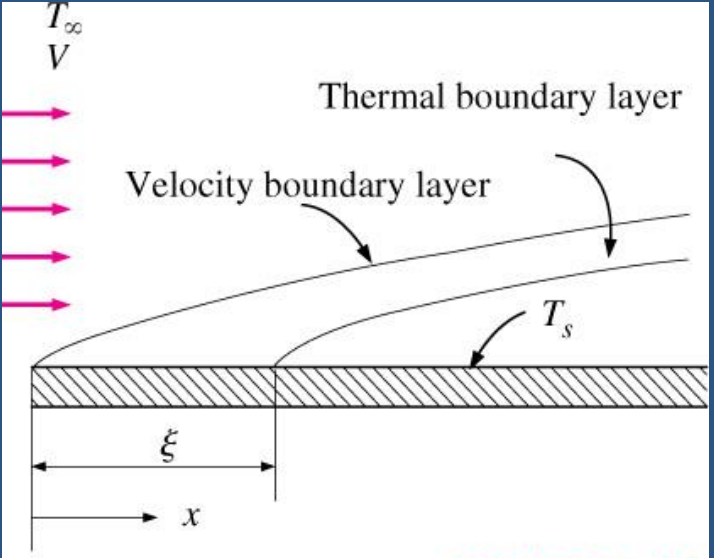


FIGURE 7–11

Flow over a flat plate with an unheated starting length.

Uniform Heat Flux

When a flat plate is subjected to *uniform heat flux* instead of uniform temperature, the local Nu number is

Laminar:	$\mathrm{Nu}_x = 0.453\ \mathrm{Re}_x^{0.5}\ \mathrm{Pr}^{1/3}$
Turbulent:	$\mathrm{Nu}_x = 0.0308\ \mathrm{Re}_x^{0.8}\ \mathrm{Pr}^{1/3}$

When heat flux $\dot{q}_s$ is prescribed, the rate of heat transfer to or from the plate and the surface temperature at a distance *x* are determined from

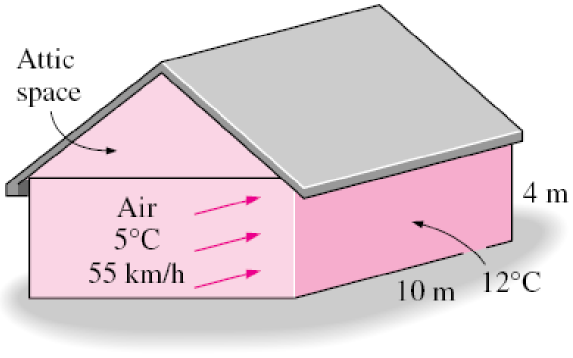
$$\dot{Q} = \dot{q}_s A_s$$

and

$$\dot{q}_s = h_x[T_s(x) - T_\infty] \quad \rightarrow \quad T_s(x) = T_\infty + \frac{\dot{q}_s}{h_x}$$

A_s : heat transfer surface area.

7–16 During a cold winter day, wind at 55 km/h is blowing parallel to a 4-m-high and 10-m-long wall of a house. If the air outside is at 5°C and the surface temperature of the wall is 12°C, determine the rate of heat loss from that wall by convection. What would your answer be if the wind velocity was doubled?

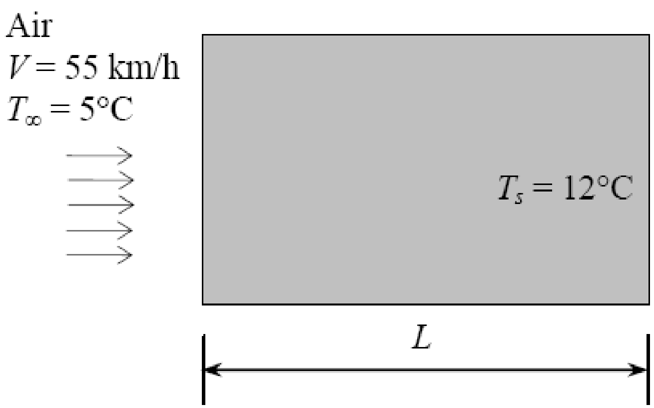


Properties The properties of air at 1 atm and the film temperature of $(T_s + T_\infty)/2 = (12+5)/2 =$ 8.5°C are (Table A-15)

$$k = 0.02428 \text{ W/m}\cdot\text{°C}$$

$$\nu = 1.413\times10^{-5} \text{ m}^2\text{/s}$$

$$\Pr = 0.7340$$



Analysis *Air flows parallel to the 10 m side:*

The Reynolds number in this case is

$$\mathrm{Re}_L = \frac{VL}{\nu} = \frac{[(55\times1000/3600)\text{m/s}](10\text{ m})}{1.413\times10^{-5}\text{ m}^2\text{/s}} = 1.081\times10^7$$

which is greater than the critical Reynolds number. Thus we have combined laminar and turbulent flow. Using the proper relation for Nusselt number, heat transfer coefficient and then heat transfer rate are determined to be

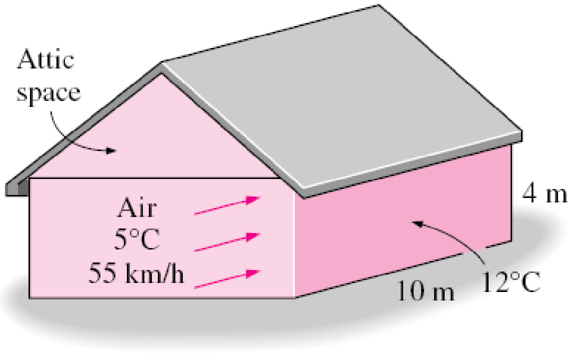
$$Nu = \frac{hL}{k} = (0.037\,\mathrm{Re}_L{}^{0.8} - 871)\Pr^{1/3} = [0.037(1.081\times10^7)^{0.8} - 871](0.7340)^{1/3} = 1.336\times10^4$$

$$h = \frac{k}{L}Nu = \frac{0.02428\text{ W/m.°C}}{10\text{ m}}(1.336\times10^4) = 32.43\text{ W/m}^2\text{.°C}$$

$$A_s = wL = (4\text{ m})(10\text{ m}) = 40\text{ m}^2$$

$$\dot{Q} = hA_s(T_\infty - T_s) = (32.43\text{ W/m}^2\text{.°C})(40\text{ m}^2)(12-5)\text{°C} = 9080\text{ W} = \mathbf{9.08\,kW}$$

7–16 During a cold winter day, wind at 55 km/h is blowing parallel to a 4-m-high and 10-m-long wall of a house. If the air outside is at 5°C and the surface temperature of the wall is 12°C, determine the rate of heat loss from that wall by convection. What would your answer be if the wind velocity was doubled?



If the wind velocity is doubled:

$$\mathrm{Re}_L = \frac{VL}{\nu} = \frac{[(110\times 1000/3600)\text{m/s}](10\,\text{m})}{1.413\times 10^{-5}\ \text{m}^2/\text{s}} = 2.162\times 10^7$$

which is greater than the critical Reynolds number. Thus we have combined laminar and turbulent flow. Using the proper relation for Nusselt number, the average heat transfer coefficient and the heat transfer rate are determined to be

$$Nu = \frac{hL}{k} = (0.037\,\mathrm{Re}_L{}^{0.8} - 871)\,\mathrm{Pr}^{1/3} = [0.037(2.162\times 10^7)^{0.8} - 871](0.7340)^{1/3} = 2.384\times 10^4$$

$$h = \frac{k}{L}Nu = \frac{0.02428\ \text{W/m.°C}}{10\,\text{m}}(2.384\times 10^4) = 57.88\ \text{W/m}^2.\text{°C}$$

$$\dot{Q} = hA_s(T_\infty - T_s) = (57.88\ \text{W/m}^2.\text{°C})(40\,\text{m}^2)(12-5)\text{°C} = 16{,}210\ \text{W} = \mathbf{16.21\,kW}$$

FLOW ACROSS CYLINDERS AND SPHERES

- Flow across cylinders and spheres is frequently encountered in many heat transfer systems
  - shell-and-tube heat exchanger,
  - pin fin heat sinks for electronic cooling.
- The characteristic length for a circular cylinder or sphere is taken to be the *external diameter* D.
- The **critical** Re number for flow across a circular cylinder or sphere is about

$$Re_{cr} = 2x10^5$$

- Cross-flow over a cylinder exhibits complex flow patterns depending on the Re number.

- At very low velocities (Re≤1), the fluid completely wraps around the cylinder.
- At higher velocities the boundary layer detaches from the surface, forming a separation region behind the cylinder.
- Flow in the wake region is characterized by periodic vortex formation and low pressures.
- The nature of the flow across a cylinder or sphere strongly affects the total C_D.
- At low Re numbers (Re<10), friction drag dominates.
- At high Re numbers (Re>5000), pressure drag dominate.
- At intermediate Re numbers — both pressure and friction drags dominate.

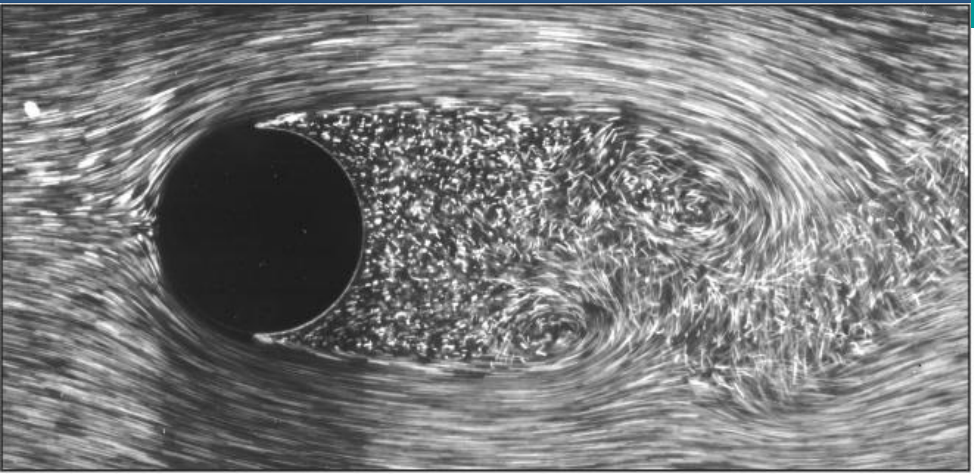


FIGURE 7–16
Laminar boundary layer separation with a turbulent wake; flow over a circular cylinder at Re = 2000.
Courtesy ONERA, photograph by Werlé.

Average C_D for circular cylinder and sphere:

- Re≤1 — creeping flow
- Re≈10 — separation starts
- Re≈90 — vortex shedding starts.
- 10^3<Re<10^5
  - in the boundary layer flow is laminar
  - in the separated region flow is highly turbulent
- 10^5<Re<10^6 — *turbulent flow*

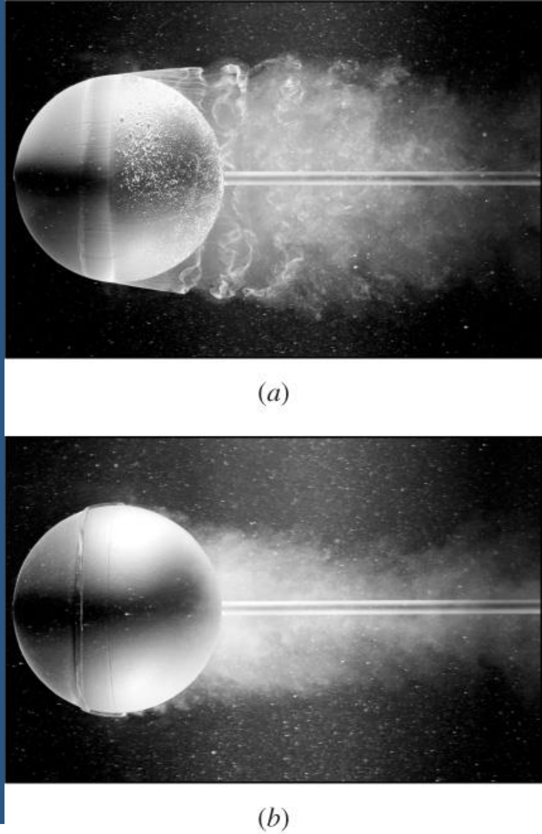


FIGURE 7–18
Flow visualization of flow over (a) a smooth sphere at Re = 15,000, and (b) a sphere at Re = 30,000 with a trip wire. The delay of boundary layer separation is clearly seen by comparing the two photographs.
Courtesy ONERA, photograph by Werlé.

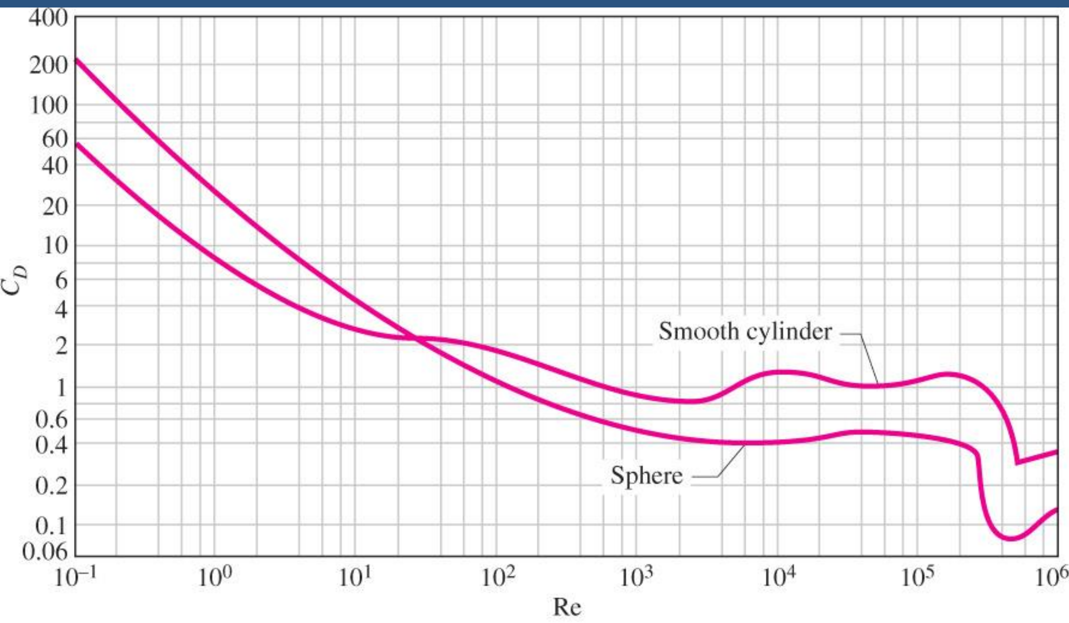


FIGURE 7–17
Average drag coefficient for cross-flow over a smooth circular cylinder and a smooth sphere.
From H. Schlichting, Boundary Layer Theory 7e. *Copyright © 1979 The McGraw-Hill Companies, Inc. Used by permission.*

Effect of Surface Roughness:

- *Surface roughness* increases the drag coefficient in turbulent flow, especially for streamlined bodies.
- For blunt bodies (e.g., a circular cylinder or sphere), an increase in the surface roughness may actually *decrease* the drag coefficient.
- This is done by tripping the boundary layer into turbulence at a lower Re number, causing the fluid to close in behind the body, narrowing the wake and reducing pressure drag considerably.

Relative roughness, ε/L	Friction coefficient C_f
0.0*	0.0029
1×10^{-5}	0.0032
1×10^{-4}	0.0049
1×10^{-3}	0.0084

*Smooth surface for Re = 10^7. Others calculated from Eq. 7–18.

FIGURE 7–8
For turbulent flow, surface roughness may cause the friction coefficient to increase severalfold.

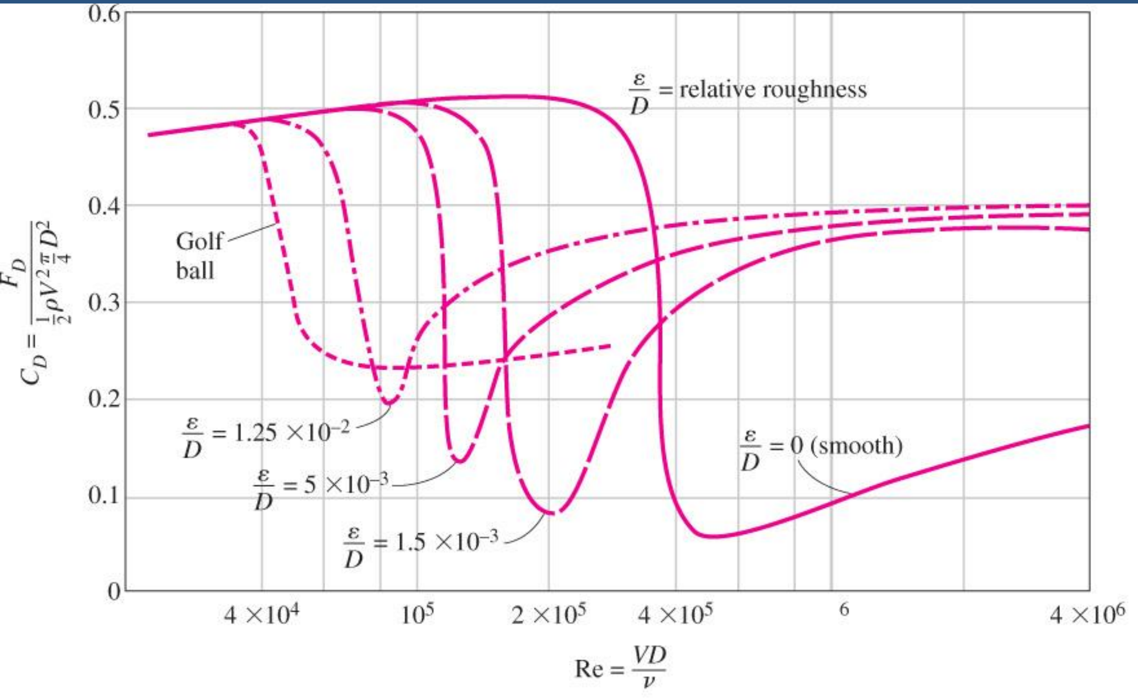


FIGURE 7–19
The effect of surface roughness on the drag coefficient of a sphere.

From Blevins (1984).

Heat Transfer Coefficient

- Flows across cylinders and spheres involve *flow separation*, which is difficult to handle analytically.
- The local Nusselt number Nu_q around the periphery of a cylinder subjected to cross flow varies considerably.

<u>Small q</u> — Nu_q decreases with increasing q as a result of the thickening of the laminar boundary layer.

<u>80°<q <90°</u> — Nu_q reaches a minimum

- low Re numbers — due to separation in laminar flow
- high Re numbers — transition to turbulent flow.

<u>q >90° laminar flow</u> — Nu_q increases with increasing q due to intense mixing in the separation zone.

<u>90°<q <140° turbulent flow</u> — Nu_q decreases due to the thickening of the boundary layer.

<u>q ≈140° turbulent flow</u> — Nu_q reaches a second minimum due to flow separation point in turbulent flow.

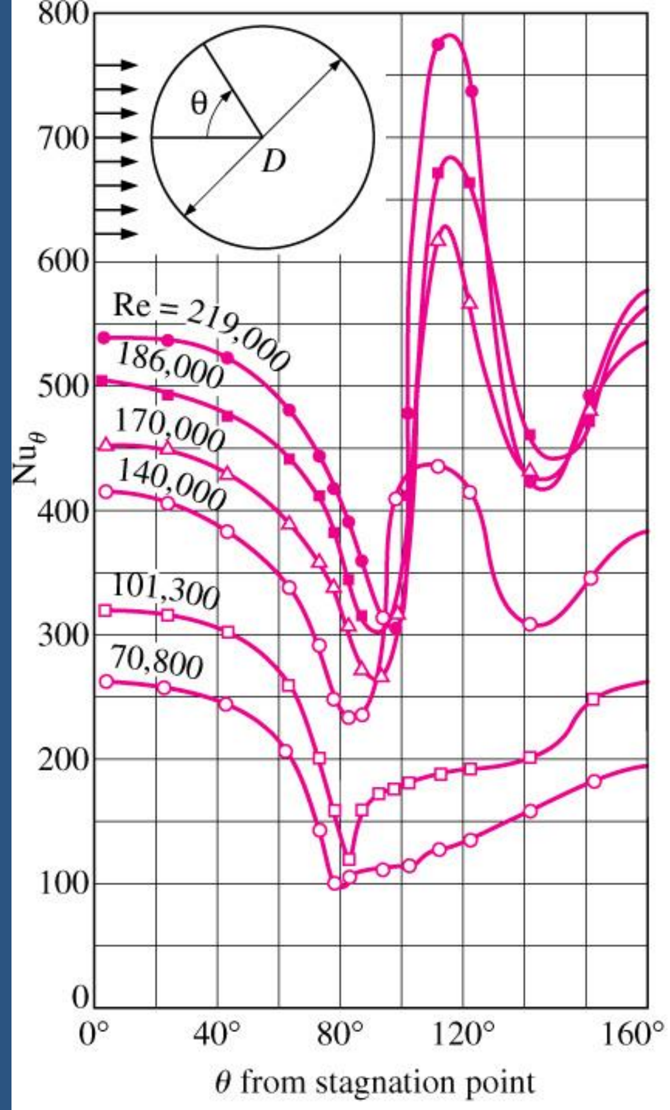


FIGURE 7–22
Variation of the local heat transfer coefficient along the circumference of a circular cylinder in cross flow of air (from Giedt, 1949).

The average Nu for cross-flow over a cylinder (by Churchill and Bernstein):

$$\mathrm{Nu}_{\mathrm{cyl}} = \frac{hD}{k} = 0.3 + \frac{0.62\,\mathrm{Re}^{1/2}\,\mathrm{Pr}^{1/3}}{[1 + (0.4/\mathrm{Pr})^{2/3}]^{1/4}}\left[1 + \left(\frac{\mathrm{Re}}{282{,}000}\right)^{5/8}\right]^{4/5}$$

for $\mathrm{Re}\ \mathrm{Pr} > 0.2$

The fluid properties are evaluated at the *film temperature:*

$$T_f = \tfrac{1}{2}(T_\infty + T_s)$$

For flow over a *sphere, the* Whitaker correlation:

$$\mathrm{Nu}_{\mathrm{sph}} = \frac{hD}{k} = 2 + [0.4\,\mathrm{Re}^{1/2} + 0.06\,\mathrm{Re}^{2/3}]\,\mathrm{Pr}^{0.4}\left(\frac{\mu_\infty}{\mu_s}\right)^{1/4}$$

The fluid properties are evaluated at the free stream temperature T_∞ except for μ_s

for $3.5 \le \mathrm{Re} \le 80{,}000$ and $0.7 \le \mathrm{Pr} \le 380$

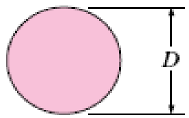
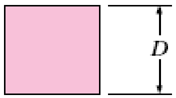
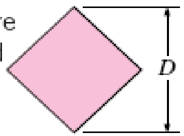
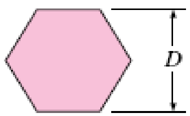
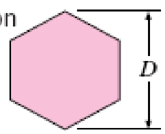
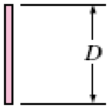
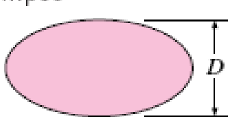
The average Nu for flow across cylinders:

$$\mathrm{Nu}_{\mathrm{cyl}} = \frac{hD}{k} = C\,\mathrm{Re}^m\,\mathrm{Pr}^n$$

n = 1/3, C and m: the experimentally determined constants (From Table 7.1)

TABLE 7–1

Empirical correlations for the average Nusselt number for forced convection over circular and noncircular cylinders in cross flow (from Zukauskas, 1972 and Jakob, 1949)

Cross-section of the cylinder	Fluid	Range of Re	Nusselt number
Circle 	Gas or liquid	0.4–4 4–40 40–4000 4000–40,000 40,000–400,000	$Nu = 0.989Re^{0.330}\ Pr^{1/3}$ $Nu = 0.911Re^{0.385}\ Pr^{1/3}$ $Nu = 0.683Re^{0.466}\ Pr^{1/3}$ $Nu = 0.193Re^{0.618}\ Pr^{1/3}$ $Nu = 0.027Re^{0.805}\ Pr^{1/3}$
Square 	Gas	5000–100,000	$Nu = 0.102Re^{0.675}\ Pr^{1/3}$
Square (tilted 45°) 	Gas	5000–100,000	$Nu = 0.246Re^{0.588}\ Pr^{1/3}$
Hexagon 	Gas	5000–100,000	$Nu = 0.153Re^{0.638}\ Pr^{1/3}$
Hexagon (tilted 45°) 	Gas	5000–19,500 19,500–100,000	$Nu = 0.160Re^{0.638}\ Pr^{1/3}$ $Nu = 0.0385Re^{0.782}\ Pr^{1/3}$
Vertical plate 	Gas	4000–15,000	$Nu = 0.228Re^{0.731}\ Pr^{1/3}$
Ellipse 	Gas	2500–15,000	$Nu = 0.248Re^{0.612}\ Pr^{1/3}$

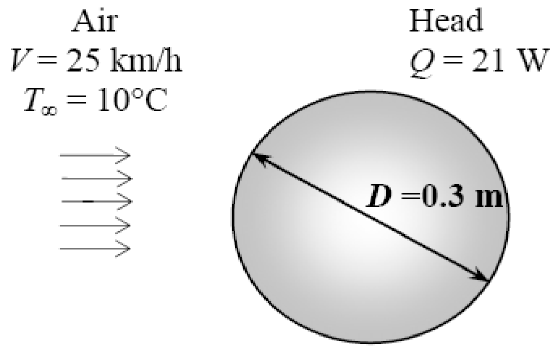
The Churchill and Bernstein equation is more accurate, and thus should be preferred in calculations whenever possible.

$$Nu_{cyl} = \frac{hD}{k} = C\,Re^m\,Pr^n$$

7–49 An average person generates heat at a rate of 84 W while resting. Assuming one-quarter of this heat is lost from the head and disregarding radiation, determine the average surface temperature of the head when it is not covered and is subjected to winds at 10°C and 25 km/h. The head can be approximated as a 30-cm-diameter sphere.

Properties We assume the surface temperature to be 15°C for viscosity. The properties of air at 1 atm pressure and the free stream temperature of 10°C are (Table A-15)

$$k = 0.02439 \text{ W/m.°C}$$
$$\nu = 1.426\times10^{-5} \text{ m}^2/\text{s}$$
$$\mu_\infty = 1.778\times10^{-5} \text{ kg/m.s}$$
$$\mu_{s,@15°C} = 1.802\times10^{-5} \text{ kg/m.s}$$
$$\Pr = 0.7336$$



Analysis The Reynolds number is

$$\text{Re} = \frac{VD}{\nu} = \frac{[(25\times1000/3600)\text{ m/s}](0.3\text{ m})}{1.426\times10^{-5}\text{ m}^2/\text{s}} = 1.461\times10^5$$

The proper relation for Nusselt number corresponding to this Reynolds number is

$$Nu = \frac{hD}{k} = 2 + \left[0.4\,\text{Re}^{0.5} + 0.06\,\text{Re}^{2/3}\right]\Pr^{0.4}\left(\frac{\mu_\infty}{\mu_s}\right)^{1/4}$$

$$= 2 + \left[0.4(1.461\times10^5)^{0.5} + 0.06(1.461\times10^5)^{2/3}\right](0.7336)^{0.4}\left(\frac{1.778\times10^{-5}}{1.802\times10^{-5}}\right)^{1/4} = 283.2$$

The heat transfer coefficient is

$$h = \frac{k}{D}Nu = \frac{0.02439\text{ W/m.°C}}{0.3\text{ m}}(283.2) = 23.02\text{ W/m}^2.°\text{C}$$

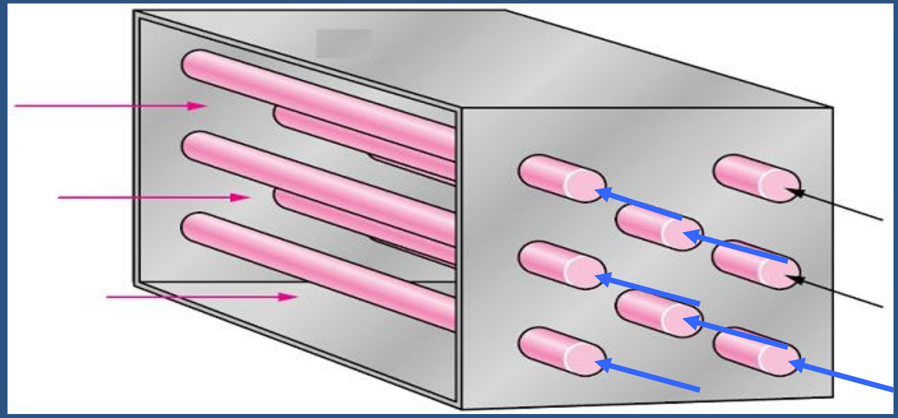
Then the surface temperature of the head is determined to be

$$A_s = \pi D^2 = \pi(0.3\text{ m})^2 = 0.2827\text{ m}^2$$

$$\dot{Q} = hA_s(T_s - T_\infty) \longrightarrow T_s = T_\infty + \frac{\dot{Q}}{hA_s} = 10\,°\text{C} + \frac{(84/4)\text{ W}}{(23.02\text{ W/m}^2.°\text{C})(0.2827\text{ m}^2)} = \mathbf{13.2\,°C}$$

FLOW ACROSS TUBE BANKS

- Cross-flow over tube banks is commonly encountered in practice in heat transfer equipment, e.g., heat exchangers.
- In such equipment, one fluid moves through the tubes while the other moves over the tubes in a perpendicular direction.
- Flow *through* the tubes can be analyzed by considering flow through a single tube, and multiplying the results by the number of tubes.
- For flow *over* the tubes the tubes affect the flow pattern and turbulence level downstream, and thus heat transfer to or from them are altered.



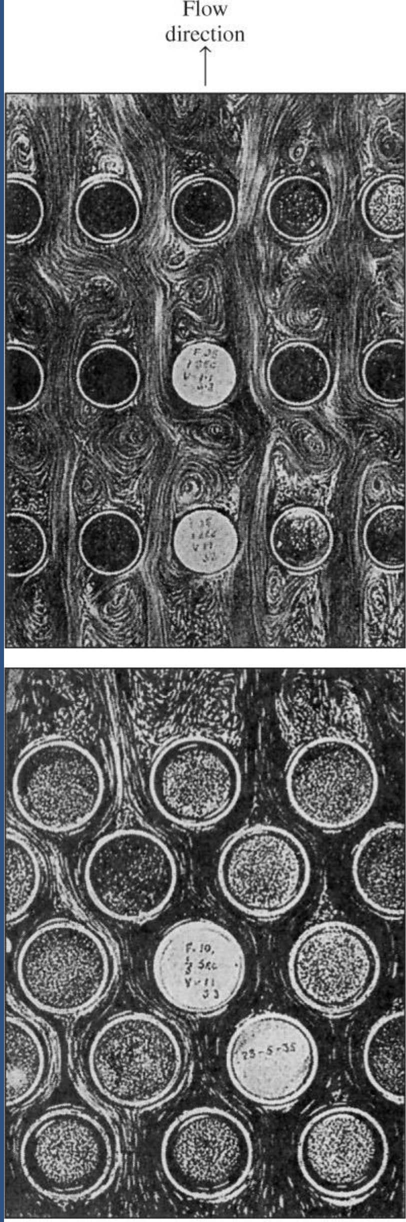
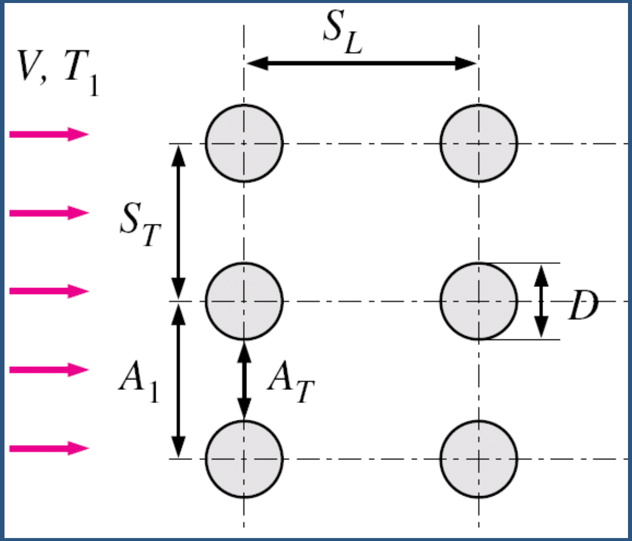


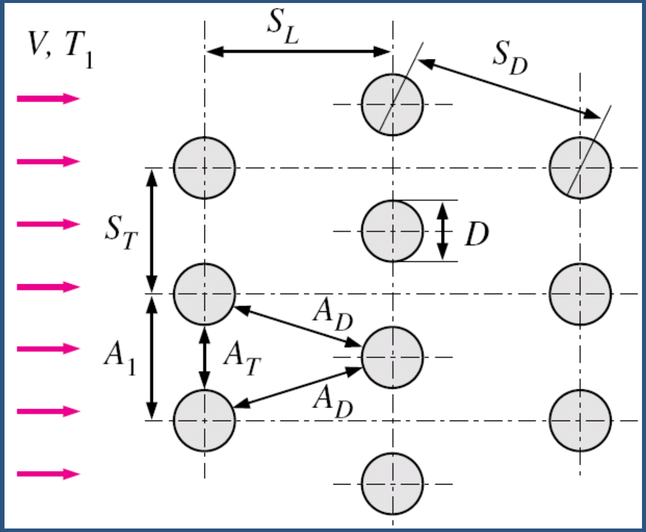
FIGURE 7–25
Flow patterns for staggered and in-line tube banks (photos by R. D. Willis).

- Typical arrangement
  - *in-line*
  - *staggered*
- The outer tube diameter *D* is the characteristic length.
- The arrangement of the tubes are characterized by the
  - *transverse pitch S_T,*
  - *longitudinal pitch S_L , and the*
  - *diagonal pitch S_D between tube centers.*

In-line

Staggered





- The diagonal pitch: $S_D = \sqrt{S_L^2 + (S_T/2)^2}$
- Re number based on max. velocity: $\mathrm{Re}_D = \dfrac{\rho V_{\max} D}{\mu} = \dfrac{V_{\max} D}{\nu}$
- Max. velocity (in-line): $V_{\max} = \dfrac{S_T}{S_T - D} V$
- Max. velocity (staggered): *Staggered and* $S_D < (S_T + D)/2$: $\quad V_{\max} = \dfrac{S_T}{2(S_D - D)} V$
- Nusselt number (Table 7-2): $\mathrm{Nu}_D = \dfrac{hD}{k} = C\,\mathrm{Re}_D^m\,\mathrm{Pr}^n(\mathrm{Pr}/\mathrm{Pr}_s)^{0.25}$
- Average temperature of inlet and exit (for property evaluation): $T_m = \dfrac{T_i + T_e}{2}$
- Nusselt number (< 16 rows): $\mathrm{Nu}_{D,N_L} = F\,\mathrm{Nu}_D$
- Log mean temp. dif. $\Delta T_{\ln} = \dfrac{(T_s - T_e) - (T_s - T_i)}{\ln[(T_s - T_e)/(T_s - T_i)]} = \dfrac{\Delta T_e - \Delta T_i}{\ln(\Delta T_e/\Delta T_i)}$
- Exit temperature: $T_e = T_s - (T_s - T_i)\exp\left(\pm\dfrac{A_s h}{\dot{m} c_p}\right)$
- Heat transfer rate: $\dot{Q} = hA_s\Delta T_{\ln} = \dot{m}c_p(T_e - T_i)$

TABLE 7–2

Nusselt number correlations for cross flow over tube banks for $N > 16$ and $0.7 < Pr < 500$ (from Zukauskas, 1987)*

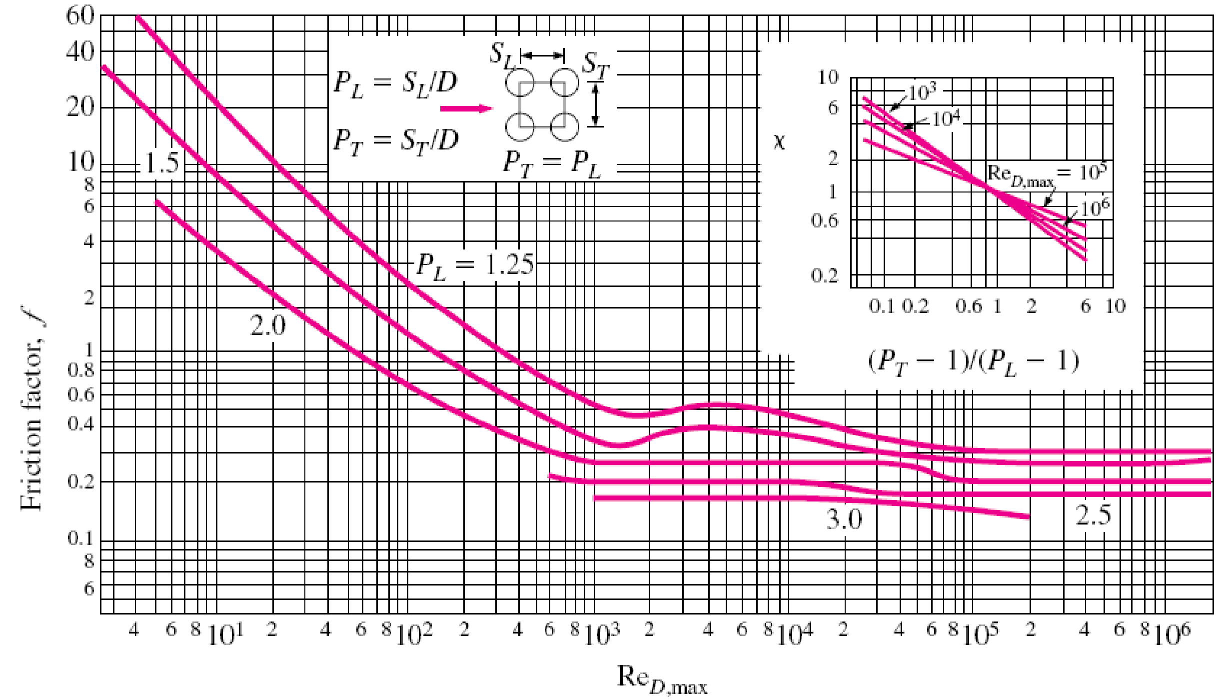
Arrangement	Range of Re_D	Correlation
In-line	0–100	$Nu_D = 0.9\,Re_D^{0.4}Pr^{0.36}(Pr/Pr_s)^{0.25}$
	100–1000	$Nu_D = 0.52\,Re_D^{0.5}Pr^{0.36}(Pr/Pr_s)^{0.25}$
	$1000–2 \times 10^5$	$Nu_D = 0.27\,Re_D^{0.63}Pr^{0.36}(Pr/Pr_s)^{0.25}$
	$2 \times 10^5–2 \times 10^6$	$Nu_D = 0.033\,Re_D^{0.8}Pr^{0.4}(Pr/Pr_s)^{0.25}$
Staggered	0–500	$Nu_D = 1.04\,Re_D^{0.4}Pr^{0.36}(Pr/Pr_s)^{0.25}$
	500–1000	$Nu_D = 0.71\,Re_D^{0.5}Pr^{0.36}(Pr/Pr_s)^{0.25}$
	$1000–2 \times 10^5$	$Nu_D = 0.35(S_T/S_L)^{0.2}\,Re_D^{0.6}Pr^{0.36}(Pr/Pr_s)^{0.25}$
	$2 \times 10^5–2 \times 10^6$	$Nu_D = 0.031(S_T/S_L)^{0.2}\,Re_D^{0.8}Pr^{0.36}(Pr/Pr_s)^{0.25}$

*All properties except Pr_s are to be evaluated at the arithmetic mean of the inlet and outlet temperatures of the fluid (Pr_s is to be evaluated at T_s).

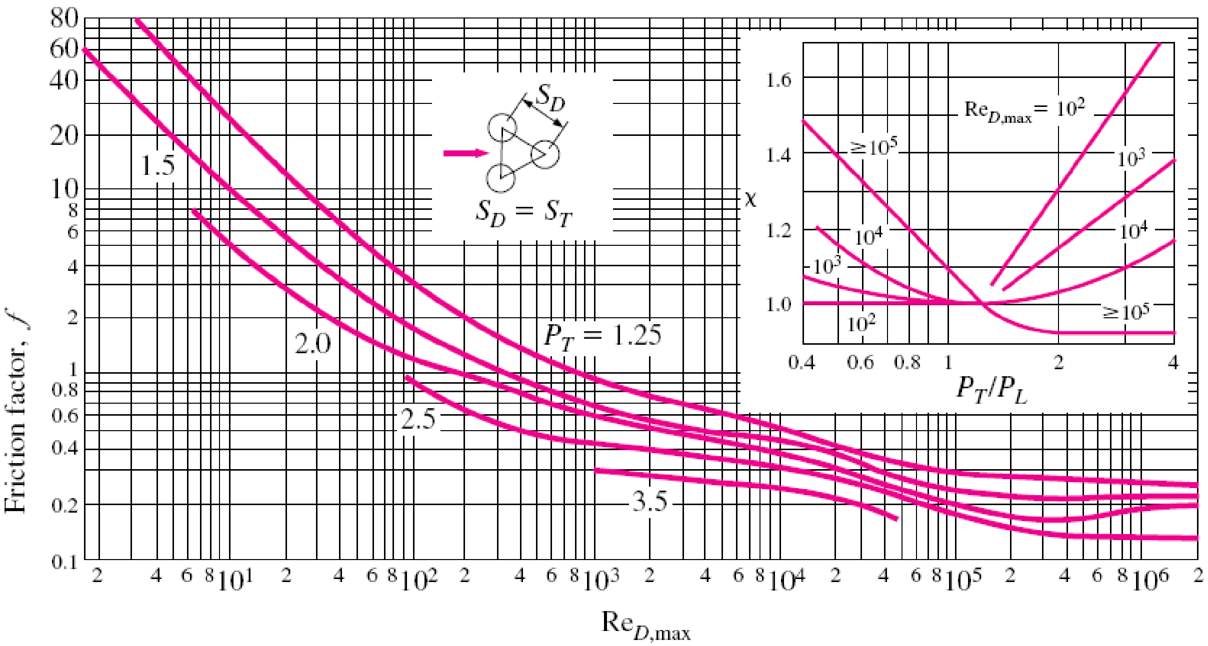
TABLE 7–3

Correction factor F to be used in $Nu_{D, N_L}, = F Nu_D$ for $N_L < 16$ and $Re_D > 1000$ (from Zukauskas, 1987)

N_L	1	2	3	4	5	7	10	13
In-line	0.70	0.80	0.86	0.90	0.93	0.96	0.98	0.99
Staggered	0.64	0.76	0.84	0.89	0.93	0.96	0.98	0.99



(*a*) In-line arrangement



(*b*) Staggered arrangement

Pressure drop:

$$\Delta P = N_L f \chi \frac{\rho V_{max}^2}{2}$$

$$\dot{W}_{pump} = \dot{V}\Delta P = \frac{\dot{m}\Delta P}{\rho}$$

$$\dot{V} = V(N_T S_T L)$$

$$\dot{m} = \rho \dot{V} = \rho V(N_T S_T L)$$

- f is the friction factor and c is the correction factor.
- The correction factor χ given is used to account for the effects of deviation from square arrangement (in-line) and from equilateral arrangement (staggered).

Concluding Points

- Drag and Heat Transfer in External Flow
- Parallel Flow over Flat Plates
- Flow across Cylinders and Spheres
- Flow across Tube Banks

HEAT AND MASS TRANSFER

Internal Forced Convection

OUTLINE

- Introduction
- Average velocity and temperature
- The entrance region
- General Thermal Analysis
- Laminar Flow in Tubes
- Turbulent Flow in Tubes
- Conclusions

Introduction

- *Pipe* — circular cross section.
- *Duct* — noncircular cross section.
- *Tubes* — small-diameter pipes.
- The fluid velocity changes from *zero* at the surface (no-slip) to a maximum at the pipe center.
- It is convenient to work with an *average* velocity, which remains constant in incompressible flow

  when the cross-sectional area is constant.

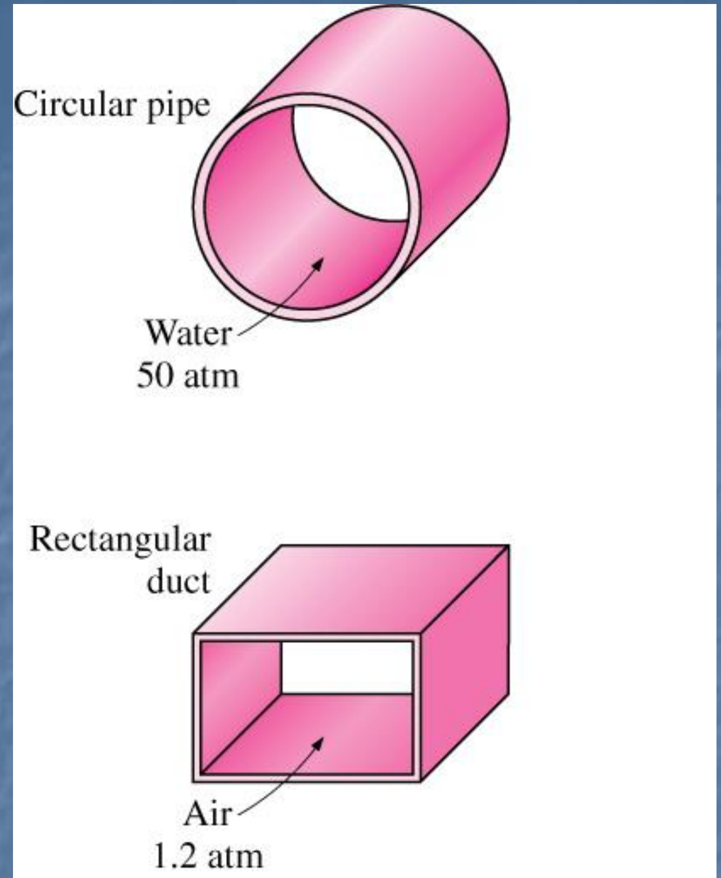


FIGURE 8–1
Circular pipes can withstand large pressure differences between the inside and the outside without undergoing any significant distortion, but noncircular pipes cannot.

Average Velocity

- Average velocity from the *conservation of mass* principle

$$\dot{m} = \rho V_{avg} A_c = \int_{A_c} \rho u(r)\, dA_c$$

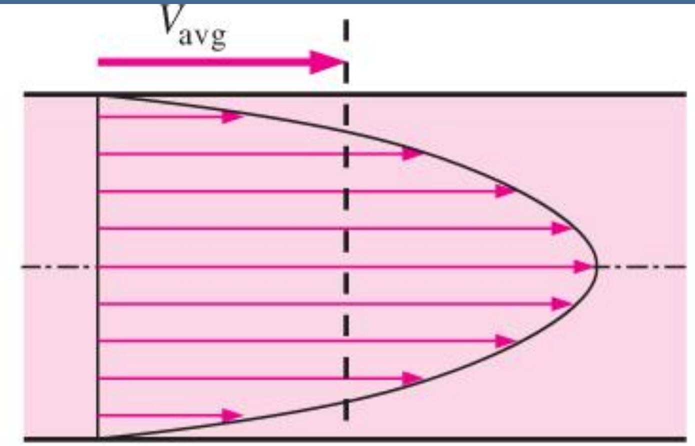


FIGURE 8–2

Average velocity V_{avg} is defined as the average speed through a cross section. For fully developed laminar pipe flow, V_{avg} is half of maximum velocity.

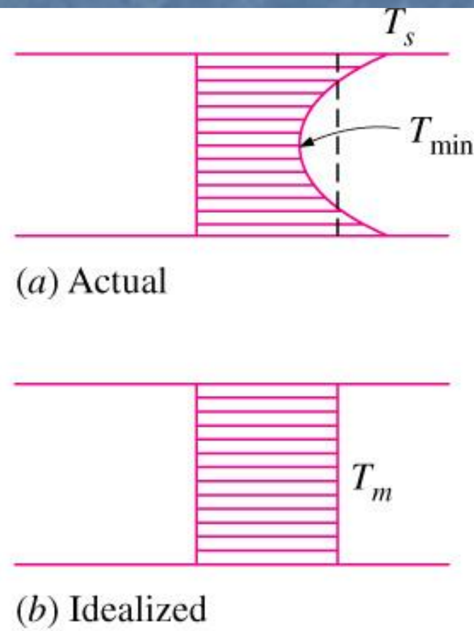


FIGURE 8–3

Actual and idealized temperature profiles for flow in a tube (the rate at which energy is transported with the fluid is the same for both cases).

Average Temperature

- Average temperature from the *conservation of energy*

$$T_m = \frac{\int_{\dot{m}} c_p T(r) \delta \dot{m}}{\dot{m} c_p} = \frac{\int_0^R c_p T(r) \rho u(r) 2\pi r dr}{\rho V_{avg}(\pi R^2) c_p} = \frac{2}{V_{avg} R^2} \int_0^R T(r) u(r)\, r dr$$

Laminar and Turbulent Flow in Tubes

- For flow in a circular tube, the Reynolds number is defined as

$$\mathrm{Re} = \frac{\rho V_{\mathrm{avg}} D}{\mu} = \frac{V_{\mathrm{avg}} D}{\nu}$$

- For flow through noncircular tubes D is replaced by the hydraulic diameter D_h.

$$D_h = \frac{4A_c}{p}$$

- laminar flow: Re<2300
- fully turbulent: Re>10,000.

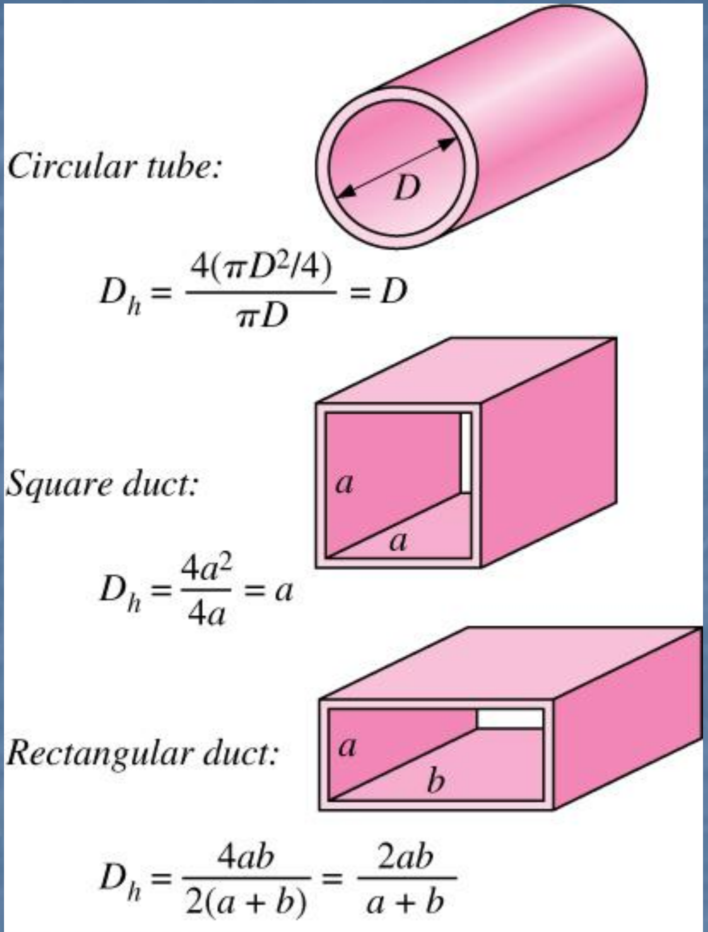
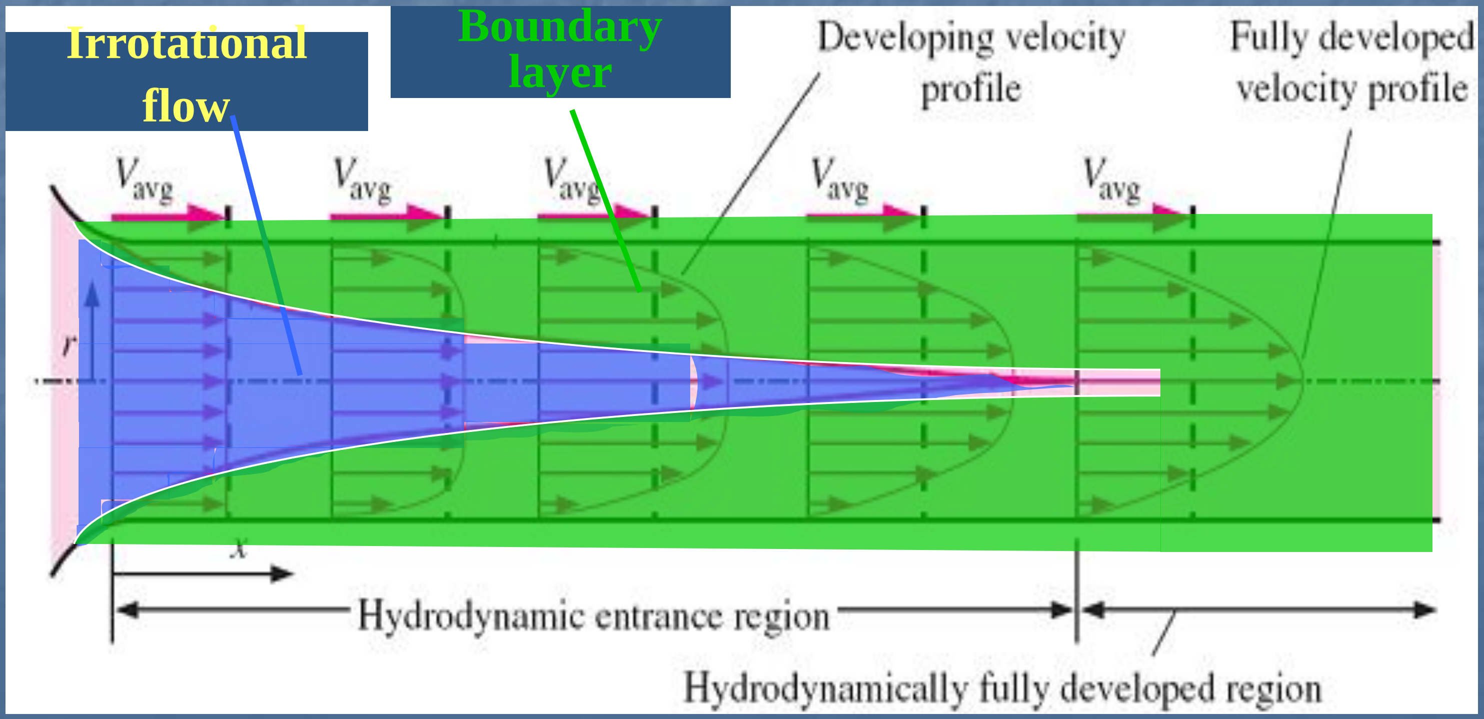


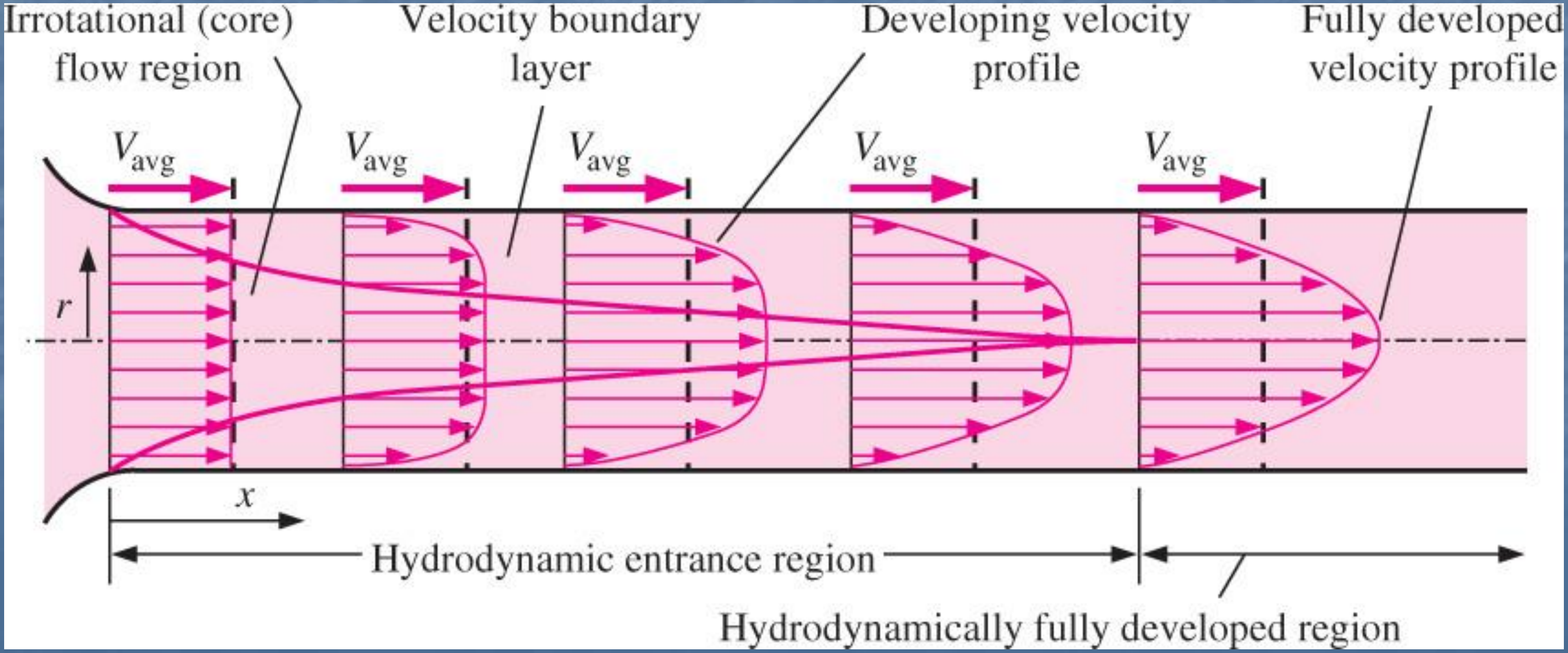
FIGURE 8–4
The hydraulic diameter $D_h = 4A_c/p$ is defined such that it reduces to ordinary diameter for circular tubes.

The Entrance Region

- Consider a fluid entering a circular pipe at a uniform velocity.
- Because of the no-slip condition a velocity gradient develops along the pipe.
- The flow in a pipe is divided into two regions:
  - the **boundary layer region**, and
  - the and the **irrotational (core) flow region**.
- The thickness of this boundary layer increases in the flow direction until it reaches the pipe center.



- **Hydrodynamic entrance region** — the region from the pipe inlet to the point at which the boundary layer merges at the centerline.
- **Hydrodynamically fully developed region** — the region beyond the entrance region in which the velocity profile is fully developed and remains unchanged.
- The velocity profile in the fully developed region is
  - *parabolic* in laminar flow, and
  - somewhat *flatter* or fuller in turbulent flow.



Thermal Entrance Region

The Thermal Entrance Region: the region of flow over which the thermal boundary layer develops and reaches the tube center.

The Thermal Entry Length (*Lt*) : the length of the thermal entrance region.

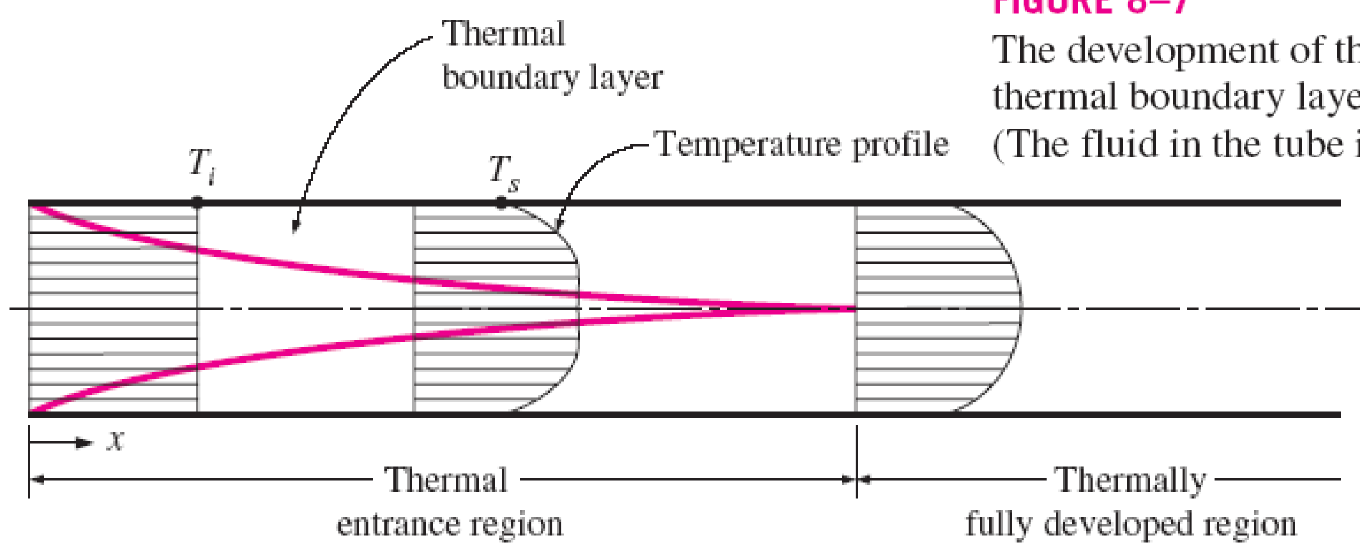


FIGURE 8–7
The development of the thermal boundary layer in a tube. (The fluid in the tube is being cooled.)

Fully Developed Flow: the region in which the flow is both hydrodynamically and thermally developed and thus both the velocity and dimensionless temperature profiles remain unchanged.

Both the friction and convection coefficients remain constant in the fully developed region of a tube.

In laminar flow, the hydrodynamic and thermal entry lengths are given approximately

$$L_{h,\,\text{laminar}} \approx 0.05\ \text{Re}\ D$$

$$L_{t,\,\text{laminar}} \approx 0.05\ \text{Re}\ \text{Pr}\ D = \text{Pr}\ L_{h,\,\text{laminar}}$$

The hydrodynamic and thermal entry lengths are taken to be:

$$L_{h,\,\text{turbulent}} \approx L_{t,\,\text{turbulent}} \approx 10D$$

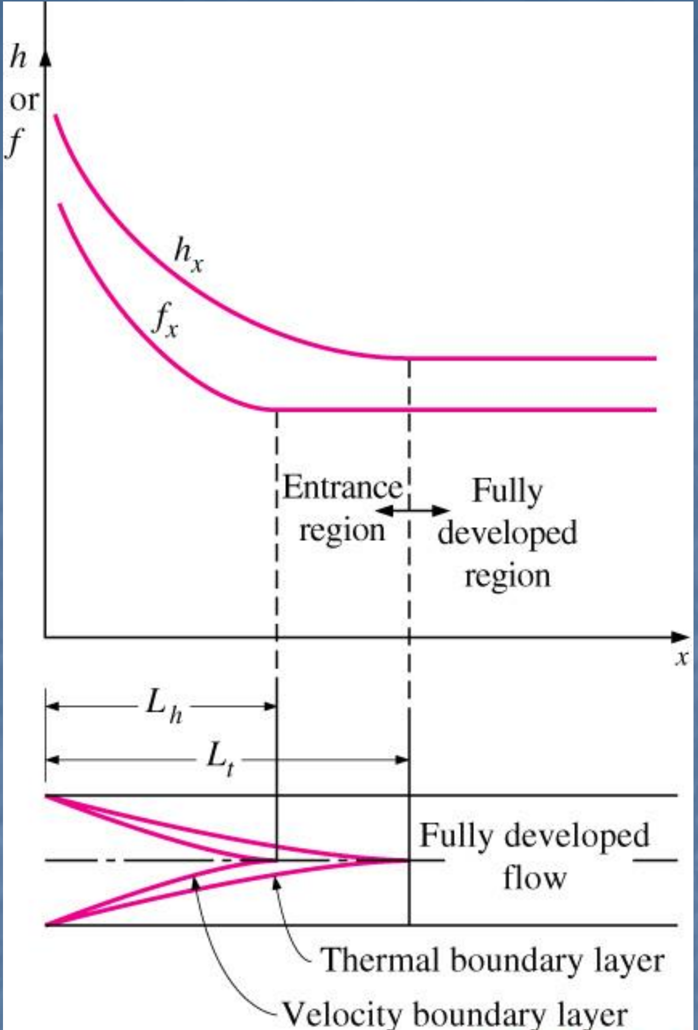


FIGURE 8–8

Variation of the friction factor and the convection heat transfer coefficient in the flow direction for flow in a tube (Pr > 1).

GENERAL THERMAL ANALYSIS

The conservation of energy equation for the steady flow of a fluid in a tube is:

$$\dot{Q} = \dot{m} c_p (T_e - T_i) \qquad \text{(W)}$$

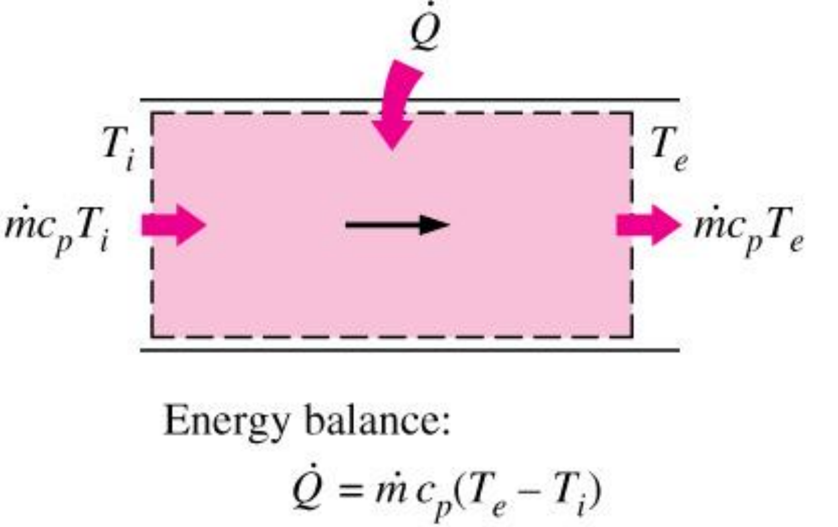


FIGURE 8–10
The heat transfer to a fluid flowing in a tube is equal to the increase in the energy of the fluid.

The surface heat flux is

$$\dot{q}_s = h_x (T_s - T_m) \qquad \text{(W/m}^2\text{)}$$

h_x : the local heat transfer coefficient
T_s : the surface temperature
T_m : the fluid temperature

The bulk mean temperature of a fluid with constant density and specific heat flowing in a circular pipe of radius *R*:

$$T_b = (T_{m,\,i} + T_{m,\,e})/2$$

T_m: average or mean temperature

Constant Surface Heat Flux ($\dot{q}_s$ = constant)

The rate of heat transfer:

$$\dot{Q} = \dot{q}_s A_s = \dot{m} c_p (T_e - T_i) \qquad \text{(W)}$$

The mean fluid temperature at the tube exit:

$$T_e = T_i + \frac{\dot{q}_s A_s}{\dot{m} c_p}$$

The surface temperature in the case of constant surface heat flux:

$$\dot{q}_s = h(T_s - T_m) \longrightarrow T_s = T_m + \frac{\dot{q}_s}{h}$$

In the fully developed region, the surface temperature T_s will increase linearly in the flow direction since h is constant and thus $T_s - T_m$ = constant.

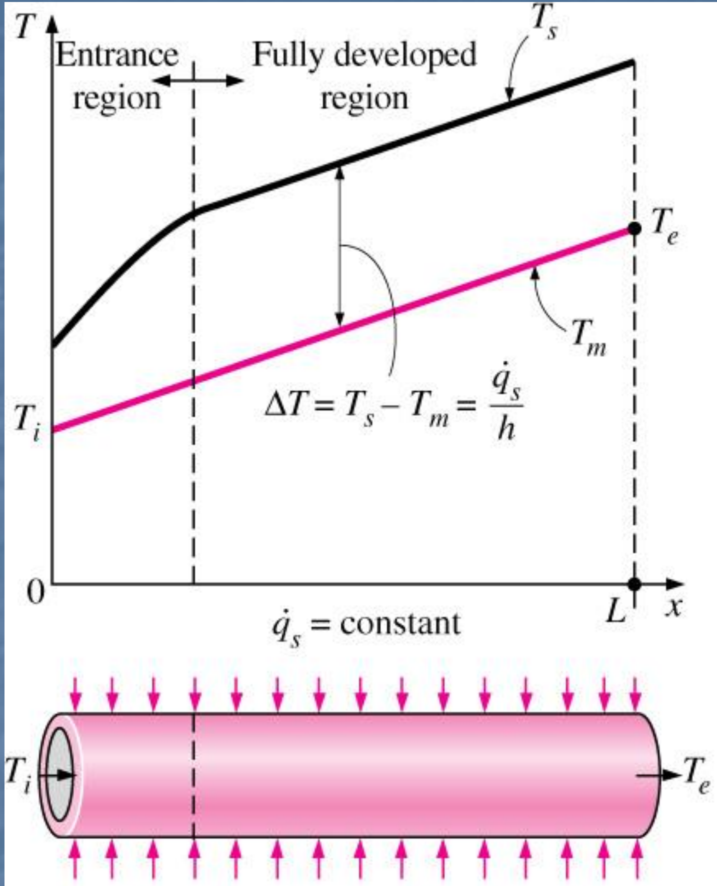


FIGURE 8–11

Variation of the *tube surface* and the *mean fluid* temperatures along the tube for the case of constant surface heat flux.

Constant Surface Temperature (T_s = constant)

From Newton's law of cooling, the rate of heat transfer to or from a fluid flowing in a tube:

$$\dot{Q} = hA_s\Delta T_{\text{ave}} = hA_s(T_s - T_m)_{\text{ave}} \qquad \text{(W)}$$

h : the average convection heat transfer coefficient

A_s : the heat transfer surface area (it is equal to πDL for a circular pipe of length L)

T_{ave} : some appropriate average temperature difference between the fluid and the surface.

$T_b = (T_i + T_e)/2$ (the bulk mean fluid temperature)

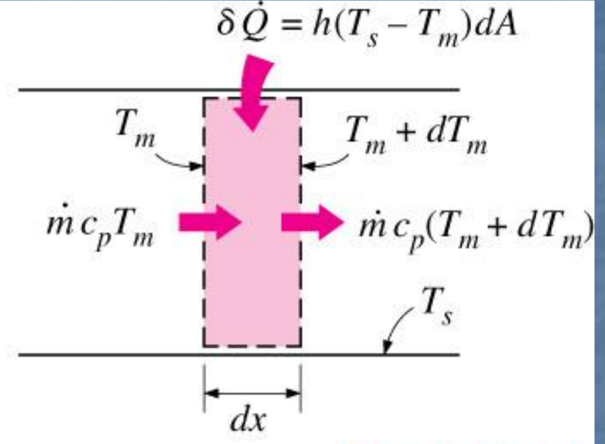


FIGURE 8–12

Energy interactions for a differential control volume in a tube.

The energy balance on a differential control volume:

$$\dot{Q} = \dot{m}C_p(T_e - T_i) \qquad \text{(W)}$$

and

$$\dot{Q} = hA_s\Delta T_{\ln}$$

The *mean fluid temperature at the tube exit:* $T_e = T_s - (T_s - T_i)\exp(-hA_s/\dot{m}c_p)$

The logarithmic mean temperature difference:

$$\Delta T_{\ln} = \frac{T_i - T_e}{\ln[(T_s - T_e)/(T_s - T_i)]} = \frac{\Delta T_e - \Delta T_i}{\ln(\Delta T_e/\Delta T_i)}$$

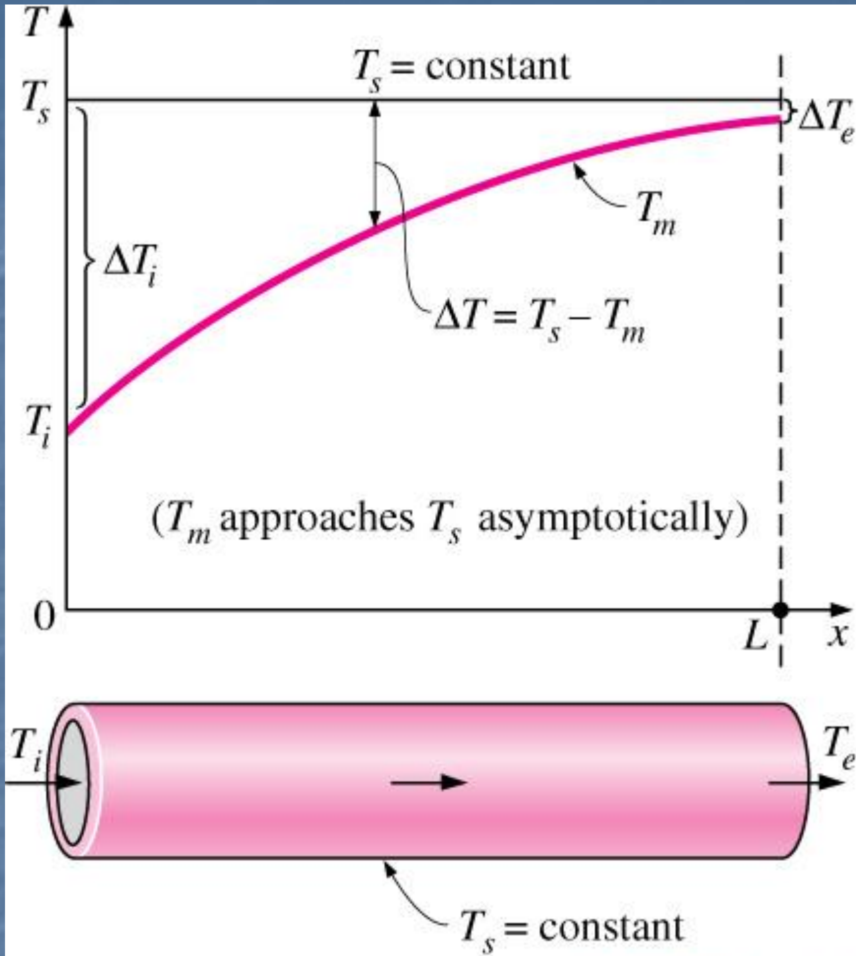
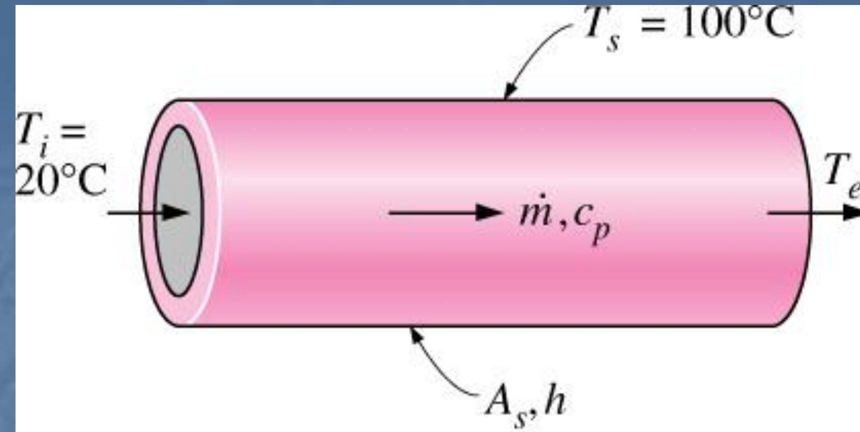


FIGURE 8–14

The variation of the *mean fluid* temperature along the tube for the case of constant temperature.



NTU = $hA_s / \dot{m}c_p$	T_e, °C
0.01	20.8
0.05	23.9
0.10	27.6
0.50	51.5
1.00	70.6
5.00	99.5
10.00	100.0

FIGURE 8–15

An NTU greater than 5 indicates that the fluid flowing in a tube will reach the surface temperature at the exit regardless of the inlet temperature.

Laminar Flow in Tubes

$$\Delta P_L = f\frac{L}{D}\frac{\rho V_{avg}^2}{2}$$

$$f = \frac{8\tau_w}{\rho V_{avg}^2}$$

Circular tube, laminar: $$f = \frac{64\mu}{\rho D V_{avg}} = \frac{64}{\text{Re}}$$

$$h_L = \frac{\Delta P_L}{\rho g} = f\frac{L}{D}\frac{V_{avg}^2}{2g}$$

$$\dot{W}_{\text{pump}, L} = \dot{V}\,\Delta P_L = \dot{V}\rho g h_L = \dot{m} g h_L$$

Horizontal tube: $$V_{avg} = \frac{(P_1 - P_2)R^2}{8\mu L} = \frac{(P_1 - P_2)D^2}{32\mu L} = \frac{\Delta P\, D^2}{32\mu L}$$

$$\dot{V} = V_{avg} A_c = \frac{(P_1 - P_2)R^2}{8\mu L}\pi R^2 = \frac{(P_1 - P_2)\pi D^4}{128\mu L} = \frac{\Delta P\, \pi D^4}{128\mu L}$$

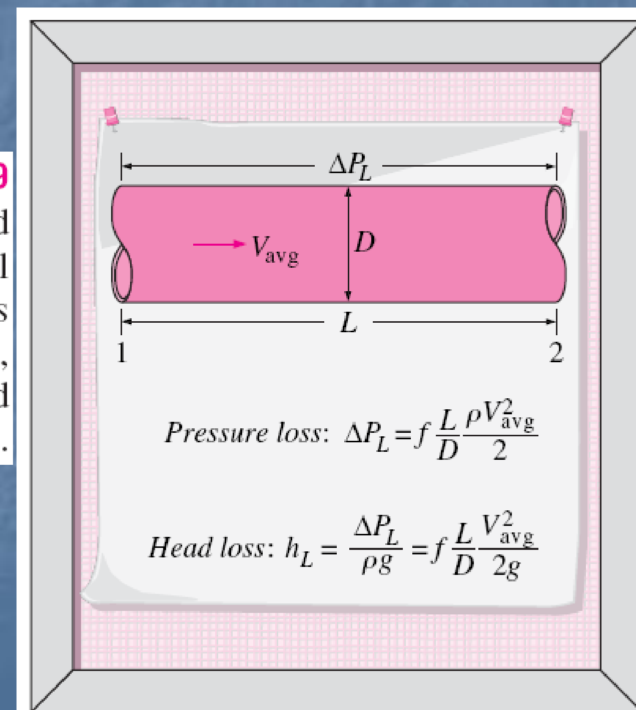


FIGURE 8–19
The relation for pressure loss (and head loss) is one of the most general relations in fluid mechanics, and it is valid for laminar or turbulent flows, circular or noncircular tubes, and pipes with smooth or rough surfaces.

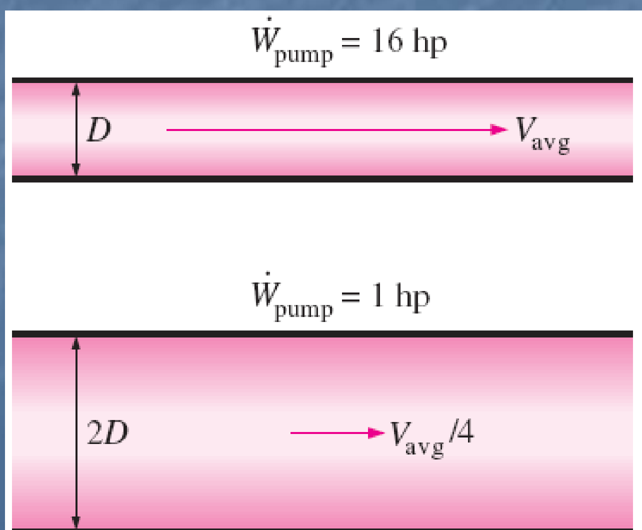


FIGURE 8–20
The pumping power requirement for a laminar flow piping system can be reduced by a factor of 16 by doubling the tube diameter.

Circular tube, laminar ($\dot{q}_s$ = constant):

$$\mathrm{Nu} = \frac{hD}{k} = 4.36$$

Circular tube, laminar (T_s = constant):

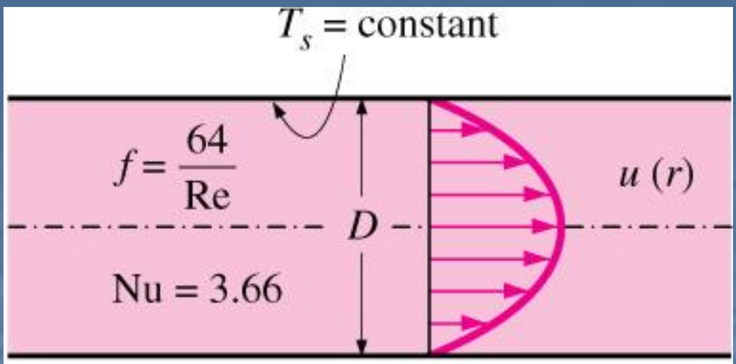
$$\mathrm{Nu} = \frac{hD}{k} = 3.66$$

Entry region, laminar:

$$\mathrm{Nu} = 3.66 + \frac{0.065\,(D/L)\,\mathrm{Re}\,\mathrm{Pr}}{1 + 0.04[(D/L)\,\mathrm{Re}\,\mathrm{Pr}]^{2/3}}$$

$$\mathrm{Nu} = 1.86\left(\frac{\mathrm{Re}\,\mathrm{Pr}\,D}{L}\right)^{1/3}\left(\frac{\mu_b}{\mu_s}\right)^{0.14}$$

$$\mathrm{Nu} = 7.54 + \frac{0.03\,(D_h/L)\,\mathrm{Re}\,\mathrm{Pr}}{1 + 0.016[(D_h/L)\,\mathrm{Re}\,\mathrm{Pr}]^{2/3}}$$



Fully developed laminar flow

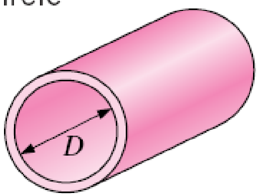
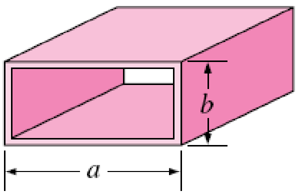
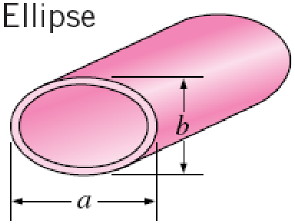
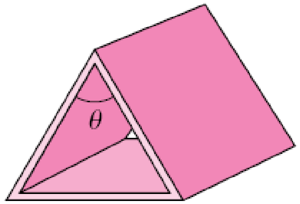
FIGURE 8–22

In laminar flow in a tube with constant surface temperature, both the *friction factor* and the *heat transfer coefficient* remain constant in the fully developed region.

Laminar Flow in Noncircular Tubes

TABLE 8–1

Nusselt number and friction factor for fully developed laminar flow in tubes of various cross sections ($D_h = 4A_c/p$, $\mathrm{Re} = V_{avg}D_h/\nu$, and $\mathrm{Nu} = hD_h/k$)

Tube Geometry	a/b or $\theta°$	Nusselt Number T_s = Const.	Nusselt Number $\dot{q}_s$ = Const.	Friction Factor f
Circle 	—	3.66	4.36	64.00/Re
Rectangle 	a/b			
	1	2.98	3.61	56.92/Re
	2	3.39	4.12	62.20/Re
	3	3.96	4.79	68.36/Re
	4	4.44	5.33	72.92/Re
	6	5.14	6.05	78.80/Re
	8	5.60	6.49	82.32/Re
	∞	7.54	8.24	96.00/Re
Ellipse 	a/b			
	1	3.66	4.36	64.00/Re
	2	3.74	4.56	67.28/Re
	4	3.79	4.88	72.96/Re
	8	3.72	5.09	76.60/Re
	16	3.65	5.18	78.16/Re
Isosceles Triangle 	θ			
	10°	1.61	2.45	50.80/Re
	30°	2.26	2.91	52.28/Re
	60°	2.47	3.11	53.32/Re
	90°	2.34	2.98	52.60/Re
	120°	2.00	2.68	50.96/Re

Turbulent Flow in Tubes

First Petukhov equation

Smooth tubes: $f = (0.790 \ln \mathrm{Re} - 1.64)^{-2}$ $\quad 3000 < \mathrm{Re} < 5 \times 10^6$

Chilton-Colburn analogy $\mathrm{Nu} = 0.125 f \mathrm{Re}\mathrm{Pr}^{1/3}$

Colburn equation $\mathrm{Nu} = 0.023\,\mathrm{Re}^{0.8}\,\mathrm{Pr}^{1/3} \quad \begin{pmatrix} 0.7 \le \mathrm{Pr} \le 160 \\ \mathrm{Re} > 10{,}000 \end{pmatrix}$ $\quad f = 0.184\,\mathrm{Re}^{-0.2}$

Dittus-Boelter equation $\mathrm{Nu} = 0.023\,\mathrm{Re}^{0.8}\,\mathrm{Pr}^{n}$

$n = 0.4$ for *heating* and 0.3 for *cooling*

Sieder and Tate $$\mathrm{Nu} = 0.027\,\mathrm{Re}^{0.8}\mathrm{Pr}^{1/3}\left(\frac{\mu}{\mu_s}\right)^{0.14} \quad \begin{pmatrix} 0.7 \le \mathrm{Pr} \le 17{,}600 \\ \mathrm{Re} \ge 10{,}000 \end{pmatrix}$$

Second Petukhov equation $$\mathrm{Nu} = \frac{(f/8)\,\mathrm{Re}\,\mathrm{Pr}}{1.07 + 12.7(f/8)^{0.5}\,(\mathrm{Pr}^{2/3} - 1)} \quad \begin{pmatrix} 0.5 \le \mathrm{Pr} \le 2000 \\ 10^4 < \mathrm{Re} < 5 \times 10^6 \end{pmatrix}$$

Gnielinski $$\mathrm{Nu} = \frac{(f/8)(\mathrm{Re} - 1000)\,\mathrm{Pr}}{1 + 12.7(f/8)^{0.5}\,(\mathrm{Pr}^{2/3} - 1)} \quad \begin{pmatrix} 0.5 \le \mathrm{Pr} \le 2000 \\ 3 \times 10^3 < \mathrm{Re} < 5 \times 10^6 \end{pmatrix}$$

Bulk mean fluid temperature $T_b = (T_i + T_e)/2$

Liquid metals

The relations given so far do not apply to liquid metals because of their very low Prandtl numbers. For liquid metals ($0.004 < \Pr < 0.01$), the following relations are recommended by Sleicher and Rouse (1975) for $10^4 < \mathrm{Re} < 10^6$:

Liquid metals, T_s = constant: $$\mathrm{Nu} = 4.8 + 0.0156\,\mathrm{Re}^{0.85}\,\mathrm{Pr}_s^{0.93} \qquad \textbf{(8–72)}$$

Liquid metals, $\dot{q}_s$ = constant: $$\mathrm{Nu} = 6.3 + 0.0167\,\mathrm{Re}^{0.85}\,\mathrm{Pr}_s^{0.93} \qquad \textbf{(8–73)}$$

Rough surfaces

$$\frac{1}{\sqrt{f}} = -2.0\log\left(\frac{\varepsilon/D}{3.7} + \frac{2.51}{\mathrm{Re}\sqrt{f}}\right)$$

Colebrook equation

$$\frac{1}{\sqrt{f}} \cong -1.8\log\left[\frac{6.9}{\mathrm{Re}} + \left(\frac{\varepsilon/D}{3.7}\right)^{1.11}\right]$$

Haaland equation

Relative Roughness, ε/D	Friction Factor, f
0.0*	0.0119
0.00001	0.0119
0.0001	0.0134
0.0005	0.0172
0.001	0.0199
0.005	0.0305
0.01	0.0380
0.05	0.0716

*Smooth surface. All values are for $\mathrm{Re} = 10^6$, and are calculated from Eq. 8–74.

FIGURE 8–25
The friction factor is minimum for a smooth pipe and increases with roughness.

Heat Transfer Enhancement

- Tubes with rough surfaces have much higher heat transfer coefficients than tubes with smooth surfaces.
- Tube surfaces are often *roughened*, *corrugated*, or *finned* in order to *enhance* the convection heat transfer coefficient and thus the convection heat transfer rate. Heat transfer in turbulent flow in a tube has been increased by as much as 400% by roughening the surface.
- Roughening the surface also increases the friction factor and thus the power requirement for the pump or the fan.
- The convection heat transfer coefficient can also be increased by inducing pulsating flow by pulse generators, by inducing swirl by inserting a twisted tape into the tube, or by inducing secondary flows by coiling the tube.

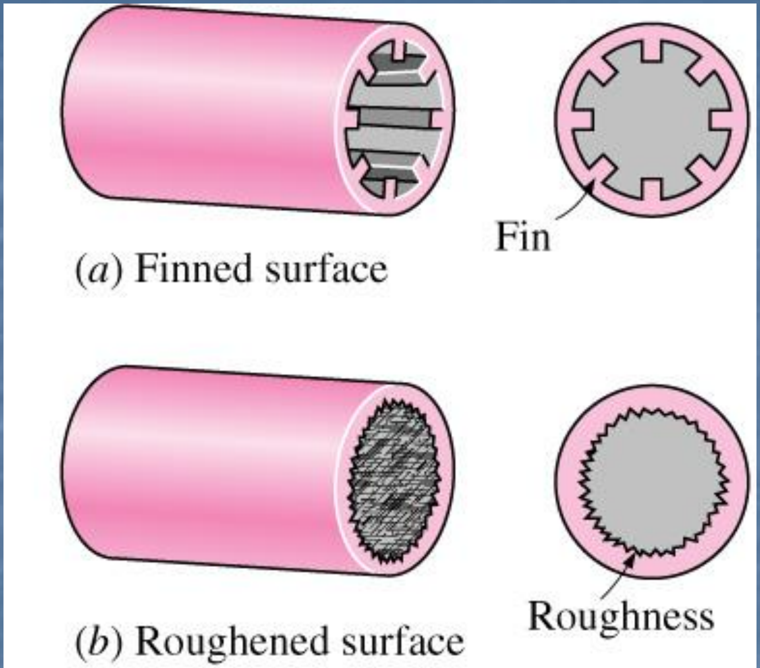
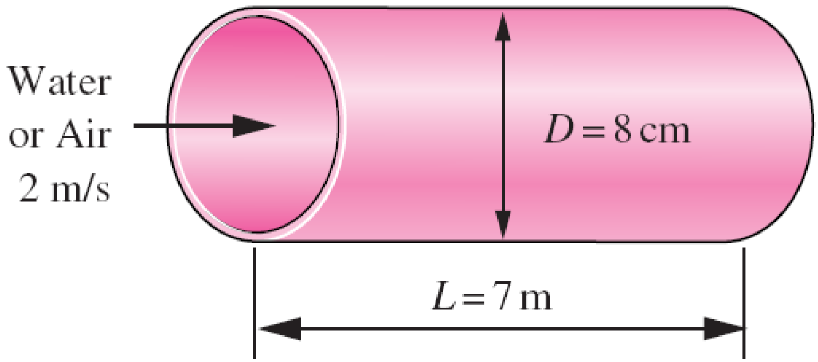


FIGURE 8–28
Tube surfaces are often *roughened*, *corrugated*, or *finned* in order to *enhance* convection heat transfer.

8–39 Determine the convection heat transfer coefficient for the flow of (*a*) air and (*b*) water at a velocity of 2 m/s in an 8-cm-diameter and 7-m-long tube when the tube is subjected to uniform heat flux from all surfaces. Use fluid properties at 25°C.



Properties The properties of air at 25°C are (Table A-15)

$$k = 0.02551 \text{ W/m.°C}$$

$$\nu = 1.562 \times 10^{-5} \text{ m}^2/\text{s}$$

$$\text{Pr} = 0.7296$$

The properties of water at 25°C are (Table A-9)

$$\rho = 997 \text{ kg/m}^3$$

$$k = 0.607 \text{ W/m.°C}$$

$$\nu = \mu / \rho = 0.891 \times 10^{-3} / 997 = 8.937 \times 10^{-7} \text{ m}^2/\text{s}$$

$$\text{Pr} = 6.14$$

Analysis The Reynolds number is

$$\text{Re} = \frac{VD}{\nu} = \frac{(2\ \text{m/s})(0.08\ \text{m})}{1.562 \times 10^{-5}\ \text{m}^2/\text{s}} = 10{,}243$$

which is greater than 10,000. Therefore, the flow is turbulent and the entry lengths in this case are roughly

$$L_h \approx L_t \approx 10D = 10(0.08\ \text{m}) = 0.8\ \text{m}$$

which is much shorter than the total length of the tube. Therefore, we can assume fully developed turbulent flow in the entire duct, and determine the Nusselt number from

$$Nu = \frac{hD}{k} = 0.023\,\text{Re}^{0.8}\,\text{Pr}^{0.4} = 0.023(10{,}243)^{0.8}(0.7296)^{0.4} = 32.76$$

Heat transfer coefficient is

$$h = \frac{k}{D}Nu = \frac{0.02551\ \text{W/m.°C}}{0.08\ \text{m}}(32.76) = \mathbf{10.45\ W/m^2.°C}$$

Repeating calculations for water:

$$\text{Re} = \frac{VD}{\nu} = \frac{(2\ \text{m/s})(0.08\ \text{m})}{8.937 \times 10^{-7}\ \text{m}^2/\text{s}} = 179{,}035$$

$$Nu = \frac{hD}{k} = 0.023\,\text{Re}^{0.8}\,\text{Pr}^{0.4} = 0.023(179{,}035)^{0.8}(6.14)^{0.4} = 757.4$$

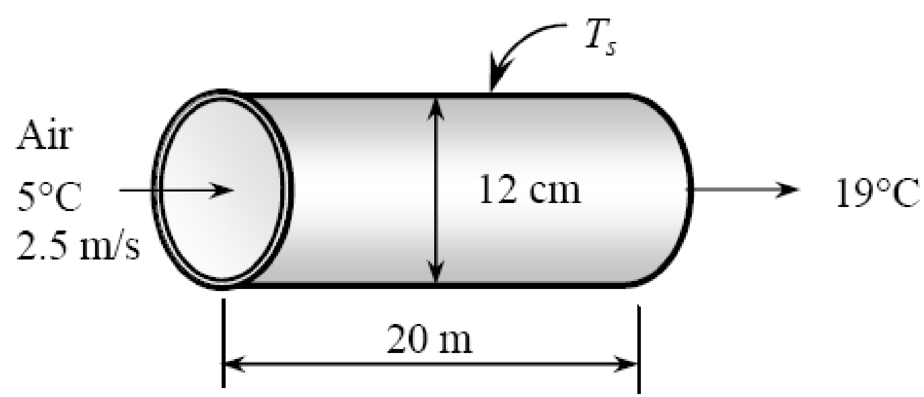
$$h = \frac{k}{D}Nu = \frac{0.607\ \text{W/m.°C}}{0.08\ \text{m}}(757.4) = \mathbf{5747\ W/m^2.°C}$$

Discussion The heat transfer coefficient for water is 550 times that of air.

8–88 Cold air at 5°C enters a 12-cm-diameter 20-m-long isothermal pipe at a velocity of 2.5 m/s and leaves at 19°C. Estimate the surface temperature of the pipe.

Properties The properties of air at 1 atm and the bulk mean temperature of (5+19)/2=12°C are (Table A-15)

$$\rho = 1.238\ \text{kg/m}^3$$
$$k = 0.02454\ \text{W/m.°C}$$
$$\nu = 1.444\times10^{-5}\ \text{m}^2\text{/s}$$
$$c_p = 1007\ \text{J/kg.°C}$$
$$\text{Pr} = 0.7331$$



Analysis The rate of heat transfer to the air is

$$\dot{m} = \rho A_c V_{\text{avg}} = (1.238\ \text{kg/m}^3)\pi\frac{(0.12\ \text{m})^2}{4}(2.5\ \text{m/s}) = 0.0350\ \text{m/s}$$

$$\dot{Q} = \dot{m}c_p\Delta T = (0.0350\ \text{kg/s})(1007\ \text{J/kg.°C})(19-5)\text{°C} = 493.1\ \text{W}$$

Reynolds number is

$$\mathrm{Re} = \frac{V_{\mathrm{avg}} D}{\nu} = \frac{(2.5\ \mathrm{m/s})(0.12\ \mathrm{m})}{1.444 \times 10^{-5}\ \mathrm{m^2/s}} = 20{,}775$$

which is greater than 10,000. Therefore, the flow is turbulent and the entry lengths in this case are roughly

$$L_h \approx L_t \approx 10D = 10(0.12\ \mathrm{m}) = 1.2\ \mathrm{m}$$

which is much shorter than the total length of the duct. Therefore, we can assume fully developed turbulent flow in the entire duct, and determine the Nusselt number from

$$Nu = \frac{hD}{k} = 0.023\,\mathrm{Re}^{0.8}\,\mathrm{Pr}^{0.4} = 0.023(20{,}775)^{0.8}(0.7331)^{0.4} = 57.79$$

Heat transfer coefficient is

$$h = \frac{k}{D} Nu = \frac{0.02454\ \mathrm{W/m.^\circ C}}{0.12\ \mathrm{m}}(57.79) = 11.82\ \mathrm{W/m^2.^\circ C}$$

The logarithmic mean temperature difference is determined from

$$\dot{Q} = hA_s \Delta T_{\ln} \longrightarrow 493.1\ \mathrm{W} = (11.82\ \mathrm{W/m^2.^\circ C})[\pi(0.12\ \mathrm{m})(20\ \mathrm{m})]\Delta T_{\ln} \longrightarrow \Delta T_{\ln} = 5.533^\circ\mathrm{C}$$

Then the pipe temperature is determined from the definition of the logarithmic mean temperature difference

$$\Delta T_{\ln} = \frac{T_e - T_i}{\ln\left(\dfrac{T_s - T_e}{T_s - T_i}\right)} \longrightarrow 5.533^\circ\mathrm{C} = \frac{19 - 5}{\ln\left(\dfrac{T_s - 19}{T_s - 5}\right)} \longrightarrow T_s = \mathbf{3.8^\circ C}$$

Conclusions

- Average velocity and temperature
- The entrance region
- General Thermal Analysis
- Laminar Flow in Tubes
- Turbulent Flow in Tubes

HEAT AND MASS TRANSFER

Natural Convection

OBJECTIVES

When you finish studying this chapter, you should be able to:

- Understand the physical mechanism of natural convection,
- Derive the governing equations of natural convection, and obtain the dimensionless Grashof number by nondimensionalizing them,
- Evaluate the Nusselt number for natural convection associated with vertical, horizontal, and inclined plates as well as cylinders and spheres,
- Examine natural convection from finned surfaces, and determine the optimum fin spacing,
- Analyze natural convection inside enclosures such as double-pane windows, and
- Consider combined natural and forced convection, and assess the relative importance of each mode.

PHYSICAL MECHANISM OF NATURAL CONVECTION

Natural Convection Heat Transfer: heat transfer as a result of this natural convection current.

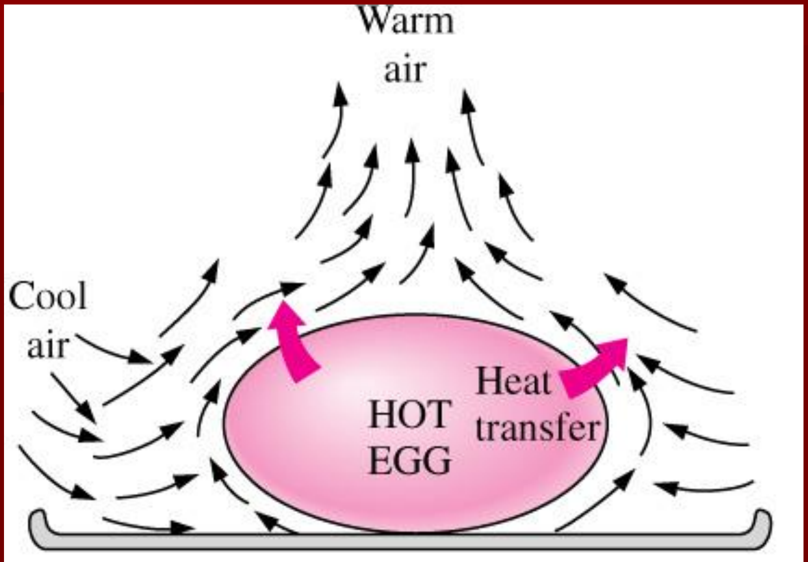


FIGURE 9–1
The cooling of a boiled egg in a cooler environment by natural convection.

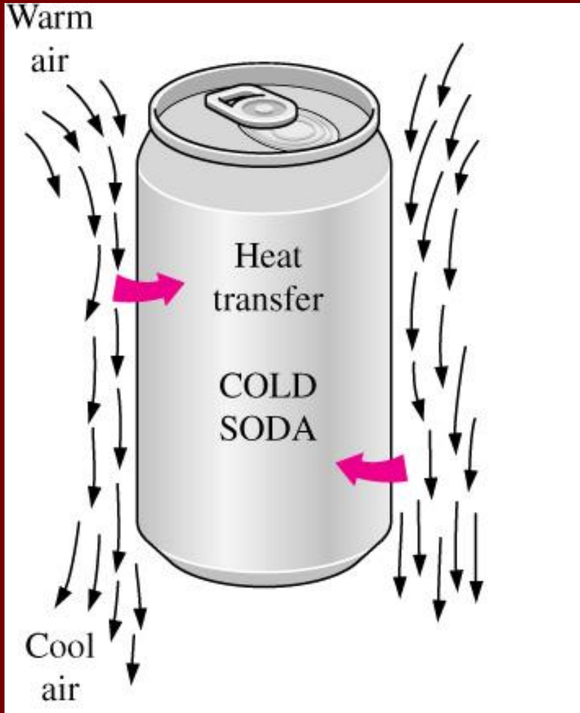


FIGURE 9–2
The warming up of a cold drink in a warmer environment by natural convection.

The magnitude of the buoyancy force is equal to the weight of the *fluid displaced* by the body.

$$F_{\text{buoyancy}} = \rho_{\text{fluid}}\, g V_{\text{body}}$$

$$\begin{aligned} F_{\text{net}} &= W - F_{\text{buoyancy}} \\ &= \rho_{\text{body}}\, g V_{\text{body}} - \rho_{\text{fluid}}\, g V_{\text{body}} \\ &= (\rho_{\text{body}} - \rho_{\text{fluid}})\, g V_{\text{body}} \end{aligned}$$

The volume expansion coefficient (β):

$$\beta = \frac{1}{\upsilon}\left(\frac{\partial \upsilon}{\partial T}\right)_P = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_P \qquad (1/\mathrm{K})$$

$$\beta \approx -\frac{1}{\rho}\frac{\Delta \rho}{\Delta T} = -\frac{1}{\rho}\frac{\rho_\infty - \rho}{T_\infty - T} \qquad (\text{at constant } P)$$

$$\rho_\infty - \rho = \rho\beta(T - T_\infty) \qquad (\text{at constant } P)$$

$$\beta_{\text{ideal gas}} = \frac{1}{T} \qquad (1/\mathrm{K})$$

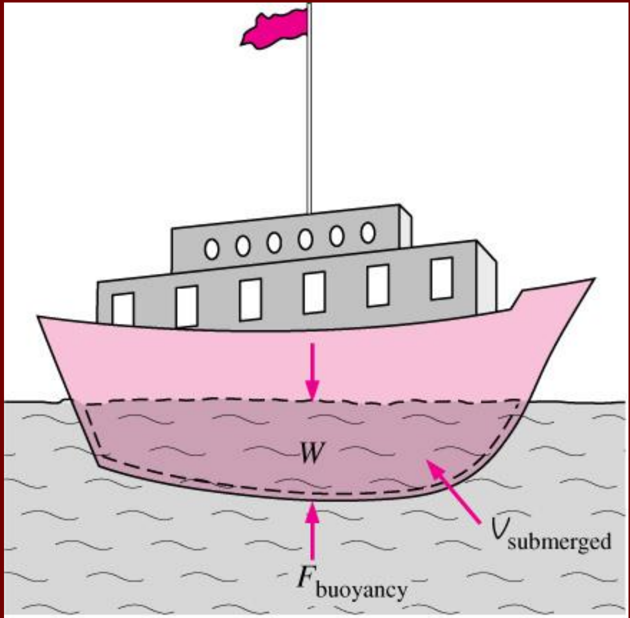


FIGURE 9–3
It is the buoyancy force that keeps the ships afloat in water ($W = F_{\text{buoyancy}}$ for floating objects).

The buoyancy force is proportional to the *density difference*, which is proportional to the *temperature difference* at constant pressure.

The *larger* the buoyancy force and the *stronger* the natural convection currents, and thus the *higher* the heat transfer rate.

Whenever two bodies in contact (solid-solid, solid-fluid, or fluid-fluid) move relative to each other, a *friction force* develops at the contact surface in the direction opposite to that of the motion.

The Grashof Number (Gr)

An arbitrary reference velocity: $\mathcal{V} = \mathrm{Re}_L\, \upsilon / L_c$

The Grashof number (Gr_L):
showing the effect of NC

$$\mathrm{Gr}_L = \frac{g\beta(T_s - T_\infty)L_c^3}{\upsilon^2}$$

g = gravitational acceleration, m/s^2
β = coefficient of volume expansion, 1/K ($\beta = 1/T$ for ideal gases)
T_s = temperature of the surface, °C
T_∞ = temperature of the fluid sufficiently far from the surface, °C
L_c = characteristic length of the geometry, m
υ = kinematic viscosity of the fluid, m^2/s

For vertical plates, the critical Gr is observed to be about 10^9.

The flow regime on a vertical plate becomes *turbulent* at Gr greater than 10^9.

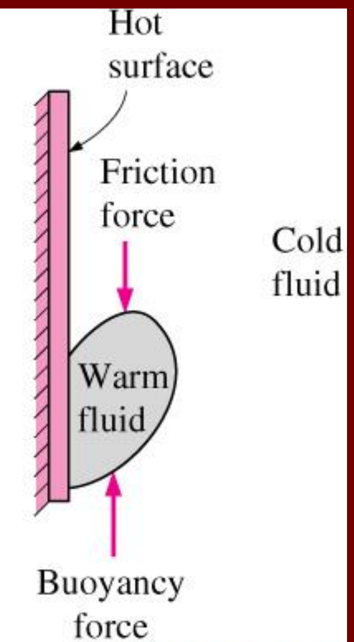


FIGURE 9–8
The Grashof number Gr is a measure of the relative magnitudes of the *buoyancy force* and the opposing *viscous force* acting on the fluid.

Natural convection effects are negligible if $\mathrm{Gr}_L/\mathrm{Re}_L^2 \ll 1$

Free convection dominates and the forced convection effects are negligible if $\mathrm{Gr}_L/\mathrm{Re}_L^2 \gg 1$

Both effects are significant and must be considered if $\mathrm{Gr}_L/\mathrm{Re}_L^2 \approx 1$

NATURAL CONVECTION OVER SURFACES

The average Nusselt number (Nu): $\mathrm{Nu} = \dfrac{hL_c}{k} = C(\mathrm{Gr}_L\,\mathrm{Pr})^n = C\,\mathrm{Ra}_L^n$

The **Rayleigh number (RaL)**: $\mathrm{Ra}_L = \mathrm{Gr}_L\,\mathrm{Pr} = \dfrac{g\beta(T_s - T_\infty)L_c^3}{\nu^2}\,\mathrm{Pr}$ $\quad T_f = \tfrac{1}{2}(T_s + T_\infty)$

n= 1/4 for laminar flow & n=1/3 for turbulent flow

C < 1.

$$\dot{Q}_{\text{conv}} = hA_s(T_s - T_\infty) \qquad (\mathrm{W})$$

Vertical Plates (T_s = constant): Table 20.1

$$10^{-1} < \mathrm{Ra}_L < 10^9$$

Vertical Plates **($\dot{q}_s$ = constant)**: $\dot{Q} = \dot{q}_s A_s$

$$h = \dot{q}_s/(T_{L/2} - T_\infty) \qquad \mathrm{Nu} = \frac{hL}{k} = \frac{\dot{q}_s L}{k(T_{L/2} - T_\infty)}$$

Vertical Cylinders: $D \geq \dfrac{35L}{\mathrm{Gr}_L^{1/4}}$

Horizontal Plates: $L_c = \dfrac{A_s}{p}$

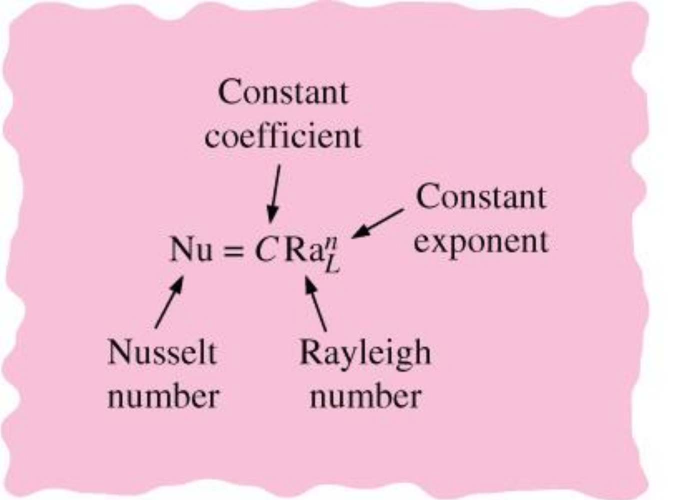
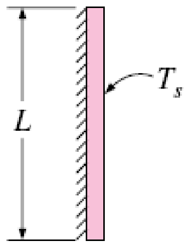
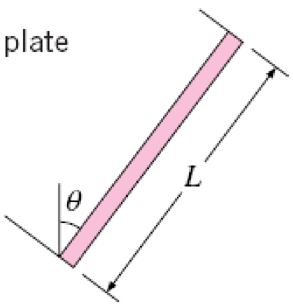
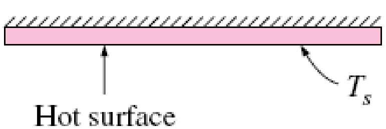
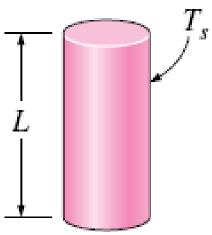
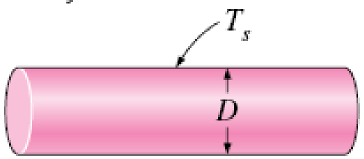
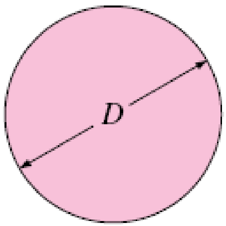


FIGURE 9–9
Natural convection heat transfer correlations are usually expressed in terms of the Rayleigh number raised to a constant *n* multiplied by another constant *C*, both of which are determined experimentally.

TABLE 9–1

Empirical correlations for the average Nusselt number for natural convection over surfaces

Geometry	Characteristic length L_c	Range of Ra	Nu
Vertical plate 	L	10^4–10^9 10^{20}–10^{13} Entire range	$\text{Nu} = 0.59\text{Ra}_L^{1/4}$ **(9–19)** $\text{Nu} = 0.1\text{Ra}_L^{1/3}$ **(9–20)** $\text{Nu} = \left\{0.825 + \dfrac{0.387\text{Ra}_L^{1/6}}{[1 + (0.492/\text{Pr})^{9/16}]^{8/27}}\right\}^2$ **(9–21)** (complex but more accurate)
Inclined plate 	L		Use vertical plate equations for the upper surface of a cold plate and the lower surface of a hot plate Replace g by $g \cos\theta$ for $\text{Ra} < 10^9$
Horizontal plate (Surface area A and perimeter p) (*a*) Upper surface of a hot plate (or lower surface of a cold plate)  (*b*) Lower surface of a hot plate (or upper surface of a cold plate) 	A_s/p	10^4–10^7 10^7–10^{11} 10^5–10^{11}	$\text{Nu} = 0.54\text{Ra}_L^{1/4}$ **(9–22)** $\text{Nu} = 0.15\text{Ra}_L^{1/3}$ **(9–23)** $\text{Nu} = 0.27\text{Ra}_L^{1/4}$ **(9–24)**

Vertical cylinder 	L		A vertical cylinder can be treated as a vertical plate when $D \geq \dfrac{35L}{Gr_L^{1/4}}$
Horizontal cylinder 	D	$Ra_D \leq 10^{12}$	$Nu = \left\{0.6 + \dfrac{0.387Ra_D^{1/6}}{[1 + (0.559/Pr)^{9/16}]^{8/27}}\right\}^2$ **(9–25)**
Sphere 	D	$Ra_D \leq 10^{11}$ $(Pr \geq 0.7)$	$Nu = 2 + \dfrac{0.589Ra_D^{1/4}}{[1 + (0.469/Pr)^{9/16}]^{4/9}}$ **(9–26)**

9–19 A 10-m-long section of a 6-cm-diameter horizontal hot-water pipe passes through a large room whose temperature is 27°C. If the temperature and the emissivity of the outer surface of the pipe are 73°C and 0.8, respectively, determine the rate of heat loss from the pipe by (*a*) natural convection and (*b*) radiation.

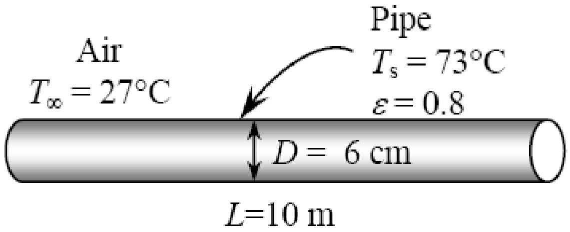
Properties The properties of air at 1 atm and the film temperature of $(T_s+T_\infty)/2 = (73+27)/2 = 50°C$ are (Table A-15)

$$k = 0.02735 \text{ W/m.°C}$$

$$\nu = 1.798\times10^{-5} \text{ m}^2\text{/s}$$

$$\text{Pr} = 0.7228$$

$$\beta = \frac{1}{T_f} = \frac{1}{(50+273)\text{K}} = 0.003096 \text{ K}^{-1}$$



Analysis (*a*) The characteristic length in this case is the outer diameter of the pipe, $L_c = D = 0.06$ m. Then,

$$Ra = \frac{g\beta(T_s - T_\infty)D^3}{\nu^2}\text{Pr} = \frac{(9.81 \text{ m/s}^2)(0.003096 \text{ K}^{-1})(73-27 \text{ K})(0.06 \text{ m})^3}{(1.798\times10^{-5} \text{ m}^2\text{/s})^2}(0.7228) = 6.747\times10^5$$

$$Nu = \left\{0.6 + \frac{0.387Ra^{1/6}}{\left[1+(0.559/\text{Pr})^{9/16}\right]^{8/27}}\right\}^2 = \left\{0.6 + \frac{0.387(6.747\times10^5)^{1/6}}{\left[1+(0.559/0.7228)^{9/16}\right]^{8/27}}\right\}^2 = 13.05$$

$$h = \frac{k}{D}Nu = \frac{0.02735 \text{ W/m.°C}}{0.06 \text{ m}}(13.05) = 5.950 \text{ W/m}^2\text{.°C}$$

$$A_s = \pi DL = \pi(0.06 \text{ m})(10 \text{ m}) = 1.885 \text{ m}^2$$

$$\dot{Q} = hA_s(T_s - T_\infty) = (5.950 \text{ W/m}^2\text{.°C})(1.885 \text{ m}^2)(73-27)\text{°C} = \mathbf{516\ W}$$

(*b*) The radiation heat loss from the pipe is

$$\dot{Q}_{rad} = \varepsilon A_s\sigma(T_s^{\,4} - T_{surr}^{\,4})$$
$$= (0.8)(1.885 \text{ m}^2)(5.67\times10^{-8} \text{ W/m}^2\text{.K}^4)\left[(73+273 \text{ K})^4 - (27+273 \text{ K})^4\right] = \mathbf{533\ W}$$

NATURAL CONVECTION FROM FINNED SURFACES AND PCBs

Natural Convection Cooling of Finned Surfaces

(T_s = constant)

PCBs: Printed circuit boards

The Rayleigh number:

$$\mathrm{Ra}_S = \frac{g\beta(T_s - T_\infty)S^3}{\upsilon^2}\mathrm{Pr} \qquad \text{and} \qquad \mathrm{Ra}_L = \frac{g\beta(T_s - T_\infty)L^3}{\upsilon^2}\mathrm{Pr} = \mathrm{Ra}_S\frac{L^3}{S^3}$$

The average Nu for vertical isothermal parallel plates:

$$T_s = \text{constant:} \qquad \mathrm{Nu} = \frac{hS}{k} = \left[\frac{576}{(\mathrm{Ra}_S S/L)^2} + \frac{2.873}{(\mathrm{Ra}_S S/L)^{0.5}}\right]^{-0.5}$$

The optimum fin spacing for a vertical heat sink:

$$T_s = \text{constant:} \qquad S_{\mathrm{opt}} = 2.714\left(\frac{S^3L}{\mathrm{Ra}_S}\right)^{0.25} = 2.714\frac{L}{\mathrm{Ra}_L^{0.25}}$$

$$S = S_{\mathrm{opt}}: \qquad \mathrm{Nu} = \frac{hS_{\mathrm{opt}}}{k} = 1.307$$

$$\dot{Q} = h(2nLH)(T_s - T_\infty) \qquad n = W/(S + t) \approx W/S$$

$$T_{\mathrm{ave}} = (T_s + T_\infty)/2$$

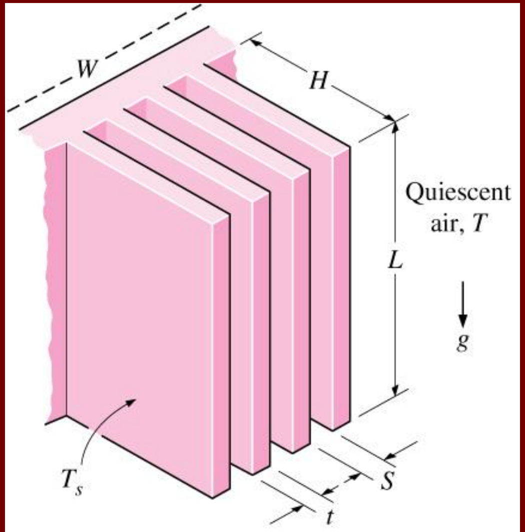


FIGURE 9–18
Various dimensions of a finned surface oriented vertically.

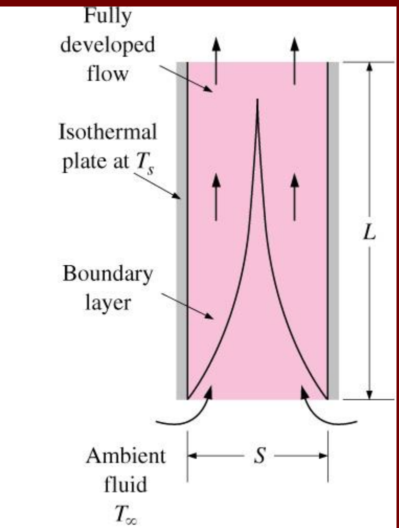


FIGURE 9–16
Natural convection flow through a channel between two isothermal vertical plates.

Natural Convection Cooling of Vertical PCBs ($\dot{q}_s$ = constant)

The modified Rayleigh number for uniform heat flux on both plates:

$$\mathrm{Nu}_L = \frac{h_L S}{k} = \left[\frac{48}{\mathrm{Ra}_L^* S/L} + \frac{2.51}{(\mathrm{Ra}_L^* S/L)^{0.4}}\right]^{-0.5}$$

The optimum fin spacing for the case of uniform heat flux on both plates:

$$\dot{q}_s = \text{constant:} \qquad S_{\text{opt}} = 2.12\left(\frac{S^4 L}{\mathrm{Ra}_L^*}\right)^{0.2}$$

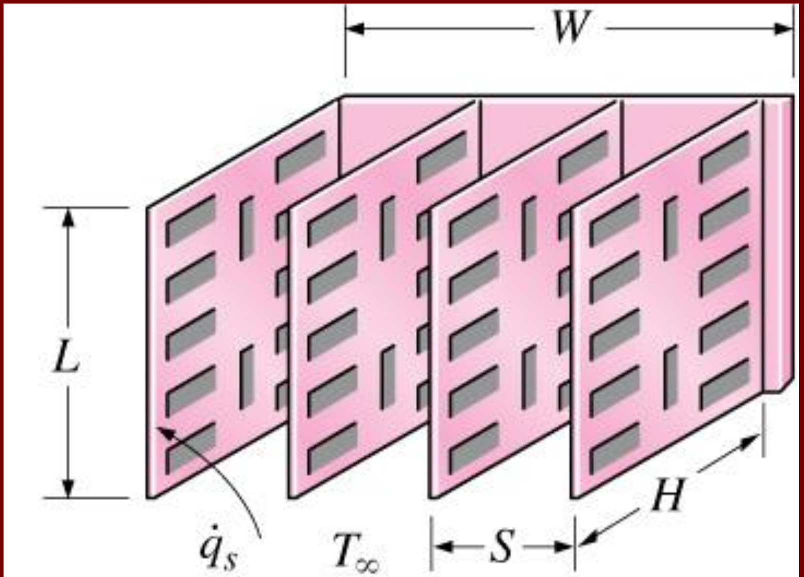


FIGURE 9–19
Arrays of vertical printed circuit boards (PCBs) cooled by natural convection.

The total rate of heat transfer from the plates:

$$\dot{Q} = \dot{q}_s A_s = \dot{q}_s(2nLH)$$

$$n = W/(S + t) \approx W/S$$

$$\dot{q}_s = h_L(T_L - T_\infty) \qquad T_{\text{ave}} = (T_L + T_\infty)/2.$$

NATURAL CONVECTION INSIDE ENCLOSURES

The Rayleigh number: $$\mathrm{Ra}_L = \frac{g\beta(T_1 - T_2)L_c^3}{\upsilon^2}\mathrm{Pr}$$ $$T_{\mathrm{ave}} = (T_1 + T_2)/2.$$

L_c: the characteristic length (the distance between the hot and cold surfaces)
T_1 : the temperature of the hot surface
T_2 : the temperature of the cold surface

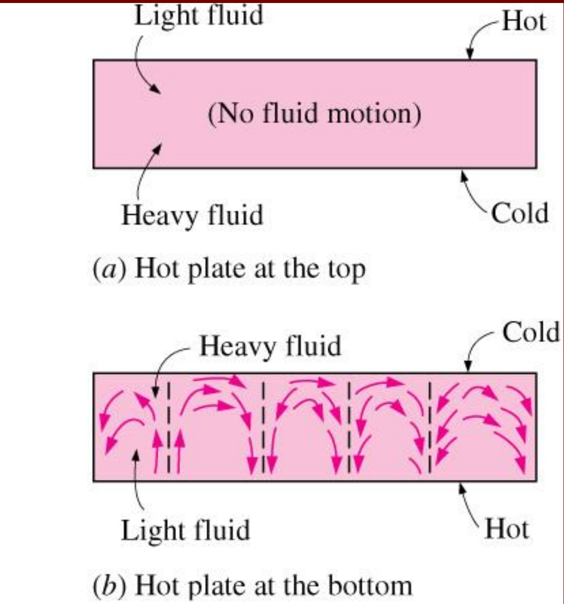


FIGURE 9–22
Convective currents in a horizontal enclosure with (*a*) hot plate at the top and (*b*) hot plate at the bottom.

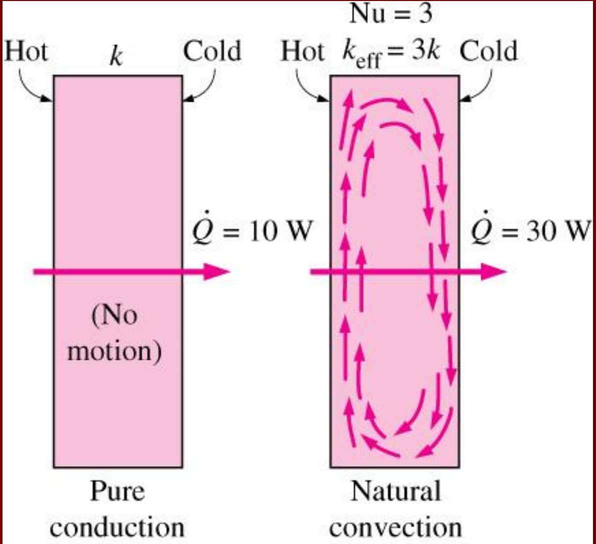


FIGURE 9–23
A Nusselt number of 3 for an enclosure indicates that heat transfer through the enclosure by *natural convection* is three times that by *pure conduction.*

$$\dot{Q} = hA_s(T_1 - T_2) = k\mathrm{Nu}A_s\frac{T_1 - T_2}{L_c}$$

$$h = k\mathrm{Nu}/L$$

$$\dot{Q}_{\mathrm{cond}} = kA_s\frac{T_1 - T_2}{L_c}$$

The effective thermal conductivity of the enclosure:

$$k_{\mathrm{eff}} = k\mathrm{Nu}$$

For the special case of Nu= 1, the effective thermal conductivity of the enclosure becomes equal to the conductivity of the fluid.

Horizontal Rectangular Enclosures

$$\mathrm{Nu} = 0.195\mathrm{Ra}_L^{1/4} \qquad 10^4 < \mathrm{Ra}_L < 4 \times 10^5$$

$$\mathrm{Nu} = 0.068\mathrm{Ra}_L^{1/3} \qquad 4 \times 10^5 < \mathrm{Ra}_L < 10^7$$

$$\mathrm{Nu} = 0.069\mathrm{Ra}_L^{1/3}\,\mathrm{Pr}^{0.074} \qquad 3 \times 10^5 < \mathrm{Ra}_L < 7 \times 10^9$$

$$\mathrm{Nu} = 1 + 1.44\left[1 - \frac{1708}{\mathrm{Ra}_L}\right]^+ + \left[\frac{\mathrm{Ra}_L^{1/3}}{18} - 1\right]^+ \qquad \mathrm{Ra}_L < 10^8$$

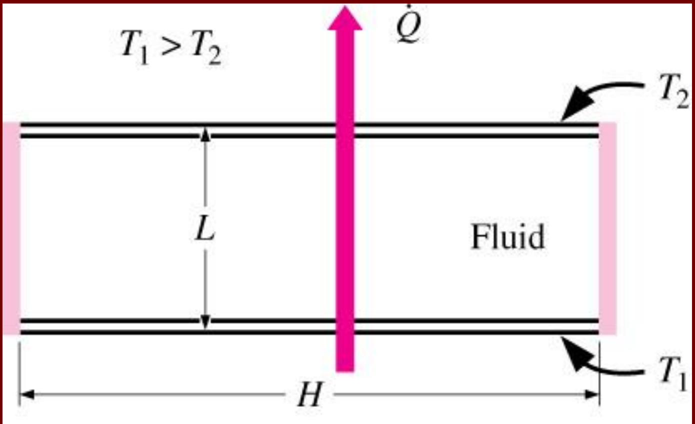


FIGURE 9–24
A horizontal rectangular enclosure with isothermal surfaces.

Inclined Rectangular Enclosures

TABLE 9–2
Critical angles for inclined rectangular enclosures

Aspect ratio, H/L	Critical angle, θ_{cr}
1	25°
3	53°
6	60°
12	67°
>12	70°

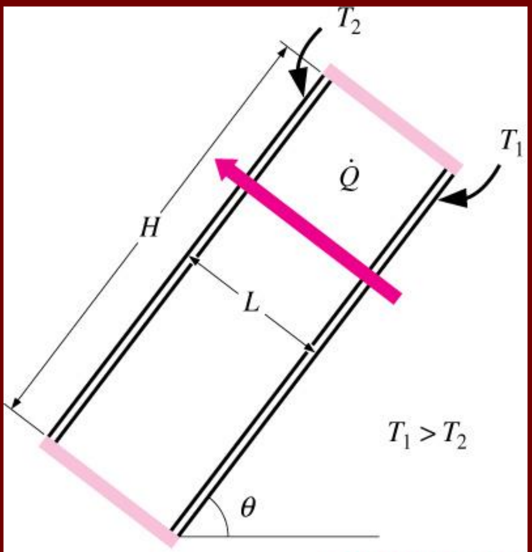


FIGURE 9–25
An inclined rectangular enclosure with isothermal surfaces.

Vertical Rectangular Enclosures

$$\text{Nu} = 0.18\left(\frac{\text{Pr}}{0.2+\text{Pr}}\text{Ra}_L\right)^{0.29}$$ $1 < H/L < 2$, any Prandtl number, $\text{Ra}_L\,\text{Pr}/(0.2+\text{Pr}) > 10^3$

$$\text{Nu} = 0.22\left(\frac{\text{Pr}}{0.2+\text{Pr}}\text{Ra}_L\right)^{0.28}\left(\frac{H}{L}\right)^{-1/4}$$ $2 < H/L < 10$, any Prandtl number, $\text{Ra}_L < 10^{10}$

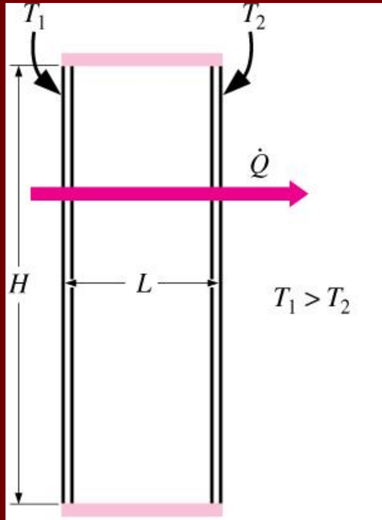


FIGURE 9–26
A vertical rectangular enclosure with isothermal surfaces.

For vertical enclosures with larger aspect ratios:

$$\text{Nu} = 0.42\text{Ra}_L^{1/4}\,\text{Pr}^{0.012}\left(\frac{H}{L}\right)^{-0.3}$$ $10 < H/L < 40$, $1 < \text{Pr} < 2\times10^4$, $10^4 < \text{Ra}_L < 10^7$

$$\text{Nu} = 0.46\text{Ra}_L^{1/3}$$ $1 < H/L < 40$, $1 < \text{Pr} < 20$, $10^6 < \text{Ra}_L < 10^9$

at $(T_1 + T_2)/2$

Concentric Cylinders

$$\dot{Q} = \frac{2\pi k_{\text{eff}}}{\ln(D_o/D_i)}(T_i - T_o) \qquad \text{(W/m)}$$

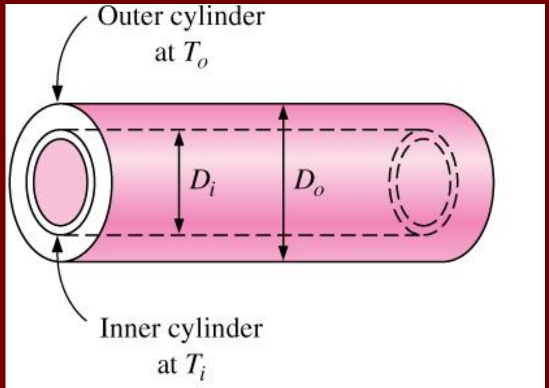


FIGURE 9–27
Two concentric horizontal isothermal cylinders.

The effective thermal conductivity:

$$\frac{k_{\text{eff}}}{k} = 0.386\left(\frac{\text{Pr}}{0.861+\text{Pr}}\right)^{1/4}(F_{\text{cyl}}\text{Ra}_L)^{1/4}$$

$0.70 \le \text{Pr} \le 6000$

$10^2 \le F_{\text{cyl}}\text{Ra}_L \le 10^7$

For $F_{\text{cyl}}\text{Ra}_L < 100$: $k_{\text{eff}} = k$

The geometric factor for concentric cylinders F_{cyl}:

$$F_{\text{cyl}} = \frac{[\ln(D_o/D_i)]^4}{L_c^3(D_i^{-3/5} + D_o^{-3/5})^5}$$

Concentric Spheres

$$\dot{Q} = k_{\text{eff}}\pi\left(\frac{D_i D_o}{L_c}\right)(T_i - T_o) \qquad \text{(W)}$$

$$L_c = (D_o - D_i)/2$$

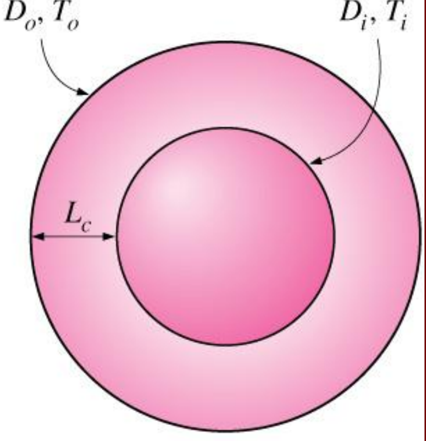


FIGURE 9–28
Two concentric isothermal spheres.

The recommended relation for effective thermal conductivity:

$$\frac{k_{\text{eff}}}{k} = 0.74\left(\frac{\text{Pr}}{0.861 + \text{Pr}}\right)^{1/4}(F_{\text{sph}}\text{Ra}_L)^{1/4}$$

$$0.70 \leq \text{Pr} \leq 4200$$

$$10^2 \leq F_{\text{sph}}\text{Ra}_L \leq 10^4$$

For $k_{\text{eff}}/k < 1$: $k_{\text{eff}} = k$

The geometric factor for concentric spheres (F_{sph}): $F_{\text{sph}} = \dfrac{L_c}{(D_i D_o)^4(D_i^{-7/5} + D_o^{-7/5})^5}$

Combined Natural Convection and Radiation

$$\dot{Q}_{\text{total}} = \dot{Q}_{\text{conv}} + \dot{Q}_{\text{rad}}$$

$$\dot{Q}_{\text{rad}} = \varepsilon\sigma A_s(T_s^4 - T_{\text{surr}}^4)$$

$$\sigma = 5.67 \times 10^{-8}\ \text{W/m}^2 \cdot \text{K}^4$$

$$\dot{Q}_{\text{rad}} = \frac{\pi A_s(T_1^4 - T_2^4)}{1/\varepsilon_1 + 1/\varepsilon_2 - 1} = \varepsilon_{\text{effective}}\,\sigma A_s(T_1^4 - T_2^4) \qquad \text{(W)}$$

ε_1 and ε_2 are the emissivities of the plates, and effective is the *effective emissivity* defined as:

$$\varepsilon_{\text{effective}} = \frac{1}{1/\varepsilon_1 + 1/\varepsilon_2 - 1}$$

9–70 Two concentric spheres of diameters 15 cm and 25 cm are separated by air at 1 atm pressure. The surface temperatures of the two spheres enclosing the air are T_1 = 350 K and T_2 = 275 K, respectively. Determine the rate of heat transfer from the inner sphere to the outer sphere by natural convection.

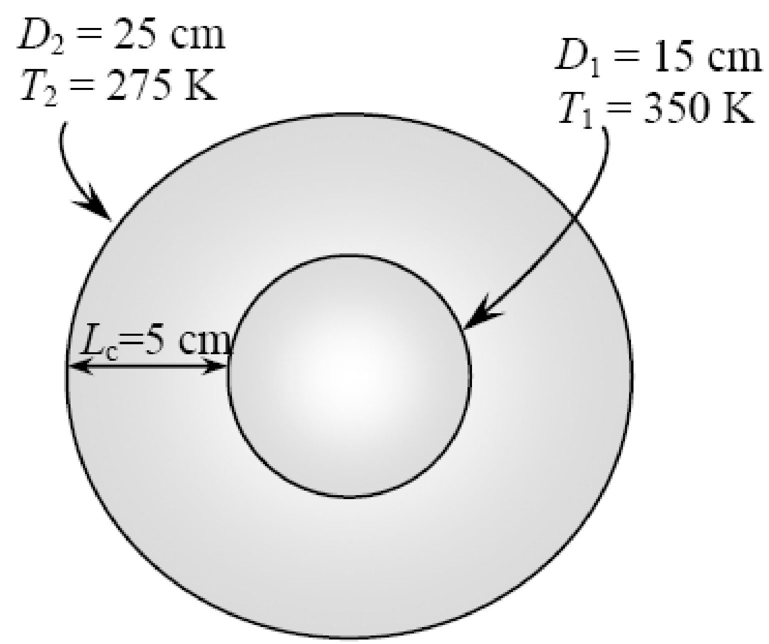
Properties The properties of air at 1 atm and the average temperature of $(T_1+T_2)/2 = (350+275)/2 = 312.5$ K = 39.5°C are (Table A-15)

$$k = 0.02658 \text{ W/m.°C}$$

$$\nu = 1.697 \times 10^{-5} \text{ m}^2/\text{s}$$

$$\Pr = 0.7256$$

$$\beta = \frac{1}{T_f} = \frac{1}{312.5 \text{ K}} = 0.003200 \text{ K}^{-1}$$



Analysis The characteristic length in this case is determined from

$$L_c = \frac{D_2 - D_1}{2} = \frac{25-15}{2} = 5 \text{ cm}.$$

Then,

$$Ra = \frac{g\beta(T_1 - T_2)L_c^3}{\nu^2}\Pr = \frac{(9.81 \text{ m/s}^2)(0.003200 \text{ K}^{-1})(350-275 \text{ K})(0.05 \text{ m})^3}{(1.697 \times 10^{-5} \text{ m}^2/\text{s})^2}(0.7256) = 7.415 \times 10^5$$

The effective thermal conductivity is

$$F_{\text{sph}} = \frac{L_c}{(D_i D_o)^4 (D_i^{-7/5} + D_o^{-7/5})^5} = \frac{0.05\text{ m}}{[(0.15\text{ m})(0.25\text{ m})]^4 \left[(0.15\text{ m})^{-7/5} + (0.25\text{ m})^{-7/5}\right]^5} = 0.005900$$

$$k_{\text{eff}} = 0.74k\left(\frac{\Pr}{0.861+\Pr}\right)^{1/4} (F_{sph} Ra)^{1/4}$$

$$= 0.74(0.02658\text{ W/m.°C})\left(\frac{0.7256}{0.861+0.7256}\right)^{1/4} \left[(0.00590)(7.415\times 10^5)\right]^{1/4} = 0.1315\text{ W/m.°C}$$

Then the rate of heat transfer between the spheres becomes

$$\dot{Q} = k_{\text{eff}}\,\pi\left(\frac{D_i D_o}{L_c}\right)(T_i - T_o) = (0.1315\text{ W/m.°C})\pi\left[\frac{(0.15\text{ m})(0.25\text{ m})}{(0.05\text{ m})}\right](350-275)\text{K} = \mathbf{23.3\ W}$$

COMBINED NATURAL AND FORCED CONVECTION

- In *assisting flow*, natural convection assists forced convection and *enhances* heat transfer. Example: upward forced flow over a hot surface.
- In *opposing flow*, natural convection resists forced convection and *decreases* heat transfer. Example: upward forced flow over a cold surface.
- In *transverse flow*, the buoyant motion is *perpendicular* to the forced motion. Transverse flow enhances fluid mixing and thus *enhances* heat transfer. Example: horizontal forced flow over a hot or cold cylinder or sphere.

$$\mathrm{Nu}_{\text{combined}} = (\mathrm{Nu}^n_{\text{forced}} \pm \mathrm{Nu}^n_{\text{natural}})^{1/n}$$

where $\mathrm{Nu}_{\text{forced}}$ and $\mathrm{Nu}_{\text{natural}}$ are determined from the correlations for *pure forced* and *pure natural convection*, respectively.

$$\dot{Q}_{\text{conv}} = hA_s(T_s - T_\infty)$$

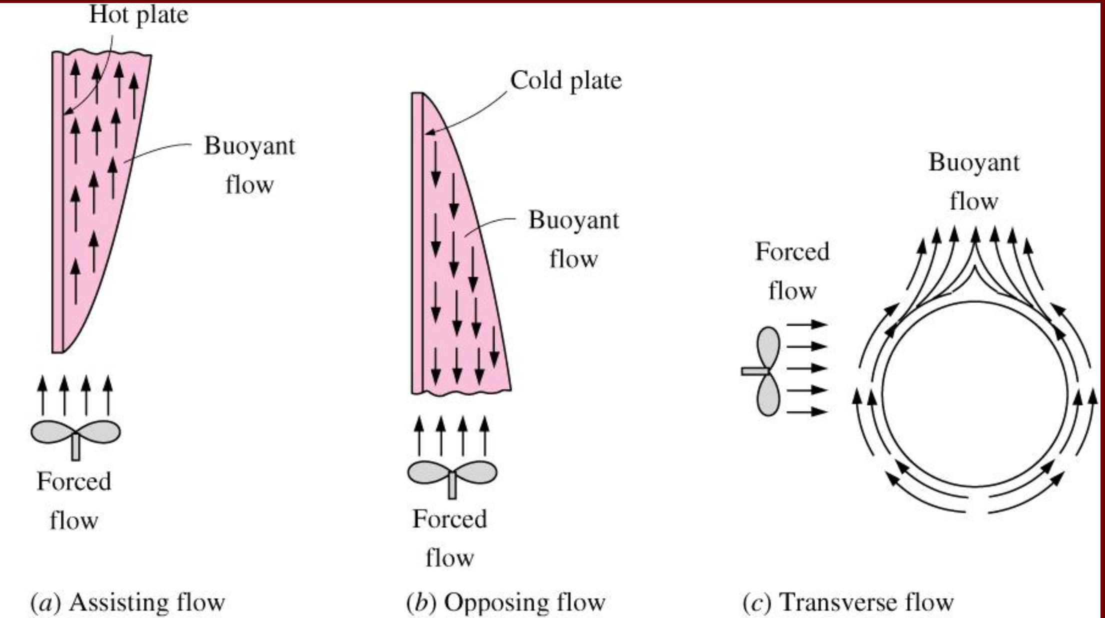


FIGURE 9–33
Natural convection can *enhance* or *inhibit* heat transfer, depending on the relative directions of *buoyancy-induced motion* and the *forced convection motion*.

Concluding Points:

- Physical Mechanism of Natural Convection?
- The Grashof Number (Gr_L)?
- The Rayleigh Number (Ra_L)?
- Natural Convection over Surfaces (Vertical Plates and Cylinders; Inclined Plates; Horizontal Plates, Cylinders and Spheres)?
- Natural Convection from Finned Surfaces and PCBs?
- Natural Convection inside Enclosures (Horizontal, Inclined and Vertical Rectangular; Concentric Cylinders and Spheres)?
- Effective Thermal Conductivity?
- Combined Natural and Forced Convection?

HEAT AND MASS TRANSFER

Boiling and Condensation

Objectives

- Differentiate between evaporation and boiling, and gain familiarity with different types of boiling,
- Develop a good understanding of the boiling curve, and the different boiling regimes corresponding to different regions of the boiling curve,
- Calculate the heat flux and its critical value associated with nucleate boiling, and examine the methods of boiling heat transfer enhancement,
- Derive a relation for the heat transfer coefficient in laminar film condensation over a vertical plate,
- Calculate the heat flux associated with condensation on inclined and horizontal plates, vertical and horizontal cylinders or spheres, and tube bundles,
- Examine dropwise condensation and understand the uncertainties associated with them.

Boiling Heat Transfer

- **Evaporation** occurs at the *liquid–vapor interface* when the vapor pressure is less than the saturation pressure of the liquid at a given temperature.
- **Boiling** occurs at the *solid–liquid interface* when a liquid is brought into contact with a surface maintained at a temperature sufficiently above the saturation temperature of the liquid.

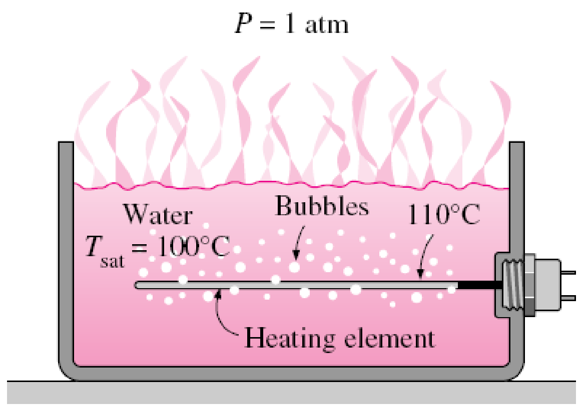
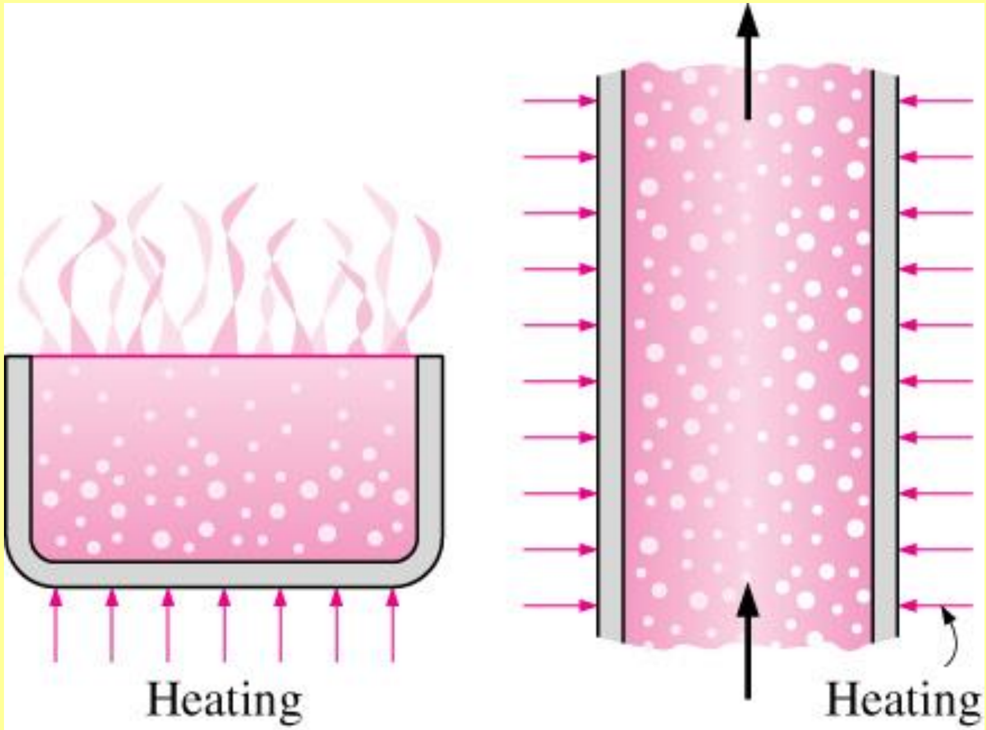


FIGURE 10–2
Boiling occurs when a liquid is brought into contact with a surface at a temperature above the saturation temperature of the liquid.

FIGURE 10–1
A liquid-to-vapor phase change process is called *evaporation* if it originates at a liquid–vapor interface and *boiling* if it occurs at a solid–liquid interface.

Classification of boiling

- Boiling is called **pool boiling** in the absence of bulk fluid flow.
- Any motion of the fluid is due to natural convection currents and the motion of the bubbles under the influence of buoyancy.
- Boiling is called **flow boiling** in the presence of bulk fluid flow.
- In flow boiling, the fluid is forced to move in a heated pipe or over a surface by external means such as a pump.



(*a*) Pool boiling (*b*) Flow boiling

FIGURE 10–3

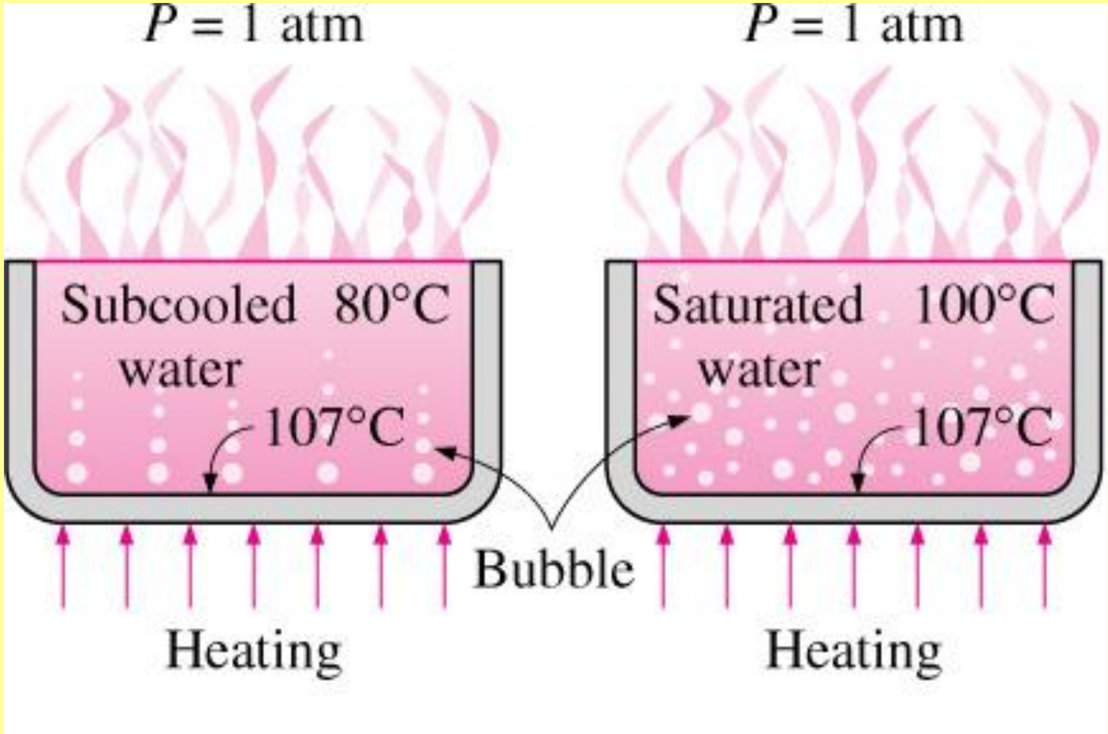
Classification of boiling on the basis of the presence of bulk fluid motion.

Subcooled Boiling

- When the temperature of the main body of the liquid is below the saturation temperature.

Saturated Boiling

- When the temperature of the liquid is equal to the saturation temperature.



(a) Subcooled boiling (b) Saturated boiling

FIGURE 10–4

Classification of boiling on the basis of the presence of bulk liquid temperature.

Pool Boiling

$$\dot{q}_{\text{boiling}} = h(T_s - T_{\text{sat}}) = h\Delta T_{\text{excess}}$$

Boiling takes different forms, depending on the $\Delta T_{excess} = T_s - T_{sat}$

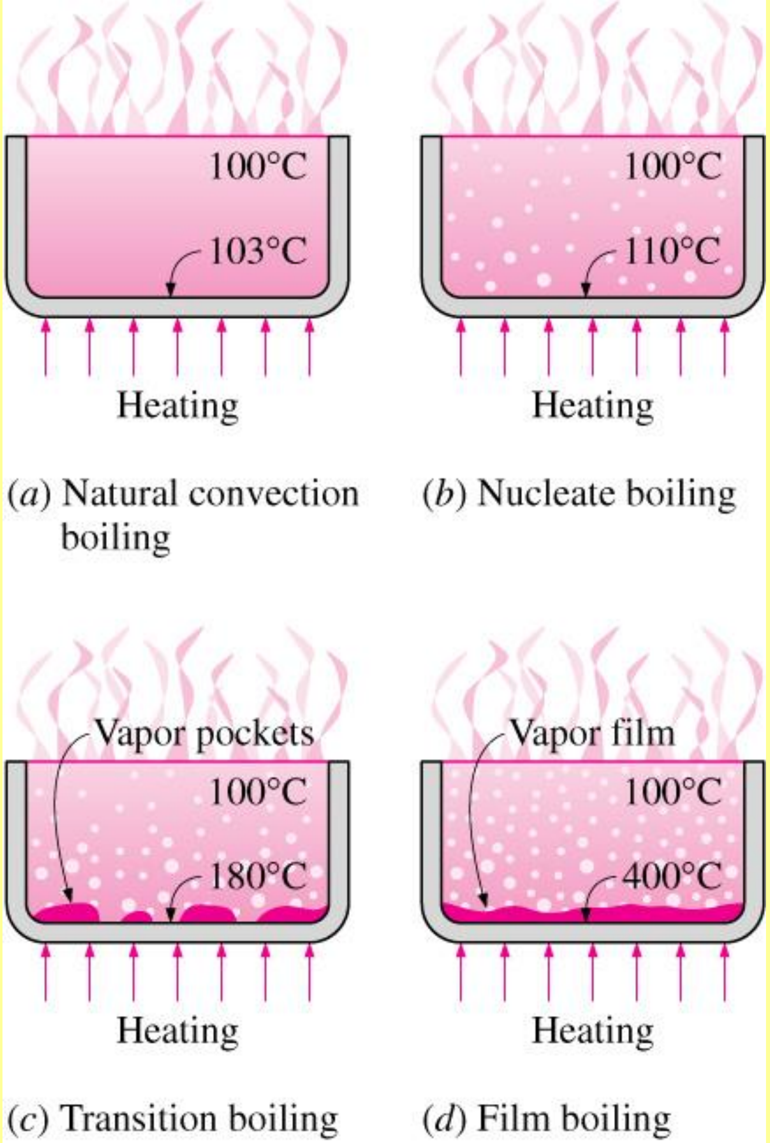
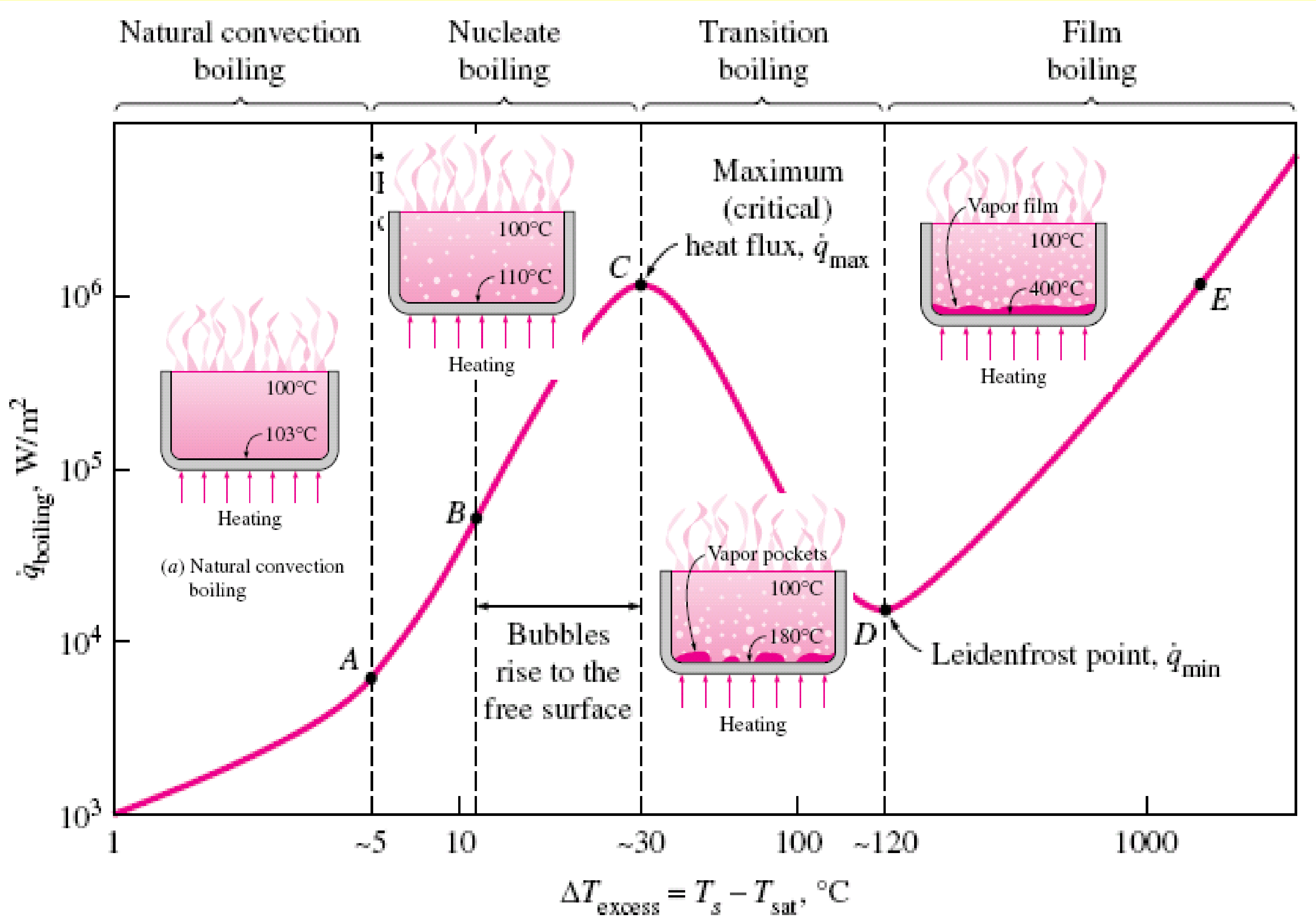


FIGURE 10–5
Different boiling regimes in pool boiling.

Pool Boiling

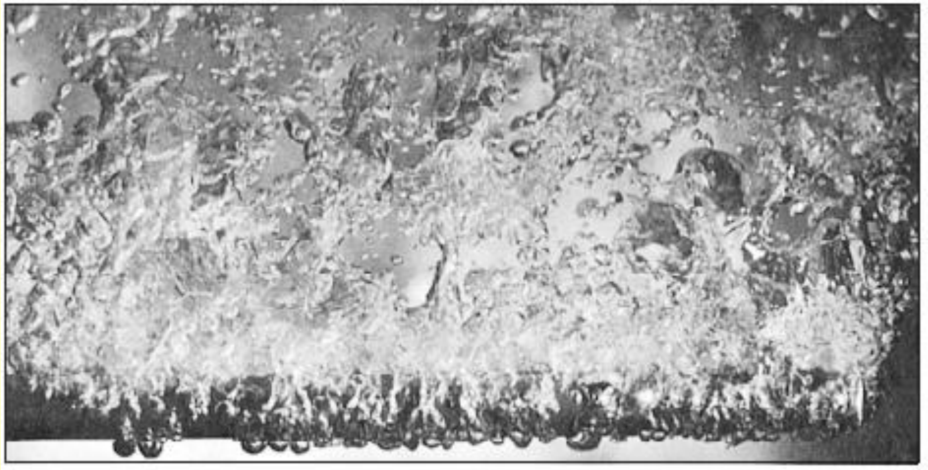


Natural Convection (to Point *A* on the Boiling Curve)

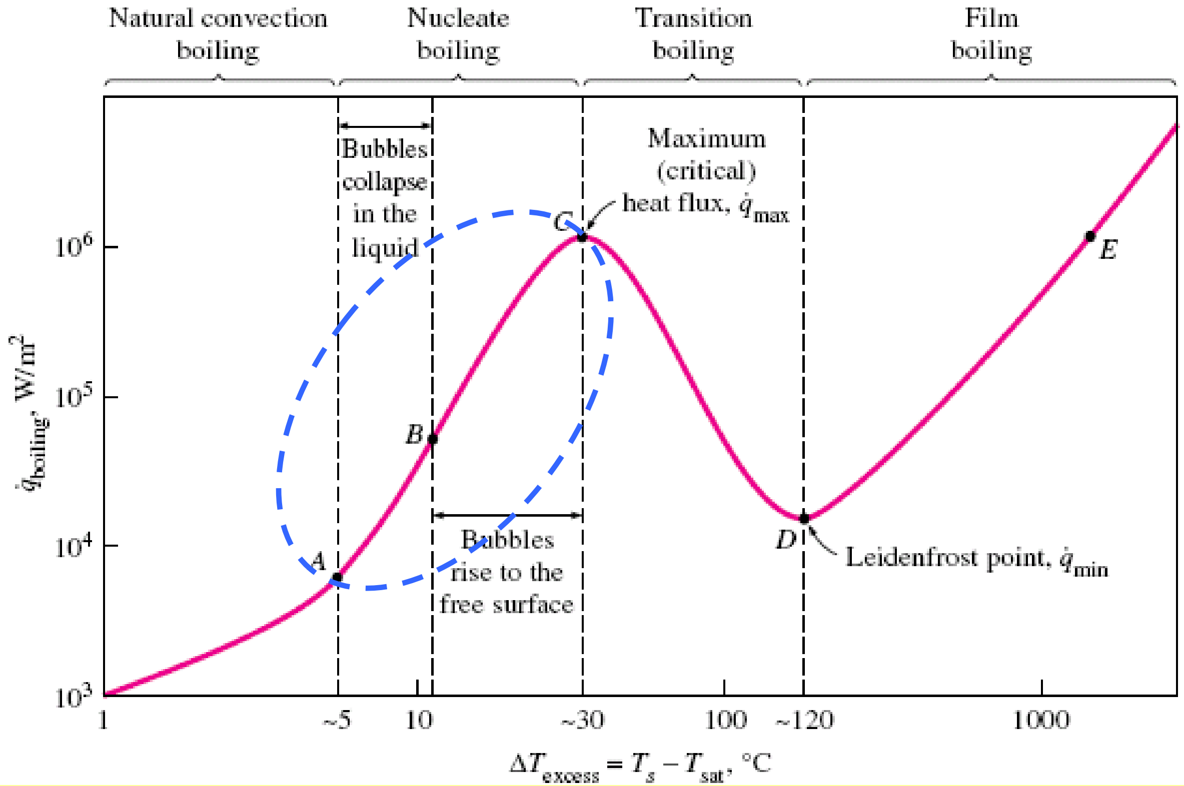
- Bubbles do not form on the heating surface until the liquid is heated a few degrees above the saturation temperature (about 2 to 6°C for water)

  the liquid is slightly *superheated* in this case (*metastable* state).
- The fluid motion in this mode of boiling is governed by natural convection currents.
- Heat transfer from the heating surface to the fluid is by natural convection.

Nucleate Boiling



- The bubbles form at an increasing rate at an increasing number of nucleation sites as we move along the boiling curve toward point *C*.
- Region *A–B* —*isolated bubbles.*
- Region *B–C* — numerous *continuous columns of vapor* in the liquid.

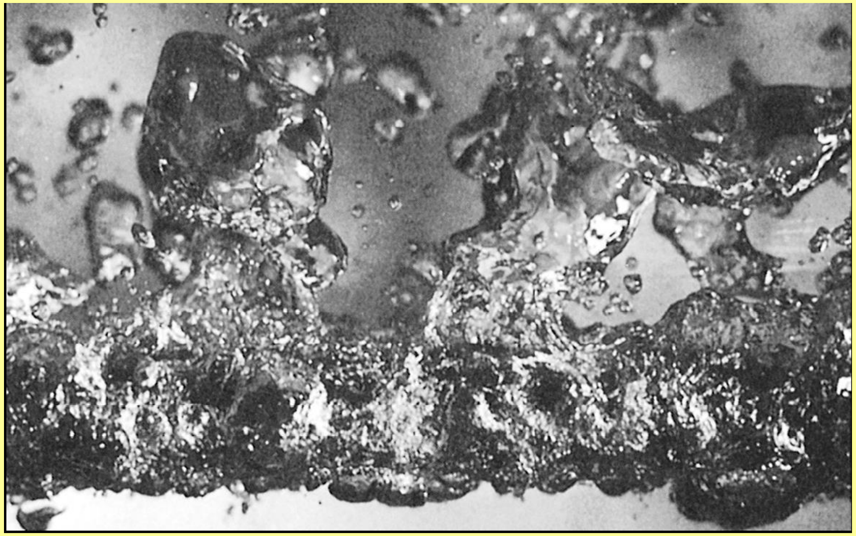


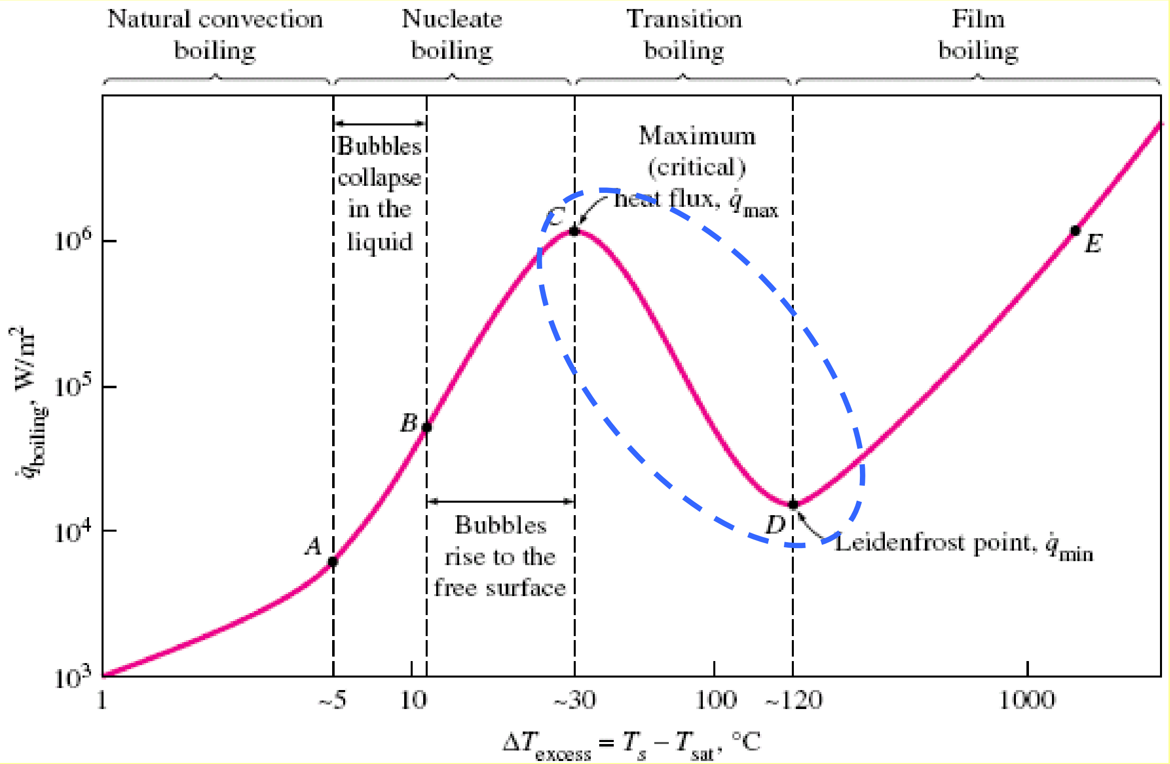
Nucleate Boiling

- In region *A–B* the stirring and agitation caused by the entrainment of the liquid to the heater surface is primarily responsible for the increased heat transfer coefficient.
- In region *A–B* the large heat fluxes obtainable in this region are caused by the combined effect of liquid entrainment and evaporation.
- After point *B* the heat flux increases at a lower rate with increasing ΔT_{excess}, and reaches a maximum at point *C*.
- The heat flux at this point is called the **critical** (or **maximum**) **heat flux**, and is of prime engineering importance.

Transition Boiling

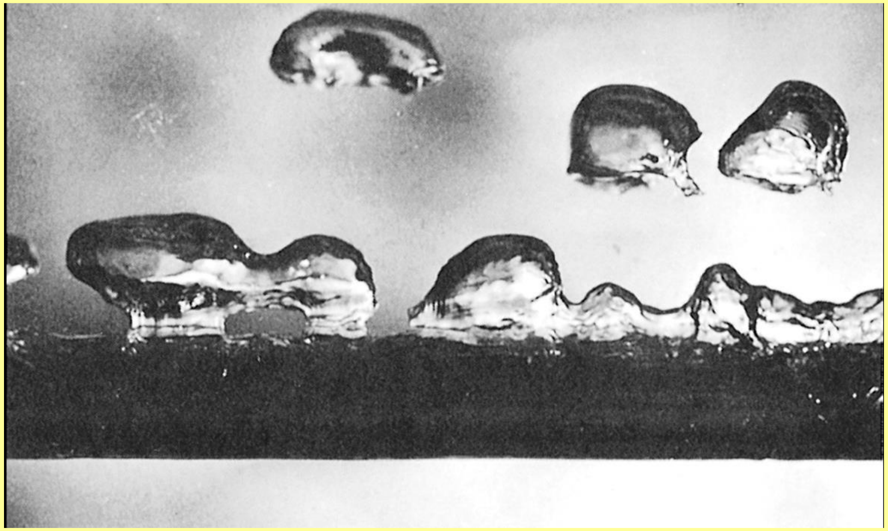
- When ΔT_{excess} is increased past point *C*, the heat flux decreases.
- This is because a large fraction of the heater surface is covered by a vapor film, which acts as an insulation.
- In the transition boiling regime, both nucleate and film boiling partially occur.

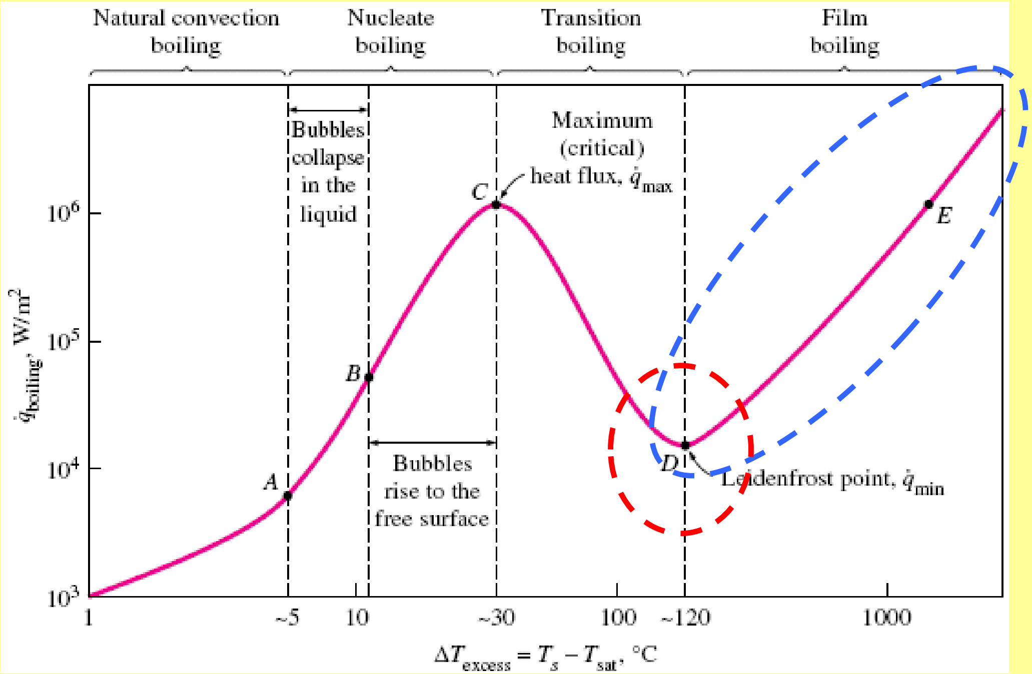




Film Boiling

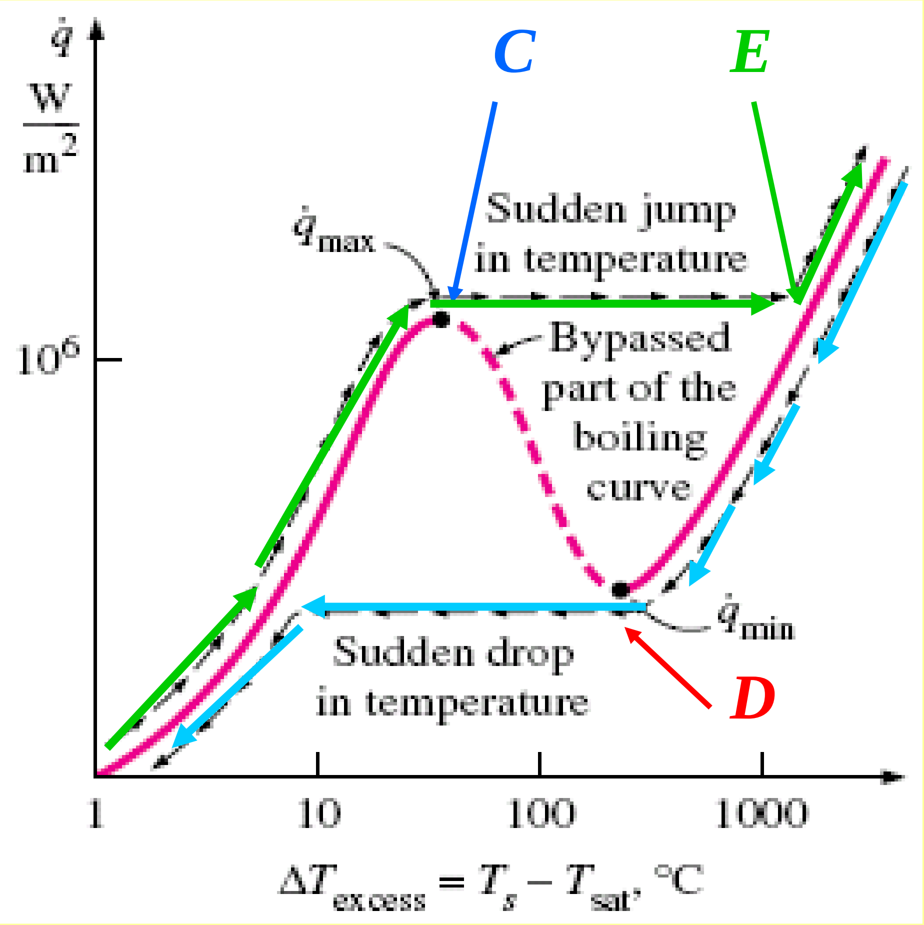
- Beyond Point D the heater surface is completely covered by a continuous stable vapor film.
- Point D, where the heat flux reaches a minimum is called the **Leidenfrost point**.
- The presence of a vapor film between the heater surface and the liquid is responsible for the low heat transfer rates in the film boiling region.
- The heat transfer rate increases with increasing excess temperature due to radiation to the liquid.





Burnout Phenomenon

- A typical boiling process does not follow the boiling curve beyond point *C*.
- When the power applied to the heated surface exceeded the value at point *C* even slightly, the surface temperature increased suddenly to point *E*.
- When the power is reduced gradually starting from point *E* the cooling curve follows Fig. 10–8 with a sudden drop in excess temperature when point *D* is reached.



Heat Transfer Correlations in Pool Boiling

- Boiling regimes differ considerably in their character.
- Different heat transfer relations need to be used for different boiling regimes.
- In the *natural convection boiling regime* heat transfer rates can be accurately determined using natural convection relations.
- No general theoretical relations for heat transfer in the nucleate boiling regime is available.
- Experimental based correlations are used.
- The rate of heat transfer strongly depends on the nature of nucleation and the type and the condition of the heated surface.

- For nucleate boiling a widely used correlation proposed in 1952 by Rohsenow:

$$\dot{q}_{\text{nucleate}} = \mu_l \, h_{fg} \left[\frac{g(\rho_l - \rho_v)}{\sigma}\right]^{1/2} \left[\frac{c_{pl}(T_s - T_{\text{sat}})}{C_{sf}\, h_{fg}\, \text{Pr}_l^n}\right]^3$$

$\dot{q}_{\text{nucleate}}$ = nucleate boiling heat flux, W/m^2
μ_l = viscosity of the liquid, kg/m · s
h_{fg} = enthalpy of vaporization, J/kg
g = gravitational acceleration, m/s^2
ρ_l = density of the liquid, kg/m^3
ρ_v = density of the vapor, kg/m^3
σ = surface tension of liquid–vapor interface, N/m
c_{pl} = specific heat of the liquid, J/kg · °C
T_s = surface temperature of the heater, °C
T_{sat} = saturation temperature of the fluid, °C
C_{sf} = experimental constant that depends on surface–fluid combination
Pr_l = Prandtl number of the liquid
n = experimental constant that depends on the fluid

TABLE 10–1

Surface tension of liquid–vapor interface for water

T, °C	σ, N/m*
0	0.0757
20	0.0727
40	0.0696
60	0.0662
80	0.0627
100	0.0589
120	0.0550
140	0.0509
160	0.0466
180	0.0422
200	0,0377
220	0.0331
240	0.0284
260	0.0237
280	0.0190
300	0.0144
320	0.0099
340	0.0056
360	0.0019
374	0.0

*Multiply by 0.06852 to convert to lbf/ft or by 2.2046 to convert to lbm/s^2.

Critical Heat Flux (CHF)

- The *maximum* (or *critical*) *heat flux* in nucleate pool boiling was determined theoretically by S. S. Kutateladze in Russia in 1948 and N. Zuber in the United States in 1958 to be:

$$\dot{q}_{\max} = C_{cr} h_{fg} \left[\sigma g \rho_v^2 \left(\rho_l - \rho_v \right) \right]^{1/4}$$

  C_{cr} is a constant whose value depends on the heater geometry, but generally is about 0.15.
- The CHF is independent of the fluid–heating surface combination, as well as the viscosity, thermal conductivity, and the specific heat of the liquid.
- The CHF increases with pressure up to about one-third of the critical pressure, and then starts to decrease and becomes zero at the critical pressure.
- The CHF is proportional to h_{fg}, and large maximum heat fluxes can be obtained using fluids with a large enthalpy of vaporization, such as water.

TABLE 10–3

Values of the coefficient C_{sf} and n for various fluid–surface combinations

Fluid–Heating Surface Combination	C_{sf}	n
Water–copper (polished)	0.0130	1.0
Water–copper (scored)	0.0068	1.0
Water–stainless steel (mechanically polished)	0.0130	1.0
Water–stainless steel (ground and polished)	0.0060	1.0
Water–stainless steel (teflon pitted)	0.0058	1.0
Water–stainless steel (chemically etched)	0.0130	1.0
Water–brass	0.0060	1.0
Water–nickel	0.0060	1.0
Water–platinum	0.0130	1.0
n-Pentane–copper (polished)	0.0154	1.7
n-Pentane–chromium	0.0150	1.7
Benzene–chromium	0.1010	1.7
Ethyl alcohol–chromium	0.0027	1.7
Carbon tetrachloride–copper	0.0130	1.7
Isopropanol–copper	0.0025	1.7

TABLE 10–4

Values of the coefficient C_{cr} for use in Eq. 10–3 for maximum heat flux (dimensionless parameter $L^* = L[g(\rho_l - \rho_v)/\sigma]^{1/2}$)

Heater Geometry	C_{cr}	Charac. Dimension of Heater, L	Range of L^*
Large horizontal flat heater	0.149	Width or diameter	$L^* > 27$
Small horizontal flat heater[1]	$18.9K_1$	Width or diameter	$9 < L^* < 20$
Large horizontal cylinder	0.12	Radius	$L^* > 1.2$
Small horizontal cylinder	$0.12L^{*-0.25}$	Radius	$0.15 < L^* < 1.2$
Large sphere	0.11	Radius	$L^* > 4.26$
Small sphere	$0.227L^{*-0.5}$	Radius	$0.15 < L^* < 4.26$

[1] $K_1 = \sigma/[g(\rho_l - \rho_v)A_{heater}]$

Minimum Heat Flux

- Minimum heat flux, which occurs at the Leidenfrost point, is of practical interest since it represents the lower limit for the heat flux in the film boiling regime.
- Zuber derived the following expression for the minimum heat flux for a *large horizontal plate*

$$\dot{q}_{\min} = 0.09\rho_v\, h_{fg}\left[\frac{\sigma g(\rho_l - \rho_v)}{(\rho_l + \rho_v)^2}\right]^{1/4}$$

- the relation above can be in error by 50% or more.

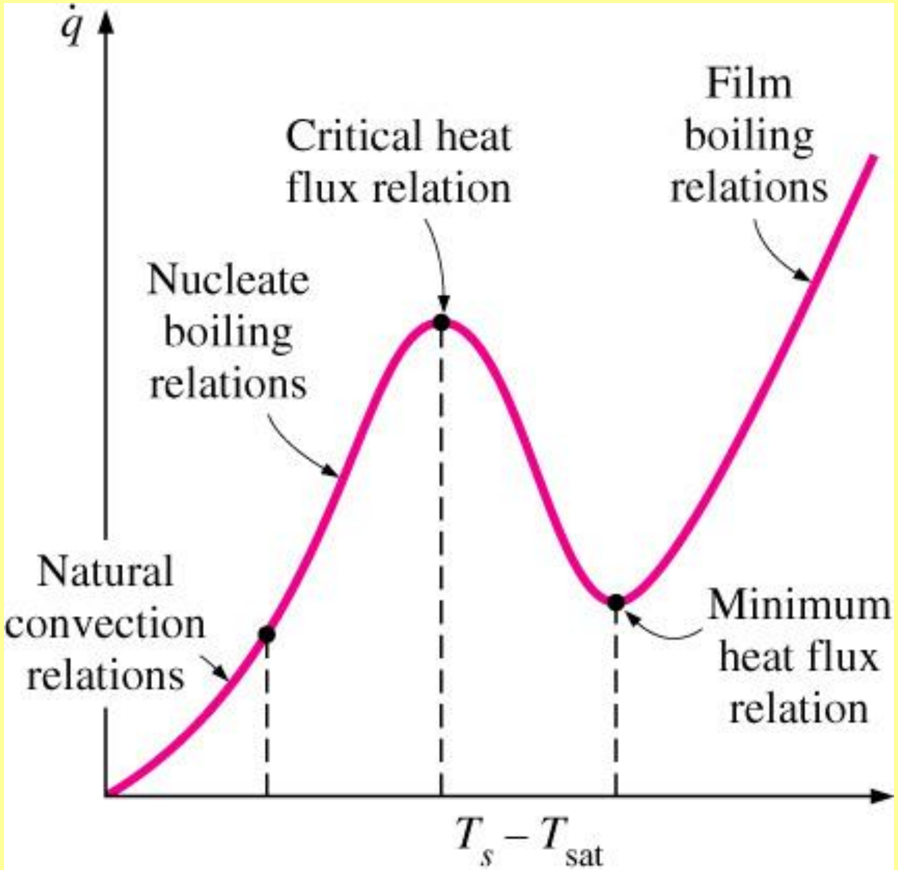


FIGURE 10–11
Different relations are used to determine the heat flux in different boiling regimes.

Film Boiling

The heat flux for film boiling on a *horizontal cylinder* or *sphere* of diameter D is given by

$$\dot{q}_{film} = C_{film}\left[\frac{g k_v^3 \rho_v (\rho_l - \rho_v)\left[h_{fg} + 0.4 C_{pv}(T_s - T_{sat})\right]}{\mu_v D (T_s - T_{sat})}\right]^{1/4}(T_s - T_{sat})$$

$$C_{film} = \begin{cases} 0.62 \text{ for horizontal cylinders} \\ 0.67 \text{ for spheres} \end{cases}$$

- At high surface temperatures (typically above 300°C), heat transfer across the vapor film by *radiation* becomes significant and needs to be considered.

$$\dot{q}_{rad} = \varepsilon\sigma\,(T_s^4 - T_{sat}^4)$$

$$\dot{q}_{total} = \dot{q}_{film} + \frac{3}{4}\dot{q}_{rad}$$

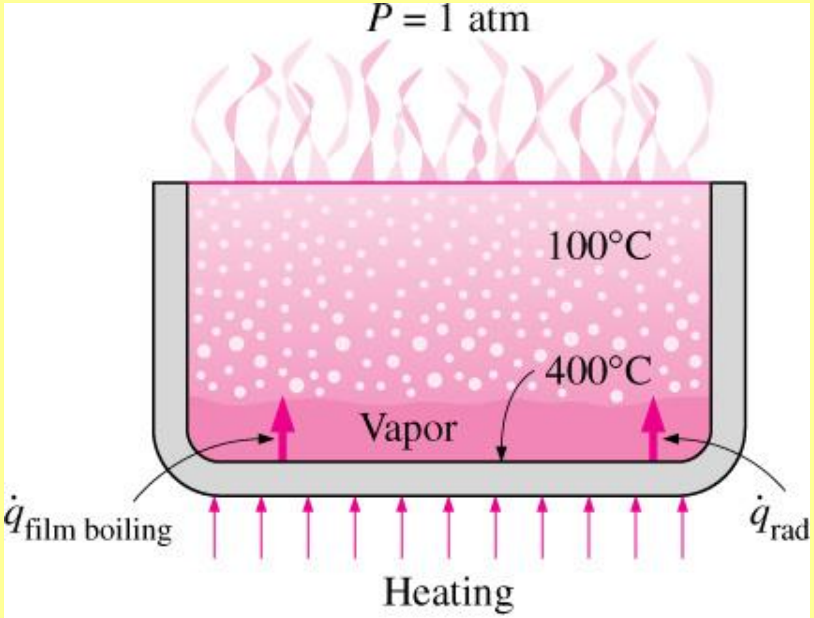


FIGURE 10–12

At high heater surface temperatures, radiation heat transfer becomes significant during film boiling.

Enhancement of Heat Transfer in Pool Boiling

- The rate of heat transfer in the nucleate boiling regime strongly depends on the number of active nucleation sites on the surface, and the rate of bubble formation at each site.
- Therefore, modification that *enhances nucleation* on the heating surface will also *enhance heat transfer* in nucleate boiling.
- *Irregularities* on the heating surface, including roughness and dirt, serve as additional nucleation sites during boiling.
- The effect of surface roughness is observed to decay with time.

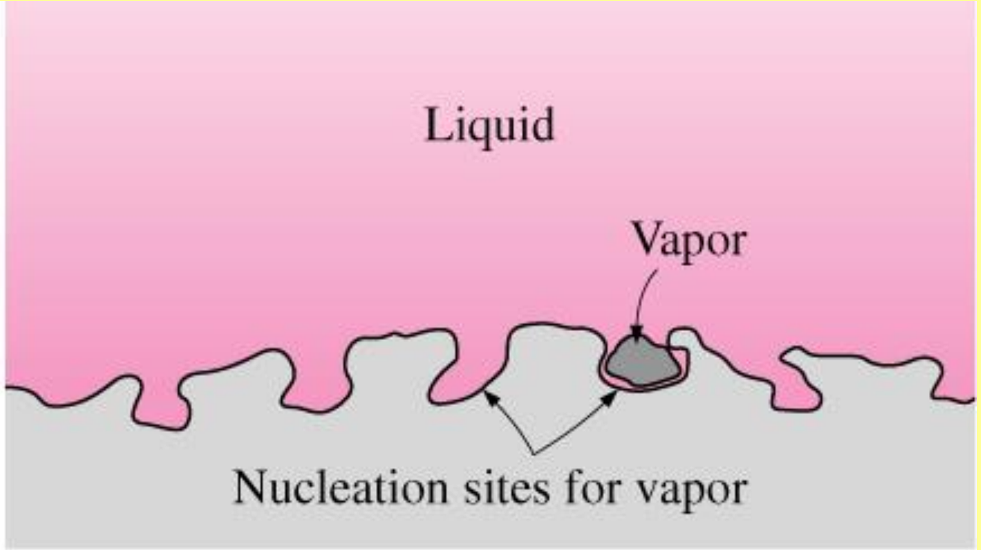


FIGURE 10–13
The cavities on a rough surface act as nucleation sites and enhance boiling heat transfer.

- Surfaces that provide enhanced heat transfer in nucleate boiling *permanently* are being manufactured and are available in the market.
- Heat transfer can be enhanced by a factor of up to 10 during nucleate boiling, and the critical heat flux by a factor of 3.

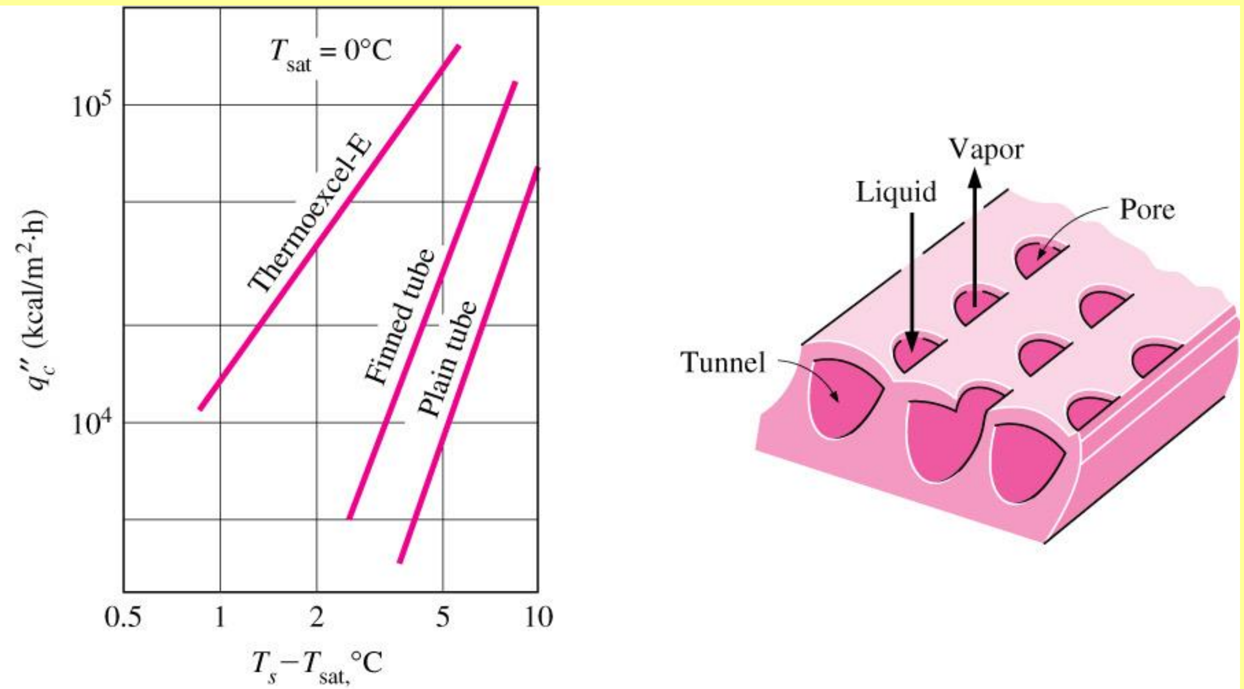


FIGURE 10–14
The enhancement of boiling heat transfer in Freon-12 by a mechanically roughened surface, thermoexcel-E.

Flow Boiling

- In flow boiling, the fluid is forced to move by an external source such as a pump as it undergoes a phase-change process.
- The boiling in this case exhibits the combined effects of convection and pool boiling.
- Flow boiling is classified as either *external* and *internal flow boiling.*
- *External flow* — the higher the velocity, the higher the nucleate boiling heat flux and the critical heat flux.

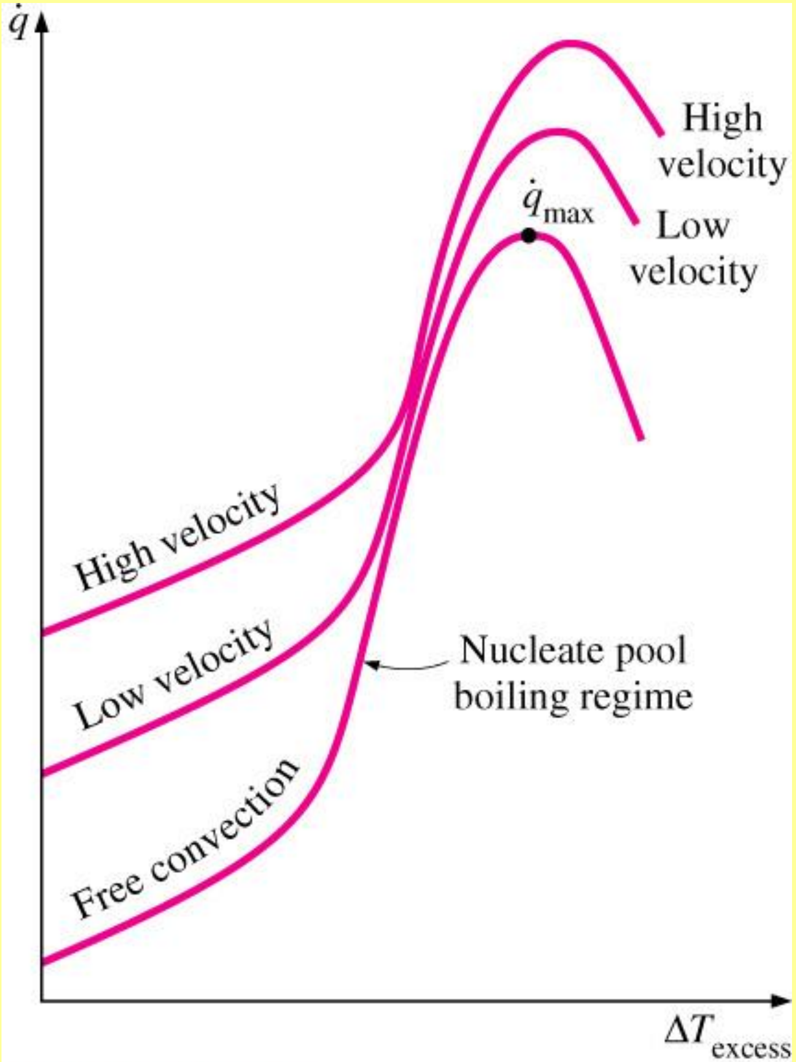
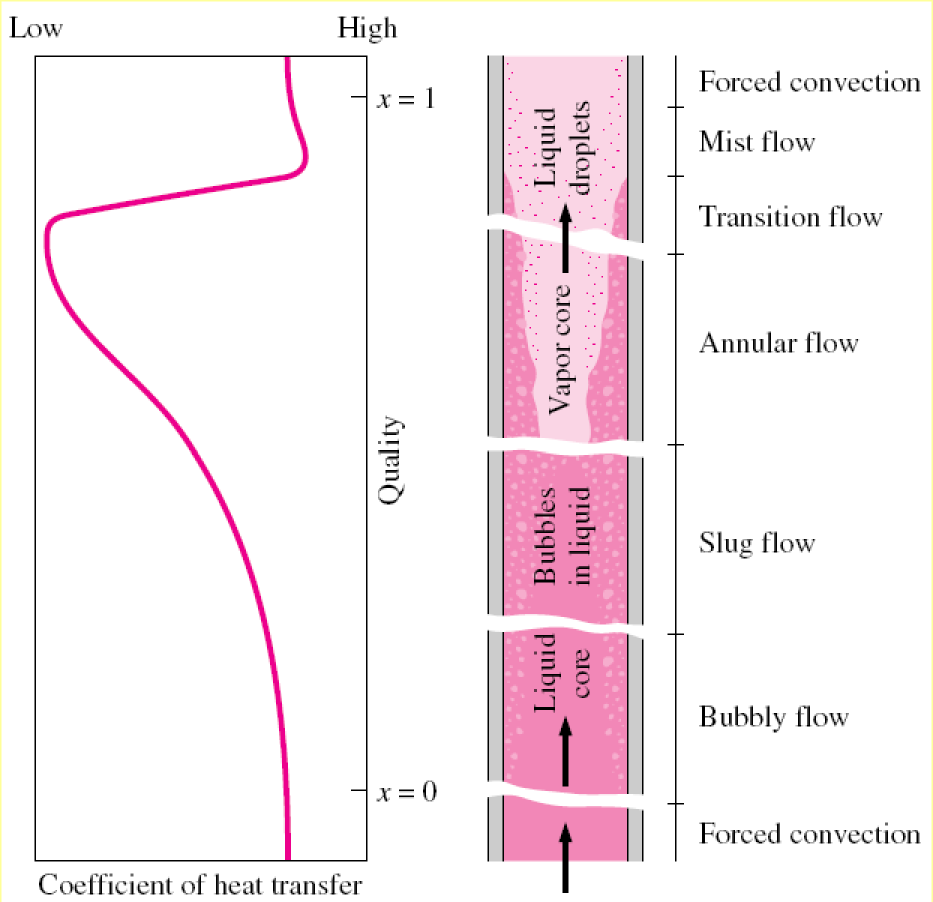


FIGURE 10–18
The effect of forced convection on external flow boiling for different flow velocities.

Flow Boiling – Internal Flow

- The two-phase flow in a tube exhibits different flow boiling regimes, depending on the relative amounts of the liquid and the vapor phases.
- Typical flow regimes:
  - Liquid single-phase flow,
  - Bubbly flow,
  - Slug flow,
  - Annular flow,
  - Mist flow,
  - Vapor single-phase flow.

Axial position in the tube

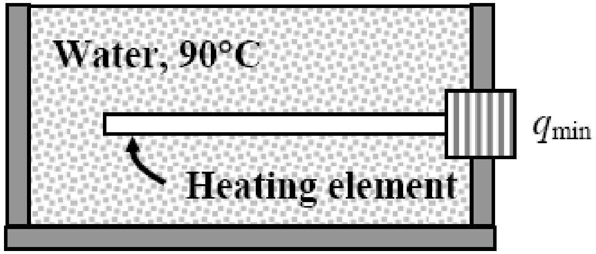


Flow Boiling – Internal Flow

- Liquid single-phase flow
  - In the inlet region the liquid is subcooled and heat transfer to the liquid is by *forced convection* (assuming no subcooled boiling).
- Bubbly flow
  - Individual bubbles
  - Low mass qualities
- Slug flow
  - Bubbles coalesce into slugs of vapor.
  - Moderate mass qualities
- Annular flow
  - Core of the flow consists of vapor only, and liquid adjacent to the walls.
  - Very high heat transfer coefficients
- Mist flow
  - a sharp decrease in the heat transfer coefficient
- Vapor single-phase flow
  - The liquid phase is completely evaporated and vapor is superheated.

10–13 Water is boiled at 90°C by a horizontal brass heating element of diameter 7 mm. Determine the surface temperature of the heater for the minimum heat flux case.

Properties The properties of water at the saturation temperature of 90°C are (Tables 10-1 and A-9)



$$\rho_l = 965.3 \text{ kg/m}^3 \qquad h_{fg} = 2283\times10^3 \text{ J/kg}$$
$$\rho_v = 0.4235 \text{ kg/m}^3 \qquad \mu_l = 0.315\times10^{-3} \text{ kg/m}\cdot\text{s}$$
$$\sigma = 0.0608 \text{ N/m} \qquad c_{pl} = 4206 \text{ J/kg}\cdot°\text{C}$$
$$\text{Pr}_l = 1.96$$

Also, $C_{sf} = 0.0060$ and $n = 1.0$ for the boiling of water on a brass heating (Table 10-3).

Analysis The minimum heat flux is determined from

$$\dot{q}_{\min} = 0.09\rho_v h_{fg}\left[\frac{\sigma g(\rho_l-\rho_v)}{(\rho_l+\rho_v)^2}\right]^{1/4}$$

$$= 0.09(0.4235)(2283\times10^3)\left[\frac{(0.0608)(9.81)(965.3-0.4235)}{(965.3+0.4235)^2}\right]^{1/4} = 13{,}715 \text{ W/m}^2$$

The surface temperature can be determined from Rohsenow equation to be

$$\dot{q}_{\text{nucleate}} = \mu_l h_{fg}\left[\frac{g(\rho_l-\rho_v)}{\sigma}\right]^{1/2}\left(\frac{c_{p,l}(T_s-T_{\text{sat}})}{C_{sf}h_{fg}\text{Pr}_l^n}\right)^3$$

$$13{,}715 \text{ W/m}^2 = (0.315\times10^{-3})(2283\times10^3)\left[\frac{9.81(965.3-0.4235)}{0.0608}\right]^{1/2}\left(\frac{4206(T_s-90)}{0.0060(2283\times10^3)1.96}\right)^3$$

$$T_s = \mathbf{92.3°C}$$

Condensation

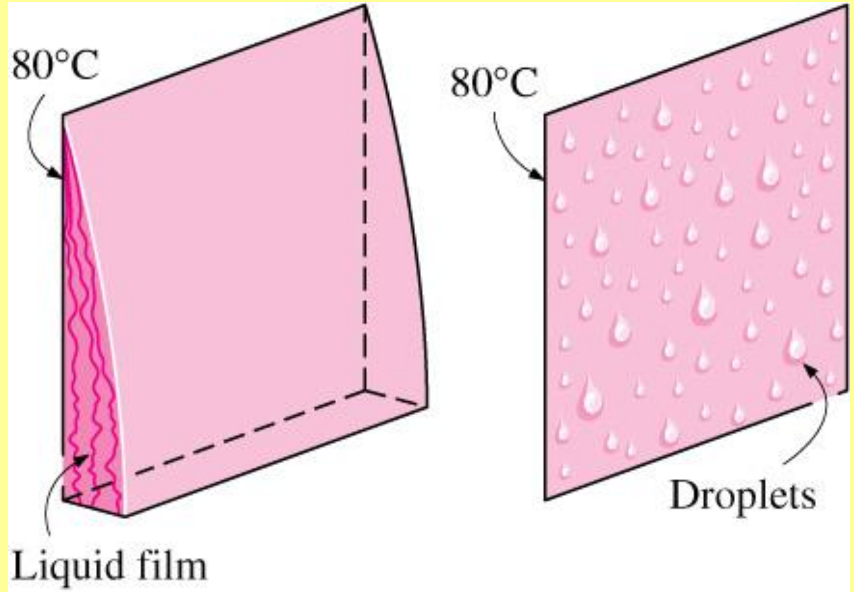
Condensation occurs when the temperature of a vapor is reduced *below* its saturation temperature.

Film condensation

- The condensate wets the surface and forms a liquid film.
- The surface is blanketed by a liquid film which serves as a *resistance* to heat transfer.

Dropwise condensation

- The condensed vapor forms droplets on the surface.
- The droplets slide down when they reach a certain size.
- No liquid film to resist heat transfer.
- *As* a result, heat transfer rates that are more than 10 times larger than with film condensation can be achieved.



(*a*) Film condensation (*b*) Dropwise condensation

FIGURE 10–20
When a vapor is exposed to a surface at a temperature below T_{sat}, condensation in the form of a liquid film or individual droplets occurs on the surface.

Film Condensation on a Vertical Plate

- Liquid film starts forming at the top of the plate and flows downward under the influence of gravity.
- δ *increases* in the flow direction x
- Heat in the amount h_{fg} is released during condensation and is *transferred* through the film to the plate surface.
- T_s must be below the saturation temperature for condensation.
- The *temperature* of the condensate is T_{sat} at the interface and decreases gradually to T_s at the wall.

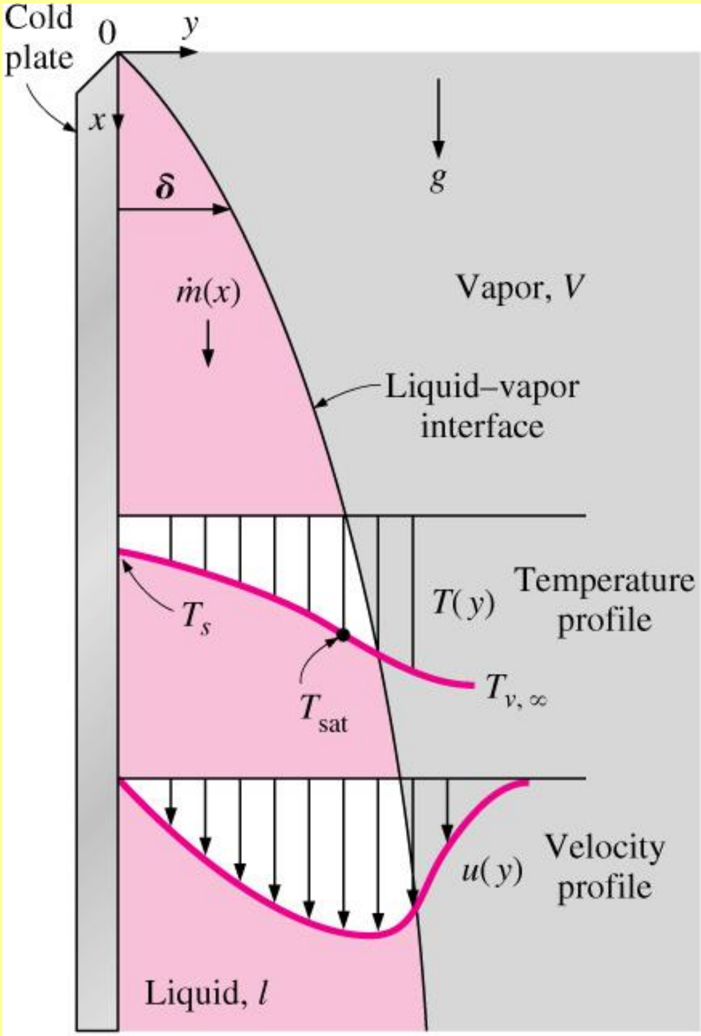


FIGURE 10–21
Film condensation on a vertical plate.

$$\mathrm{Re} = \frac{D_h \rho_l V_l}{\mu_l} = \frac{4 A_c \rho_l V_l}{p \mu_l} = \frac{4 \rho_l V_l \delta}{\mu_l} = \frac{4\dot{m}}{p \mu_l}$$

$D_h = 4A_c/p = 4\delta$ = hydraulic diameter of the condensate flow, m

p = wetted perimeter of the condensate, m

$A_c = p\delta$ = wetted perimeter × film thickness, m^2, cross-sectional area of the condensate flow at the lowest part of the flow

ρ_l = density of the liquid, kg/m^3

μ_l = viscosity of the liquid, kg/m · s

V_l = average velocity of the condensate at the lowest part of the flow, m/s

$\dot{m} = \rho_l V_l A_c$ = mass flow rate of the condensate at the lowest part, kg/s

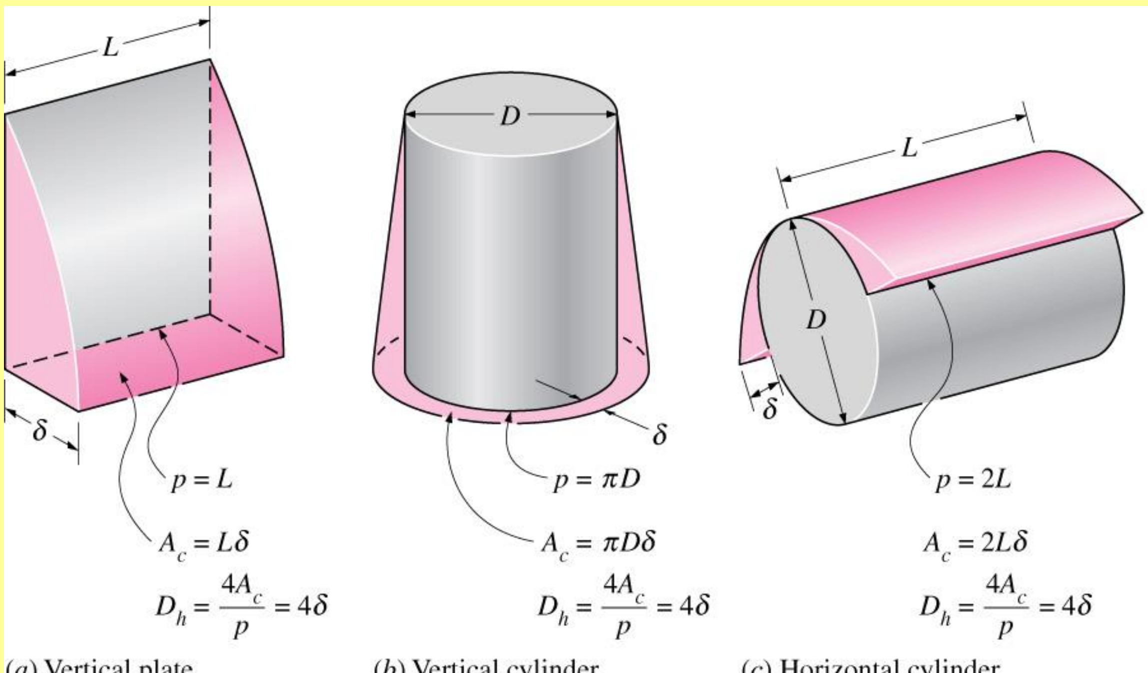


FIGURE 10–22

The wetted perimeter p, the condensate cross-sectional area A_c, and the hydraulic diameter D_h for some common geometries.

modified latent heat of vaporization h^*_{fg}, defined as

$$h^*_{fg} = h_{fg} + 0.68c_{pl}(T_{sat} - T_s) \tag{10–9a}$$

where c_{pl} is the specific heat of the liquid at the average film temperature.

We can have a similar argument for vapor that enters the condenser as **superheated vapor** at a temperature T_v instead of as saturated vapor. In this case the vapor must be cooled first to T_{sat} before it can condense, and this heat must be transferred to the wall as well. The amount of heat released as a unit mass of superheated vapor at a temperature T_v is cooled to T_{sat} is simply $c_{pv}(T_v - T_{sat})$, where c_{pv} is the specific heat of the vapor at the average temperature of $(T_v + T_{sat})/2$. The modified latent heat of vaporization in this case becomes

$$h^*_{fg} = h_{fg} + 0.68c_{pl}(T_{sat} - T_s) + c_{pv}(T_v - T_{sat}) \tag{10–9b}$$

With these considerations, the rate of heat transfer can be expressed as

$$\dot{Q}_{conden} = hA_s(T_{sat} - T_s) = \dot{m}h^*_{fg} \tag{10–10}$$

where A_s is the heat transfer area (the surface area on which condensation occurs). Solving for $\dot{m}$ from the equation above and substituting it into Eq. 10–8 gives yet another relation for the Reynolds number,

$$\text{Re} = \frac{4\dot{Q}_{conden}}{p\mu_l h^*_{fg}} = \frac{4A_s h(T_{sat} - T_s)}{p\mu_l h^*_{fg}} \tag{10–11}$$

This relation is convenient to use to determine the Reynolds number when the condensation heat transfer coefficient or the rate of heat transfer is known.

The temperature of the liquid film varies from T_{sat} on the liquid–vapor interface to T_s at the wall surface. Therefore, the properties of the liquid should be evaluated at the *film temperature* $T_f = (T_{sat} + T_s)/2$, which is approximately the *average* temperature of the liquid. The h_{fg}, however, should be evaluated at T_{sat} since it is not affected by the subcooling of the liquid.

Vertical Plate – Flow Regimes

- The dimensionless parameter controlling the transition between regimes is the Reynolds number defined as:

$$\mathrm{Re} = \frac{D_h \rho_l V_l}{\mu_l} = \frac{4 A_c \rho_l V_l}{p \mu_l} = \frac{4 \rho_l V_l \delta}{\mu_l} = \frac{4 \dot{m}}{p \mu_l}$$

- Three prime flow regimes:
  - Re<30 – Laminar (wave-free),
  - 30<Re<1800 – Wavy-laminar,
  - Re>1800 – Turbulent.
- The Reynolds number increases in the flow direction.

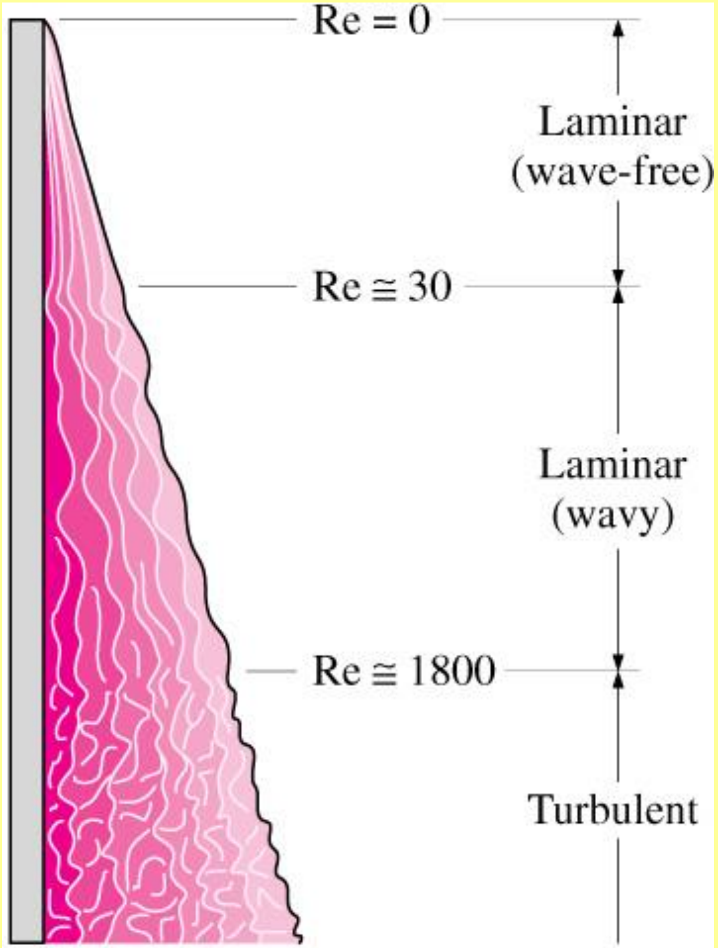


FIGURE 10–23
Flow regimes during film condensation on a vertical plate.

Heat Transfer Correlations for Film Condensation — Vertical wall

Assumptions:

1. Both the plate and the vapor are maintained at *constant temperatures* of T_s and T_{sat}, respectively, and the temperature across the liquid film varies *linearly.*
2. Heat transfer across the liquid film is by pure *conduction*.
3. The velocity of the vapor is low (or zero) so that it exerts *no drag* on the condensate (no viscous shear on the liquid–vapor interface).
4. The flow of the condensate is *laminar* (Re<30) and the properties of the liquid are constant.
5. The acceleration of the condensate layer is negligible.

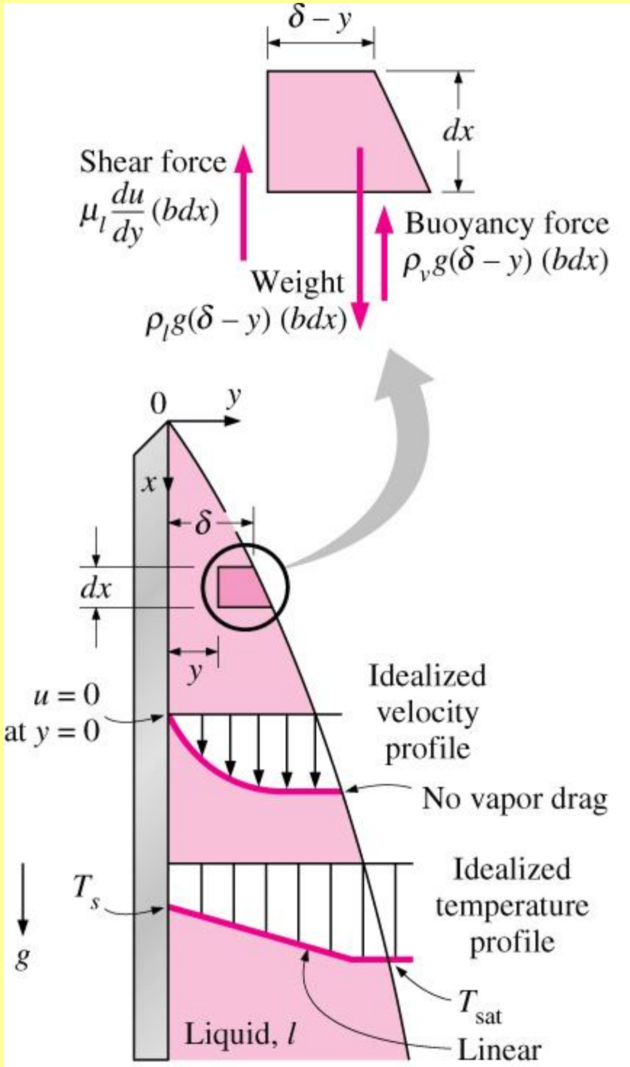


FIGURE 10–24
The volume element of condensate on a vertical plate considered in Nusselt's analysis.

The *average heat transfer coefficient* for laminar film condensation over a vertical flat plate of height L is

$$h_{\text{vert}} = 0.943\left[\frac{g\rho_l(\rho_l-\rho_v)h^*_{fg}\,k_l^3}{\mu_l\,(T_{\text{sat}}-T_s)L}\right]^{1/4} \qquad (\text{W/m}^2\cdot{}^\circ\text{C}), \quad 0<\text{Re}<30 \qquad \textbf{(10–22)}$$

where

g = gravitational acceleration, m/s^2

ρ_l, ρ_v = densities of the liquid and vapor, respectively, kg/m^3

μ_l = viscosity of the liquid, kg/m · s

$h^*_{fg} = h_{fg} + 0.68c_{pl}\,(T_{\text{sat}} - T_s)$ = modified latent heat of vaporization, J/kg

k_l = thermal conductivity of the liquid, W/m · °C

L = height of the vertical plate, m

T_s = surface temperature of the plate, °C

T_{sat} = saturation temperature of the condensing fluid, °C

At a given temperature, $\rho_v \ll \rho_l$ and thus $\rho_l - \rho_v \approx \rho_l$ except near the critical point of the substance. Using this approximation and substituting Eqs. 10–14 and 10–18 at $x = L$ into Eq. 10–8 by noting that $\delta_{x=L} = kl/h_{x=L}$ and $h_{\text{vert}} = \frac{4}{3}$ $h_{x=L}$ (Eqs. 10–19 and 10–21) give

$$\text{Re} \cong \frac{4g\rho_l(\rho_l-\rho_v)\delta^3}{3\mu_l^2} = \frac{4g\rho_l^2}{3\mu_l^2}\left(\frac{k_l}{h_{x=L}}\right)^3 = \frac{4g}{3\nu_l^2}\left(\frac{k_l}{3h_{\text{vert}}/4}\right)^3 \qquad \textbf{(10–23)}$$

Then the heat transfer coefficient h_{vert} in terms of Re becomes

$$h_{\text{vert}} \cong 1.47k_l\,\text{Re}^{-1/3}\left(\frac{g}{\nu_l^2}\right)^{1/3}, \qquad \begin{array}{l} 0<\text{Re}<30 \\ \rho_v \ll \rho_l \end{array} \qquad \textbf{(10–24)}$$

The average heat transfer coefficient in wavy laminar condensate flow for $\rho_v \ll \rho_l$ and $30 < \mathrm{Re} < 1800$

$$h_{\text{vert, wavy}} = \frac{\mathrm{Re}\, k_l}{1.08\, \mathrm{Re}^{1.22} - 5.2}\left(\frac{g}{\nu_l^2}\right)^{1/3}, \qquad \begin{array}{l} 30 < \mathrm{Re} < 1800 \\ \rho_v \ll \rho_l \end{array}$$

A simpler alternative to the relation above proposed by Kutateladze (1963) is

$$h_{\text{vert, wavy}} = 0.8\, \mathrm{Re}^{0.11}\, h_{\text{vert (smooth)}} \qquad \textbf{(10–26)}$$

which relates the heat transfer coefficient in wavy laminar flow to that in wave-free laminar flow. McAdams (1954) went even further and suggested accounting for the increase in heat transfer in the wavy region by simply increasing the heat transfer coefficient determined from Eq. 10–22 for the laminar case by 20 percent. It is also suggested using Eq. 10–22 for the wavy region also, with the understanding that this is a conservative approach that provides a safety margin in thermal design. In this book we use Eq. 10–25.

A relation for the Reynolds number in the wavy laminar region can be determined by substituting the h relation in Eq. 10–25 into the Re relation in Eq. 10–11 and simplifying. It yields

$$\mathrm{Re}_{\text{vert, wavy}} = \left[4.81 + \frac{3.70\, L k_l (T_{\text{sat}} - T_s)}{\mu_l\, h^*_{fg}}\left(\frac{g}{\nu_l^2}\right)^{1/3}\right]^{0.820}, \quad \rho_v \ll \rho_l \qquad \textbf{(10–27)}$$

Turbulent Flow on Vertical Plates

At a Reynolds number of about 1800, the condensate flow becomes turbulent. Several empirical relations of varying degrees of complexity are proposed for the heat transfer coefficient for turbulent flow. Again assuming $\rho_v \ll \rho_l$ for simplicity, Labuntsov (1957) proposed the following relation for the turbulent flow of condensate on *vertical plates:*

$$h_{\text{vert, turbulent}} = \frac{\text{Re}\, k_l}{8750 + 58\,\text{Pr}^{-0.5}(\text{Re}^{0.75} - 253)}\left(\frac{g}{\nu_l^2}\right)^{1/3}, \quad \begin{matrix}\text{Re} > 1800 \\ \rho_v \ll \rho_l\end{matrix} \qquad \textbf{(10–28)}$$

The physical properties of the condensate are again to be evaluated at the film temperature $T_f = (T_{\text{sat}} + T_s)/2$. The Re relation in this case is obtained by substituting the h relation above into the Re relation in Eq. 10–11, which gives

$$\text{Re}_{\text{vert, turbulent}} = \left[\frac{0.0690\, Lk_l\, \text{Pr}^{0.5}(T_{\text{sat}} - T_s)}{\mu_l\, h^*_{fg}}\left(\frac{g}{\nu_l^2}\right)^{1/3} - 151\,\text{Pr}^{0.5} + 253\right]^{4/3} \qquad \textbf{(10–29)}$$

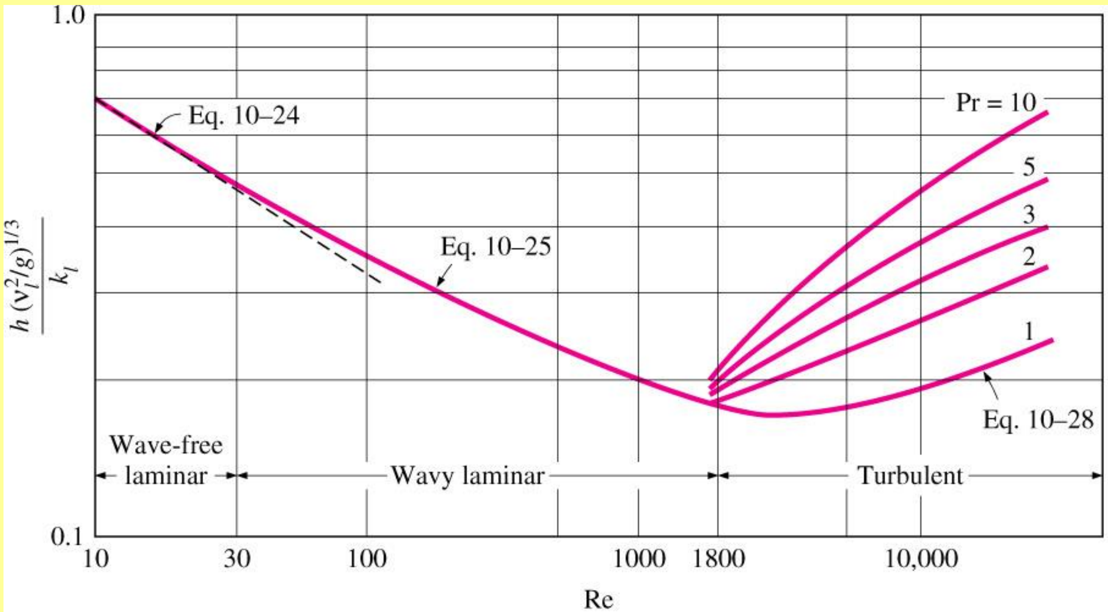


FIGURE 10–26
Nondimensionalized heat transfer coefficients for the wave-free laminar, wavy laminar, and turbulent flow of condensate on vertical plates.

Inclined Plates

$$h_{\text{inclined}} = h_{\text{vert}} (\cos\theta)^{1/4} \qquad \text{(laminar)}$$

Horizontal tubes and spheres

$$h_{\text{horiz}} = 0.729 \left[\frac{g\rho_l(\rho_l - \rho_v)\, h_{fg}^{*}\, k_l^3}{\mu_l (T_{\text{sat}} - T_s) D} \right]^{1/4}$$

Inclined Plates

$$\frac{h_{\text{vert}}}{h_{\text{horiz}}} = 1.29 \left(\frac{D}{L} \right)^{1/4}$$

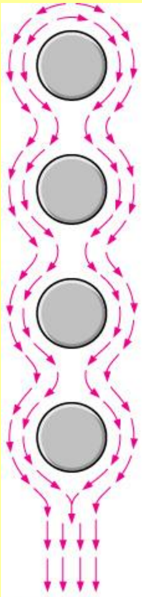


FIGURE 10–28
Film condensation on a vertical tier of horizontal tubes.

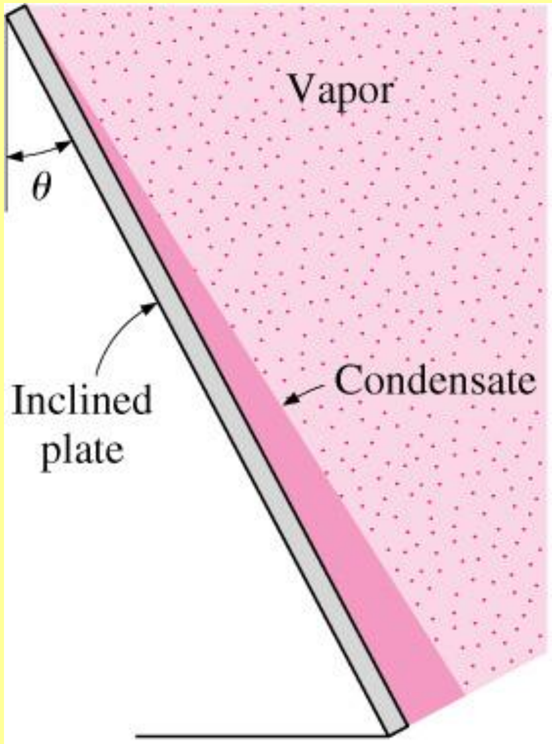


FIGURE 10–27
Film condensation on an inclined plate.

Horizontal tube banks

$$h_{\text{horiz, } N \text{ tubes}} = 0.729 \left[\frac{g\rho_l(\rho_l - \rho_v)\, h_{fg}^{*}\, k_l^3}{\mu_l (T_{\text{sat}} - T_s)\, ND} \right]^{1/4} = \frac{1}{N^{1/4}} h_{\text{horiz, 1 tube}}$$

For *low vapor velocities,* Chato (1962) recommends this expression for condensation

$$h_{\text{internal}} = 0.555\left[\frac{g\rho_l(\rho_l - \rho_v)k_l^3}{\mu_l(T_{\text{sat}} - T_s)}\left(h_{fg} + \frac{3}{8}c_{pl}(T_{\text{sat}} - T_s)\right)\right]^{1/4} \tag{10–34}$$

for

$$\text{Re}_{\text{vapor}} = \left(\frac{\rho_v V_v D}{\mu_v}\right)_{\text{inlet}} < 35{,}000 \tag{10–35}$$

Film condensation inside horizontal tubes

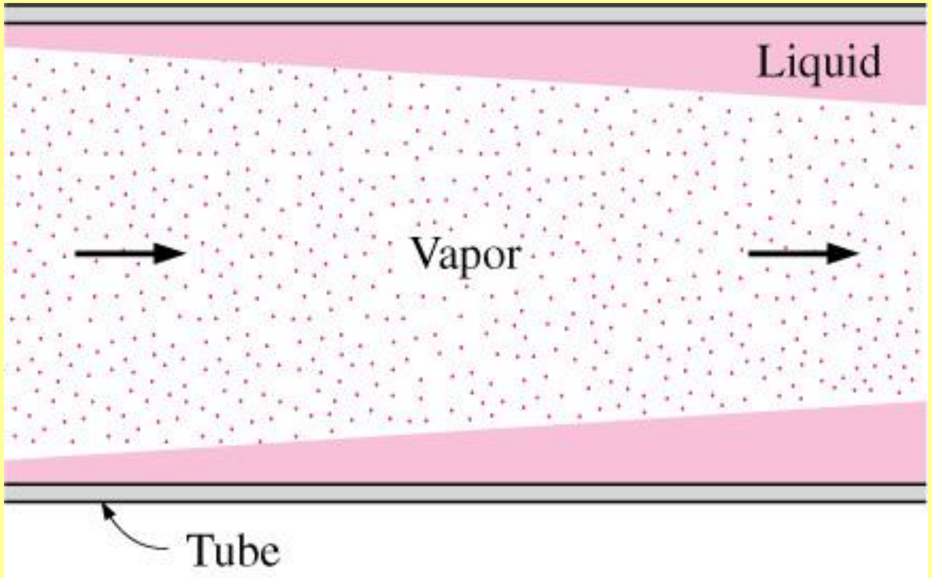


FIGURE 10–34

Condensate flow in a horizontal tube with large vapor velocities.

Dropwise Condensation

- One of the most effective mechanisms of heat transfer, and extremely large heat transfer coefficients can be achieved.
- Small droplets grow as a result of continued condensation, coalesce into large droplets, and slide down when they reach a certain size.
- Large heat transfer coefficients enable designers to achieve a specified heat transfer rate with a smaller surface area.

FIGURE 10–35
Dropwise condensation of steam on a vertical surface.
(From Hampson and Özişik.)

- The challenge in dropwise condensation is not to achieve it, but rather, to *sustain* it for prolonged periods of time.
- Dropwise condensation has been studied experimentally for a number of surface–fluid combinations.
- Griffith (1983) recommends these simple correlations for dropwise condensation of *steam* on *copper surfaces:*

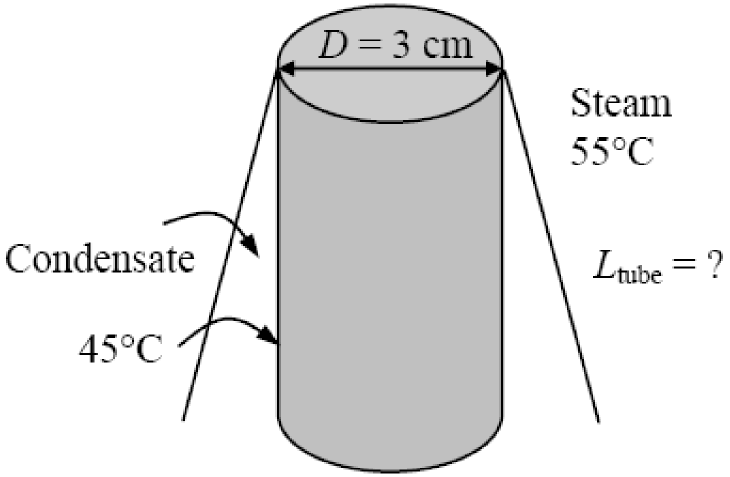
$$h_{\text{dropwise}} = \begin{cases} 51{,}104 + 2044T_{\text{sat}}, & 22^\circ\text{C} < T_{\text{sat}} < 100^\circ\text{C} \quad \text{(10–36)} \\ 255{,}310 & T_{\text{sat}} > 100^\circ\text{C} \quad \text{(10–37)} \end{cases}$$

where T_{sat} is in °C and the heat transfer coefficient h_{dropwise} is in W/m^2 · °C.

10–54 Saturated steam at 55°C is to be condensed at a rate of 10 kg/h on the outside of a 3-cm-outer-diameter vertical tube whose surface is maintained at 45°C by the cooling water. Determine the required tube length.

Properties The properties of water at the saturation temperature of 55°C are $h_{fg} = 2371\times10^3$ J/kg and ρ_v = 0.1045 kg/m^3. The properties of liquid water at the film temperature of $T_f = (T_{sat} + T_s)/2 = (55 + 45)/2 =$ 50°C are (Table A-9),

$$\rho_l = 988.1 \text{ kg/m}^3$$
$$\mu_l = 0.547\times10^{-3} \text{ kg/m}\cdot\text{s}$$
$$\nu_l = \mu_l/\rho_l = 0.554\times10^{-6} \text{ m}^2\text{/s}$$
$$c_{pl} = 4181 \text{ J/kg}\cdot°\text{C}$$
$$k_l = 0.644 \text{ W/m}\cdot°\text{C}$$



Analysis The modified latent heat of vaporization is

$$h^*_{fg} = h_{fg} + 0.68c_{pl}(T_{sat} - T_s)$$
$$= 2371\times10^3 \text{ J/kg} + 0.68\times4181 \text{ J/kg}\cdot°\text{C}(55-45)°\text{C} = 2399\times10^3 \text{ J/kg}$$

The Reynolds number is determined from its definition to be

$$\text{Re} = \frac{4\dot{m}}{p\mu_l} = \frac{4(10/3600\ \text{kg/s})}{\pi(0.03\ \text{m})(0.547\times10^{-3}\ \text{kg/m}\cdot\text{s})} = 215.5$$

which is between 30 and 1800. Therefore the condensate flow is wavy laminar, and the condensation heat transfer coefficient is determined from

$$h = h_{\text{vertical,wavy}} = \frac{\text{Re}\,k_l}{1.08\,\text{Re}^{1.22} - 5.2}\left(\frac{g}{\nu_l^2}\right)^{1/3}$$

$$= \frac{215.5\times(0.644\ \text{W/m}\cdot{}^\circ\text{C})}{1.08(215.5)^{1.22} - 5.2}\left(\frac{9.8\ \text{m/s}^2}{(0.554\times10^{-6}\ \text{m}^2/\text{s})^2}\right)^{1/3} = 5644\ \text{W/m}^2\cdot{}^\circ\text{C}$$

The rate of heat transfer during this condensation process is

$$\dot{Q} = \dot{m}h_{fg}^{*} = (10/3600\ \text{kg/s})(2399\times10^3\ \text{J/kg}) = 6{,}664\ \text{W}$$

Heat transfer can also be expressed as

$$\dot{Q} = hA_s(T_{\text{sat}} - T_s) = h(\pi DL)(T_{\text{sat}} - T_s)$$

Then the required length of the tube becomes

$$L = \frac{\dot{Q}}{h(\pi D)(T_{\text{sat}} - T_s)} = \frac{6664\ \text{W}}{(5844\ \text{W/m}^2\cdot{}^\circ\text{C})\pi(0.03\ \text{m})(55-45)^\circ\text{C}} = \mathbf{1.21\ m}$$

Concluding Points

- Boiling heat transfer
- Pool boiling
  - Boiling regimes and the boiling curve
- Flow boiling
- Condensation heat transfer
- Film condensation
- Film condensation inside horizontal tubes
- Dropwise condensation

HEAT AND MASS TRANSFER

Heat Exchangers

Objectives

- Recognize numerous types of heat exchangers, and classify them,
- Develop an awareness of fouling on surfaces, and determine the overall heat transfer coefficient for a heat exchanger,
- Perform a general energy analysis on heat exchangers,
- Obtain a relation for the logarithmic mean temperature difference for use in the LMTD method, and modify it for different types of heat exchangers using the correction factor,
- Develop relations for effectiveness, and analyze heat exchangers when outlet temperatures are not known using the effectiveness-NTU method,
- Know the primary considerations in the selection of heat exchangers.

TYPES OF HEAT EXCHANGERS

Heat exchanger: a device used to transfer **heat** between fluids that are at different temperatures and separated by a solid wall.

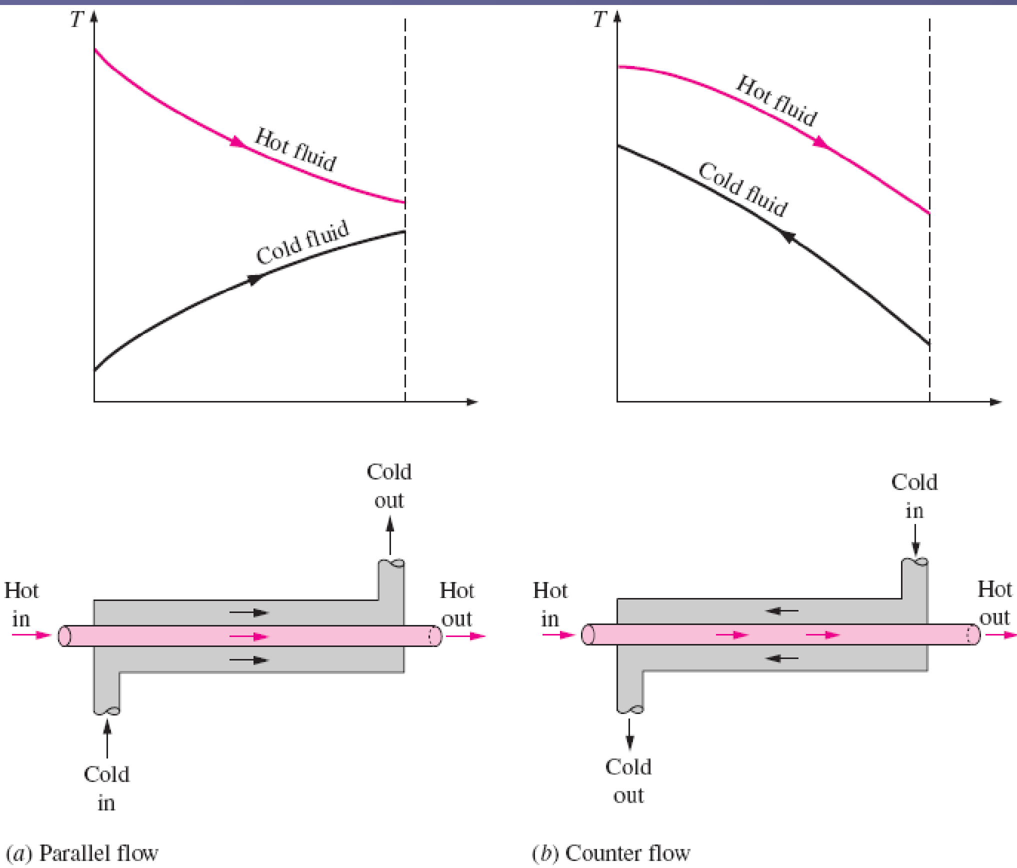


FIGURE 11–1
Different flow regimes and associated temperature profiles in a double-pipe heat exchanger.

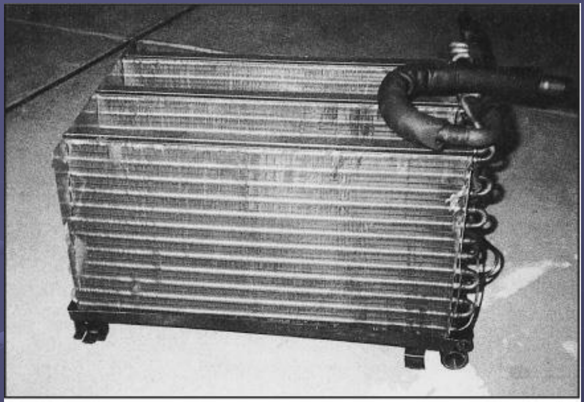


FIGURE 11–2
A gas-to-liquid compact heat exchanger for a residential air-conditioning system.
(© Yunus Çengel)

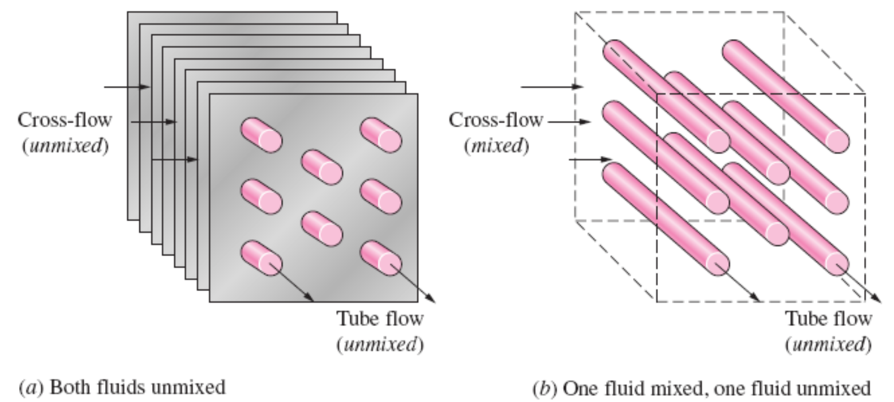


FIGURE 11–3
Different flow configurations in cross-flow heat exchangers.

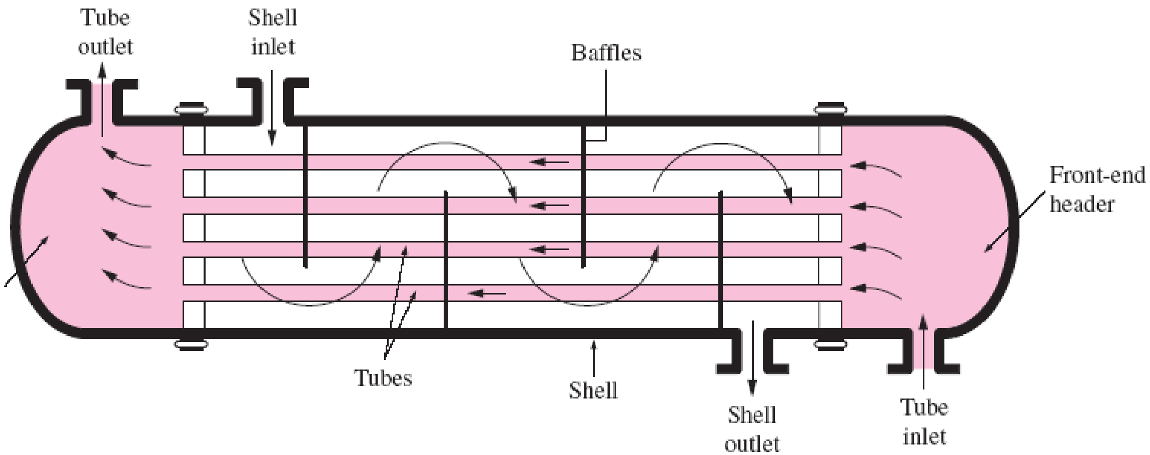


FIGURE 11–4
The schematic of a shell-and-tube heat exchanger (one-shell pass and one-tube pass).

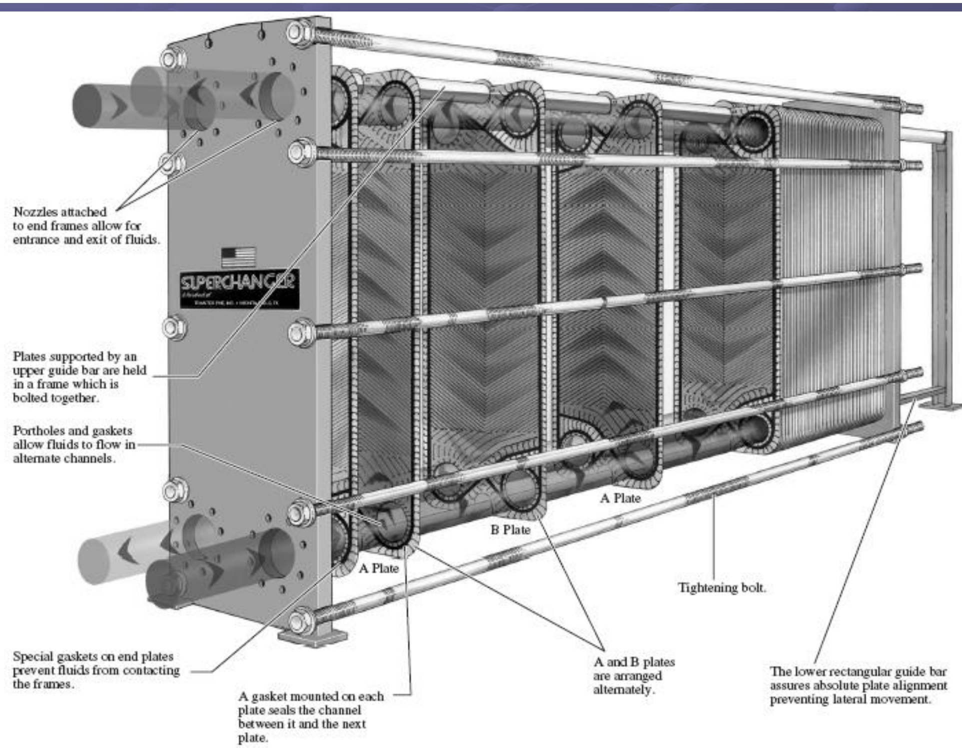
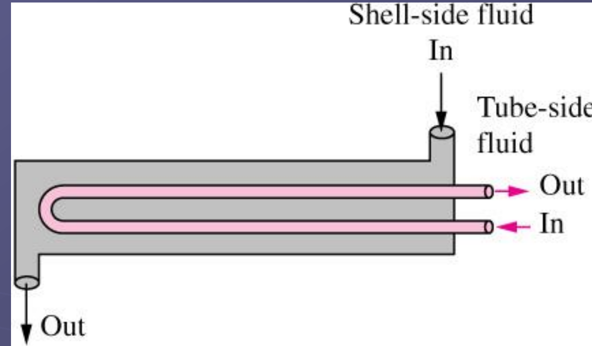
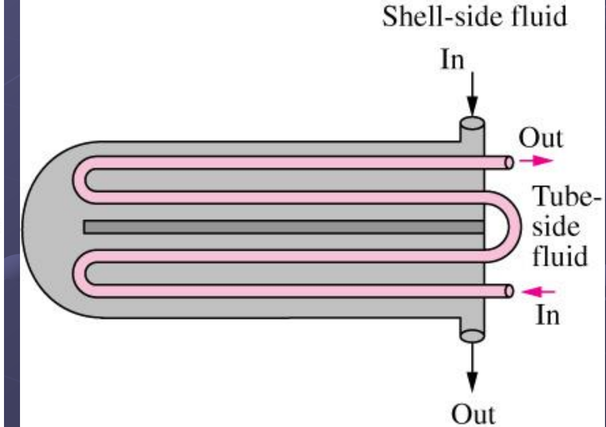


FIGURE 11–6
A plate-and-frame liquid-to-liquid heat exchanger.
(Courtesy of Tranter PHE, Inc.)



(*a*) One-shell pass and two-tube passes



(*b*) Two-shell passes and four-tube passes

FIGURE 11–5
Multipass flow arrangements in shell-and-tube heat exchangers.

A Plate Heat Exchanger Used in a Geothermal District Heating System in Izmir, Turkey


Scaling/fouling

THE OVERALL HEAT TRANSFER COEFFICIENT

For a double-pipe heat exchanger:

$$R_{\text{wall}} = \frac{\ln(D_o/D_i)}{2\pi kL} \qquad A_i = \pi D_i L \qquad A_o = \pi D_o L$$

The total thermal resistance:

$$R = R_{\text{total}} = R_i + R_{\text{wall}} + R_o = \frac{1}{h_i A_i} + \frac{\ln(D_o/D_i)}{2\pi kL} + \frac{1}{h_o A_o}$$

The rate of heat transfer between the two fluids:

$$\dot{Q} = \frac{\Delta T}{R} = UA\,\Delta T = U_i A_i\,\Delta T = U_o A_o\,\Delta T$$

The overall heat transfer coefficient (U): $\mathrm{W/m^2 \cdot {}^\circ C}$

$$\frac{1}{UA_s} = \frac{1}{U_i A_i} = \frac{1}{U_o A_o} = R = \frac{1}{h_i A_i} + R_{\text{wall}} + \frac{1}{h_o A_o}$$

$$U_i A_i = U_o A_o$$

$$U_i \neq U_o \text{ unless } A_i = A_o$$

For $(R_{\text{wall}} \approx 0)$ and $(A_i \approx A_o \approx A_s)$

The overall heat transfer coefficient becomes:

$$\frac{1}{U} \approx \frac{1}{h_i} + \frac{1}{h_o} \qquad U \approx U_i \approx U_o$$

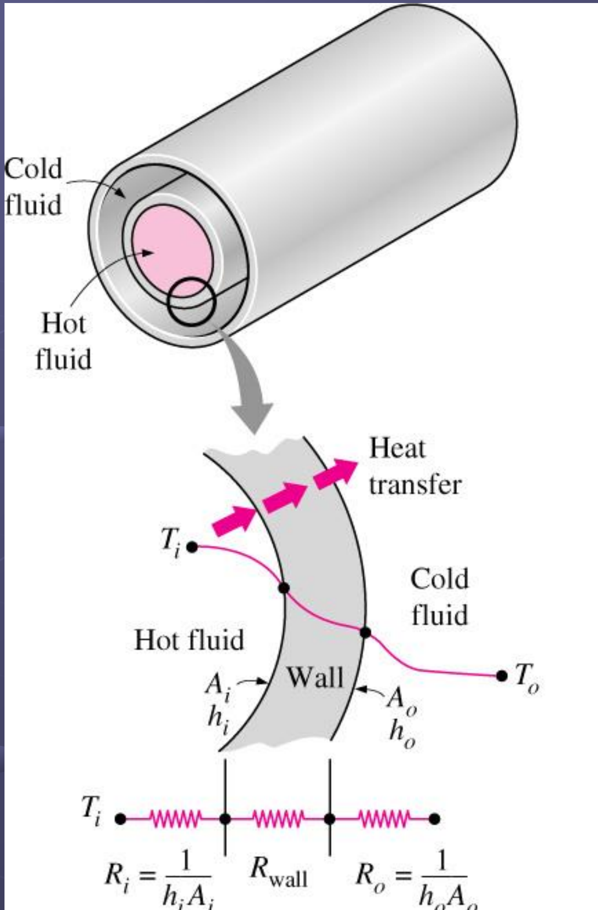


FIGURE 11–7
Thermal resistance network associated with heat transfer in a double-pipe heat exchanger.

When the tube is *finned* on one side to enhance heat transfer, the total heat transfer surface area on the finned side becomes

$$A_s = A_{total} = A_{fin} + A_{unfinned}$$

$$A_s = A_{unfinned} + \eta_{fin} A_{fin}$$

A_{fin} : surface area of the fins

$A_{unfinned}$: area of the unfinned portion of the tube surface

TABLE 11–1

Representative values of the overall heat transfer coefficients in heat exchangers

Type of heat exchanger	U, W/m$^2 \cdot$ °C*
Water-to-water	850–1700
Water-to-oil	100–350
Water-to-gasoline or kerosene	300–1000
Feedwater heaters	1000–8500
Steam-to-light fuel oil	200–400
Steam-to-heavy fuel oil	50–200
Steam condenser	1000–6000
Freon condenser (water cooled)	300–1000
Ammonia condenser (water cooled)	800–1400
Alcohol condensers (water cooled)	250–700
Gas-to-gas	10–40
Water-to-air in finned tubes (water in tubes)	30–60†
	400–850‡
Steam-to-air in finned tubes (steam in tubes)	30–300†
	400–4000‡

*Multiply the listed values by 0.176 to convert them to Btu/h · ft^2 · °F.

†Based on air-side surface area.

‡Based on water- or steam-side surface area.

Fouling Factor

The layer of deposits represents *additional resistance* to heat transfer and causes the rate of heat transfer in a heat exchanger to decrease. It is represented by a **fouling factor** R_f, as a measure of the *thermal resistance* introduced by fouling.

For an unfinned shell-and-tube heat exchanger:

$$\frac{1}{UA_s} = \frac{1}{U_iA_i} = \frac{1}{U_oA_o} = R = \frac{1}{h_iA_i} + \frac{R_{f,i}}{A_i} + \frac{\ln(D_o/D_i)}{2\pi kL} + \frac{R_{f,o}}{A_o} + \frac{1}{h_oA_o}$$

$R_{f,i}$: the fouling factors at A_i

$R_{f,o}$: the fouling factor at A_o

$$A_i = \pi D_i L$$

$$A_o = \pi D_o L$$

FIGURE 11–9
Precipitation fouling of ash particles on superheater tubes.
(From Steam: Its Generation, and Use, Babcock and Wilcox Co., 1978. Reprinted by permission.)

TABLE 11–2

Representative fouling factors (thermal resistance due to fouling for a unit surface area)

Fluid	R_f, m$^2 \cdot$ °C/W
Distilled water, sea-water, river water, boiler feedwater:	
Below 50°C	0.0001
Above 50°C	0.0002
Fuel oil	0.0009
Steam (oil-free)	0.0001
Refrigerants (liquid)	0.0002
Refrigerants (vapor)	0.0004
Alcohol vapors	0.0001
Air	0.0004

(*Source:* Tubular Exchange Manufacturers Association.)

ANALYSIS OF HEAT EXCHANGERS

Assumptions:

- The *kinetic* and *potential energy changes* are negligible.
- The *specific heats* of the fluids is constant
- *Axial heat conduction* along the tube is considered negligible.
- The heat loss from the outer surface to the surroundings is negligible.

The rate of heat transfer from the hot fluid is equal to the rate of heat transfer to the cold one:

$$\dot{Q} = \dot{m}_c c_{pc}(T_{c,\text{out}} - T_{c,\text{in}}) \quad \text{and} \quad \dot{Q} = \dot{m}_h c_{ph}(T_{h,\text{in}} - T_{h,\text{out}})$$

The subscripts *c* and *h* stand for *cold* and *hot* fluids, respectively.

$$\begin{aligned} \dot{m}_c, \dot{m}_h &= \text{mass flow rates} \\ c_{pc}, c_{ph} &= \text{specific heats} \\ T_{c,\text{out}}, T_{h,\text{out}} &= \text{outlet temperatures} \\ T_{c,\text{in}}, T_{h,\text{in}} &= \text{inlet temperatures} \end{aligned}$$

The **heat capacity rate** for the hot and cold fluid streams:

$$C_h = \dot{m}_h c_{ph} \quad \text{and} \quad C_c = \dot{m}_c c_{pc}$$

$$\dot{Q} = C_c(T_{c,\text{out}} - T_{c,\text{in}}) \qquad \dot{Q} = C_h(T_{h,\text{in}} - T_{h,\text{out}})$$

The rate of heat transfer in a condenser or a boiler undergoes a phase-change process:

$$\dot{Q} = \dot{m} h_{fg}$$

$\dot{m}$: the rate of evaporation or condensation of the fluid

h_{fg} : the enthalpy of vaporization of the fluid at the specified temperature or pressure

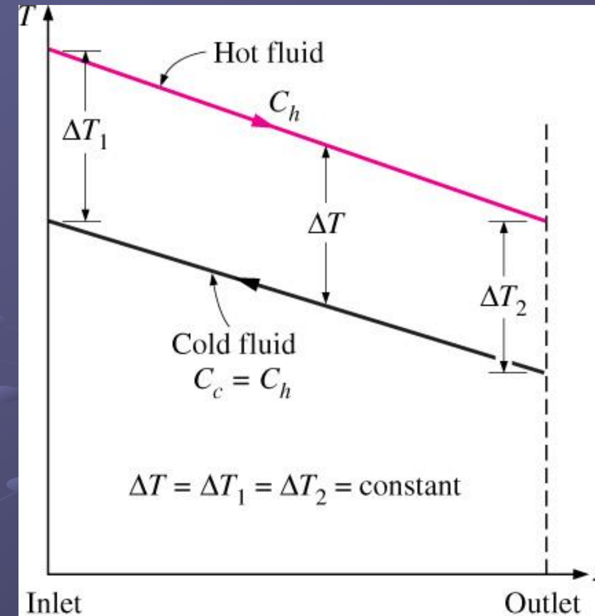
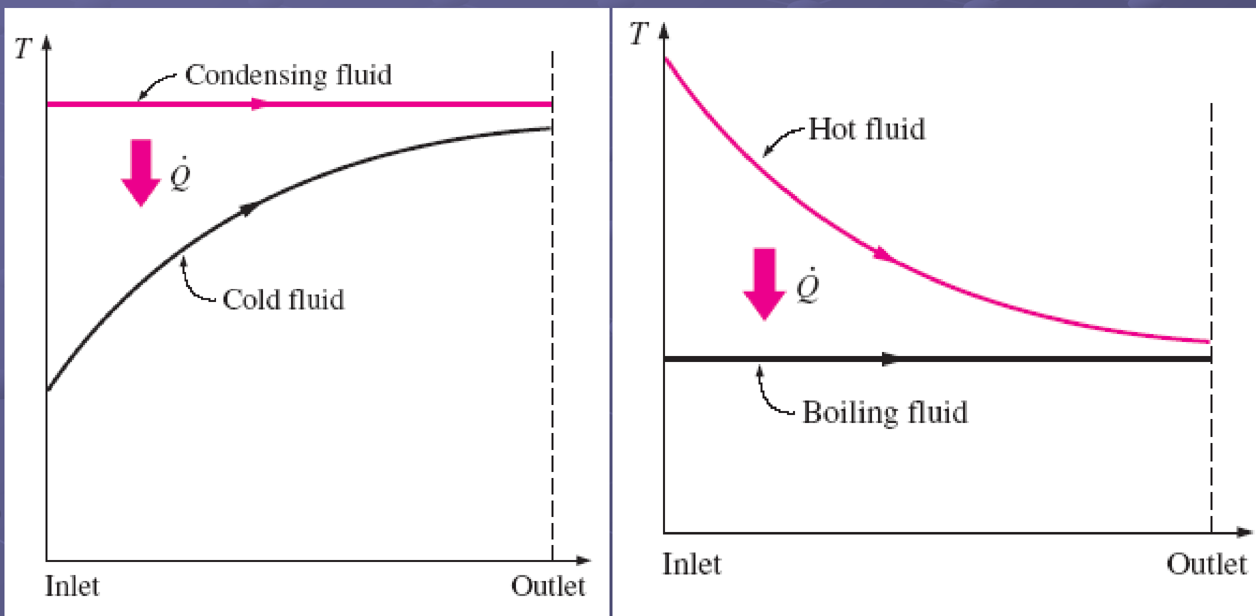


FIGURE 11–12

Two fluid streams that have the same capacity rates experience the same temperature change in a well-insulated heat exchanger.



(a) Condenser ($C_h \to \infty$)

(b) Boiler ($C_c \to \infty$)

FIGURE 11–13

Variation of fluid temperatures in a heat exchanger when one of the fluids condenses or boils.

$$\dot{Q} = UA_s \Delta T_m$$

U : the overall heat transfer coefficient

A_s : the heat transfer area,

ΔT_m : average temperature difference between the two fluids

THE LOG MEAN TEMPERATURE DIFFERENCE METHOD (ΔT_{lm})

$$\delta \dot{Q} = -\dot{m}_h C_{ph}\, dT_h$$

$$\delta \dot{Q} = \dot{m}_c C_{pc}\, dT_c$$

$$dT_h = -\frac{\delta \dot{Q}}{\dot{m}_h C_{ph}}$$

$$dT_c = \frac{\delta \dot{Q}}{\dot{m}_c C_{pc}}$$

$$dT_h - dT_c = d(T_h - T_c) = -\delta \dot{Q}\left(\frac{1}{\dot{m}_h C_{ph}} + \frac{1}{\dot{m}_c C_{pc}}\right)$$

$$\delta \dot{Q} = U(T_h - T_c)\, dA_s$$

$$\frac{d(T_h - T_c)}{T_h - T_c} = -U\, dA_s\left(\frac{1}{\dot{m}_h C_{ph}} + \frac{1}{\dot{m}_c C_{pc}}\right)$$

$$\ln \frac{T_{h,\,\mathrm{out}} - T_{c,\,\mathrm{out}}}{T_{h,\,\mathrm{in}} - T_{c,\,\mathrm{in}}} = -UA_s\left(\frac{1}{\dot{m}_h C_{ph}} + \frac{1}{\dot{m}_c C_{pc}}\right)$$

$$\dot{Q} = UA_s\, \Delta T_{\mathrm{lm}}$$

$$\Delta T_{\mathrm{lm}} = \frac{\Delta T_1 - \Delta T_2}{\ln(\Delta T_1/\Delta T_2)}$$

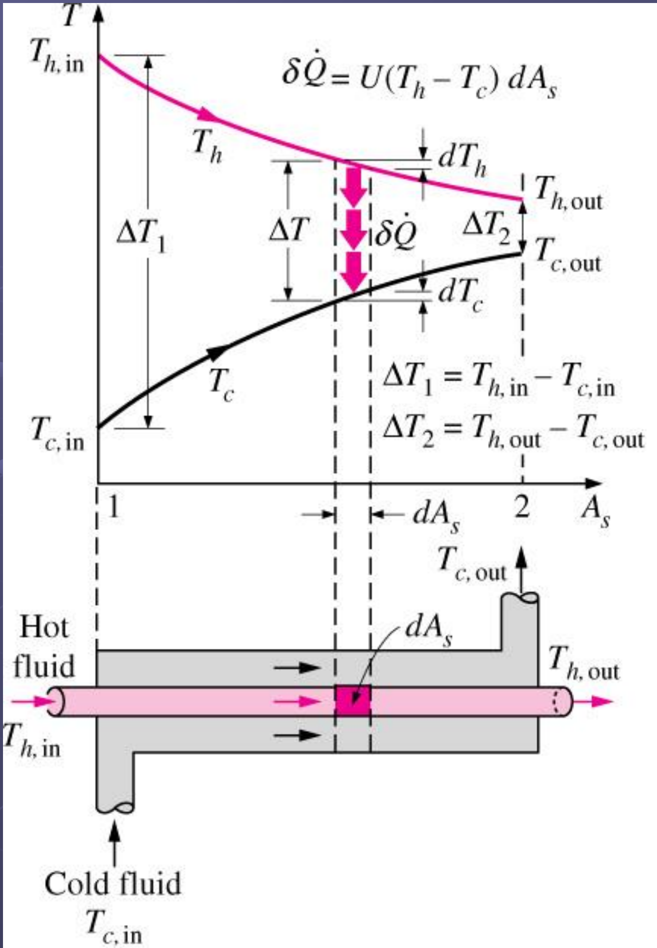
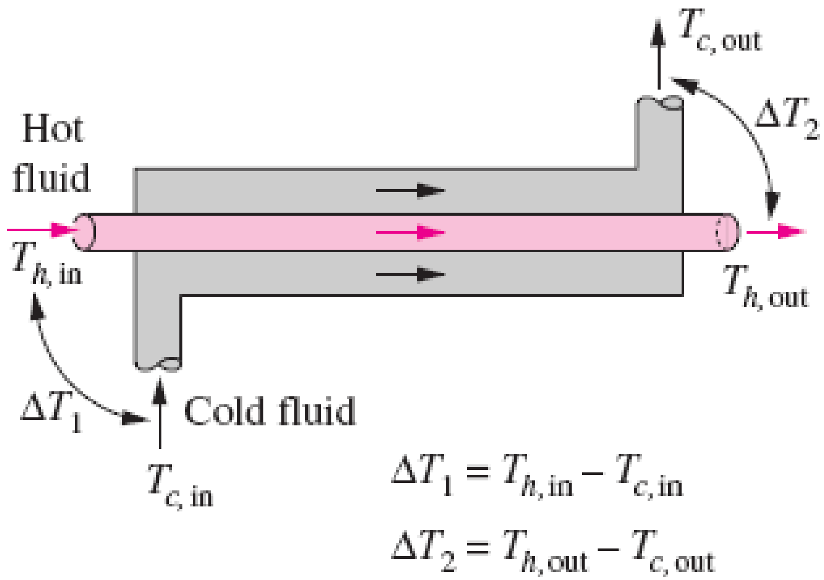
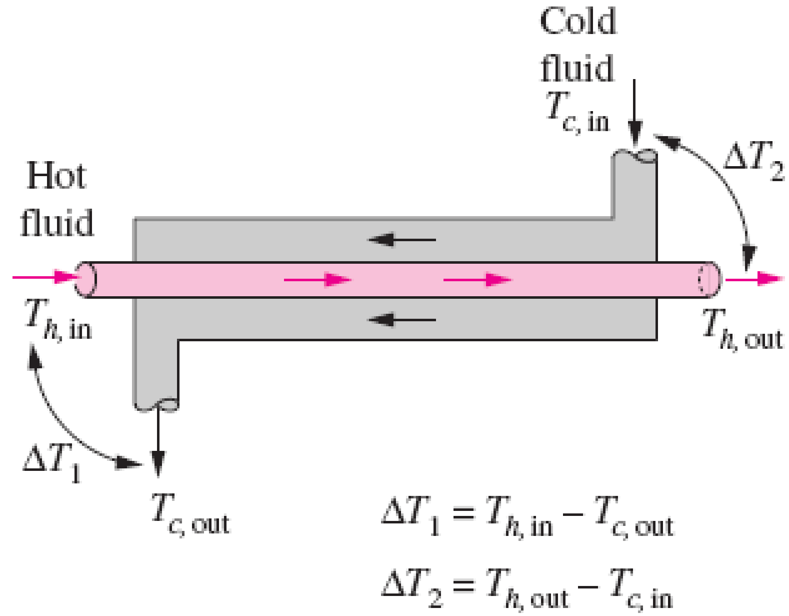


FIGURE 11–14
Variation of the fluid temperatures in a parallel-flow double-pipe heat exchanger.

ΔT_{lm} : the log mean temperature difference



(*a*) Parallel-flow heat exchangers



(*b*) Counter-flow heat exchangers

FIGURE 11–15
The ΔT_1 and ΔT_2 expressions in parallel-flow and counter-flow heat exchangers.

$\Delta T_{am} = \frac{1}{2}(\Delta T_1 + \Delta T_2)$ arithmetic mean temperature

ΔT_{lm} is obtained by tracing the actual temperature profile of the fluids along the heat exchanger and is an *exact representation* of the *average temperature difference* between the hot and cold fluids.

ΔT_{lm} is always less than ΔT_{am}.

Counter-Flow Heat Exchangers

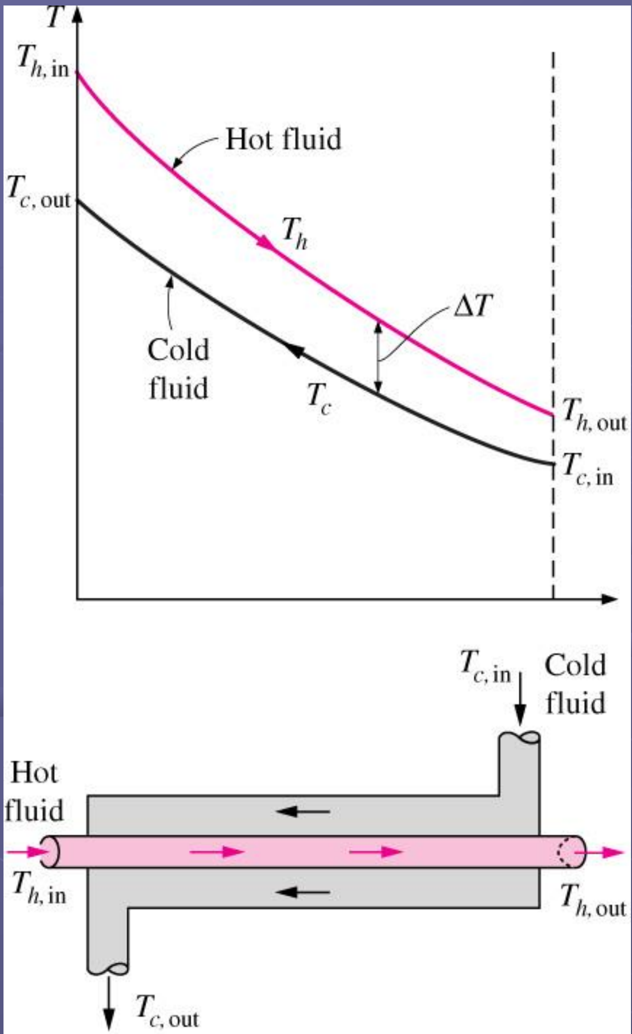


FIGURE 11–16

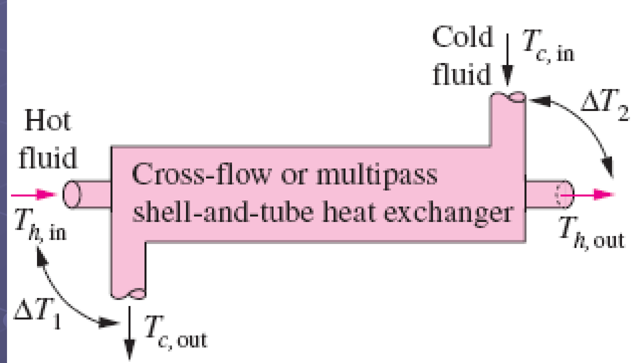
The variation of the fluid temperatures in a counter-flow double-pipe heat exchanger.

Multipass and Cross-Flow Heat Exchangers

Use of a Correction Factor (F)

$$\Delta T_{lm} = F\,\Delta T_{lm,\,CF}$$

$\Delta T_{lm,\,CF}$: the log mean temperature difference for *counter-flow* heat exchangers with the same inlet and outlet temperatures



Heat transfer rate:

$$\dot{Q} = UA_sF\Delta T_{lm,CF}$$

where $$\Delta T_{lm,CF} = \frac{\Delta T_1 - \Delta T_2}{\ln(\Delta T_1/\Delta T_2)}$$

$$\Delta T_1 = T_{h,in} - T_{c,out}$$

$$\Delta T_2 = T_{h,out} - T_{c,in}$$

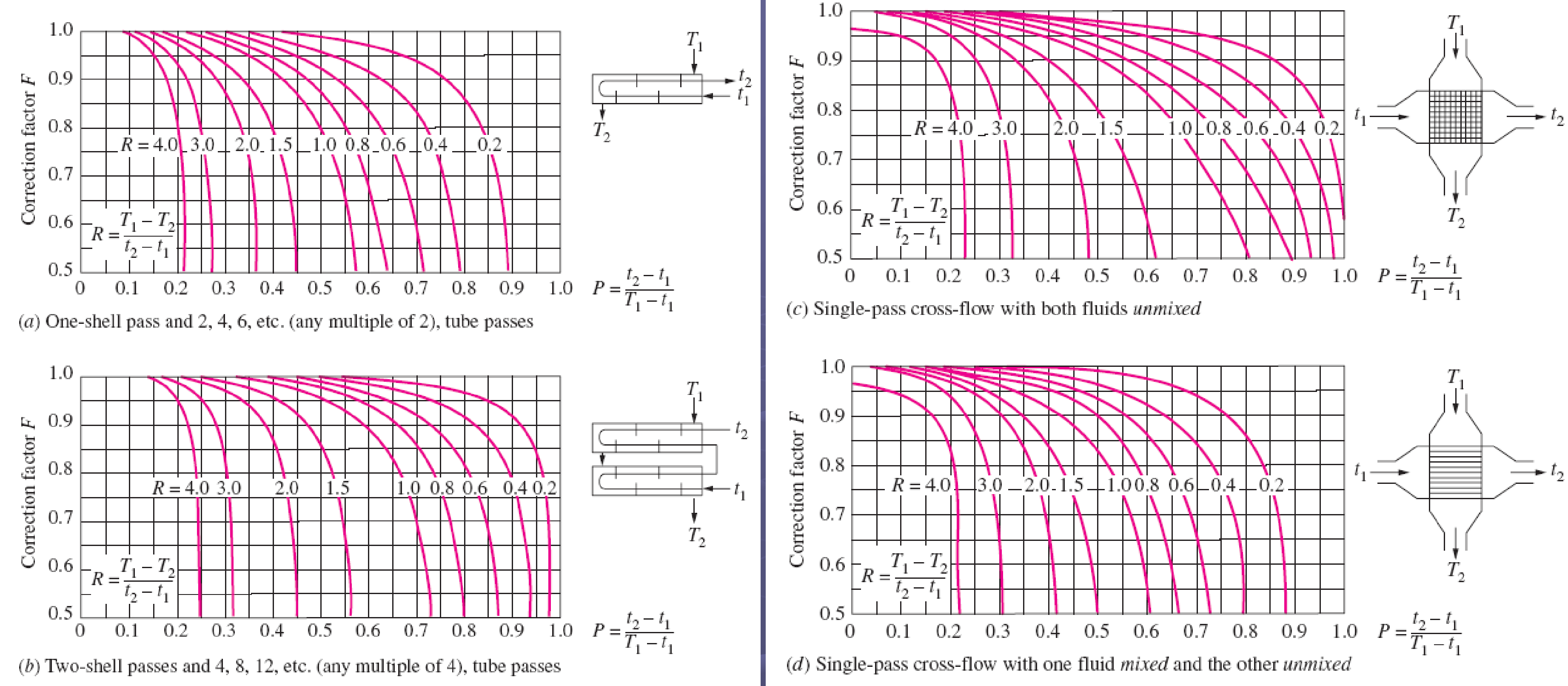
and $F = \ldots$ (Fig. 23–18)

Temperature ratios:

$$R = \frac{T_1 - T_2}{t_2 - t_1} = \frac{(\dot{m}C_p)_{\text{tube side}}}{(\dot{m}C_p)_{\text{shell side}}}$$

$$P = \frac{t_2 - t_1}{T_1 - t_1}$$

F for a heat exchanger is *a measure of deviation of the* ΔT_{lm} *from the corresponding values for the counter-flow case.*



(*a*) One-shell pass and 2, 4, 6, etc. (any multiple of 2), tube passes

(*b*) Two-shell passes and 4, 8, 12, etc. (any multiple of 4), tube passes

(*c*) Single-pass cross-flow with both fluids *unmixed*

(*d*) Single-pass cross-flow with one fluid *mixed* and the other *unmixed*

FIGURE 11–18

Correction factor F charts for common shell-and-tube and cross-flow heat exchangers. *(From Bowman, Mueller, and Nagle)*

The correction factor for a *condenser* or *boiler* is *F*= 1, regardless of the configuration of the heat exchanger.

11–44 A stream of hydrocarbon ($c_p = 2.2$ kJ/kg · K) is cooled at a rate of 720 kg/h from 150°C to 40°C in the tube side of a double-pipe counter-flow heat exchanger. Water ($c_p = 4.18$ kJ/kg · K) enters the heat exchanger at 10°C at a rate of 540 kg/h. The outside diameter of the inner tube is 2.5 cm, and its length is 6.0 m. Calculate the overall heat transfer coefficient.

Analysis The rate of heat transfer is

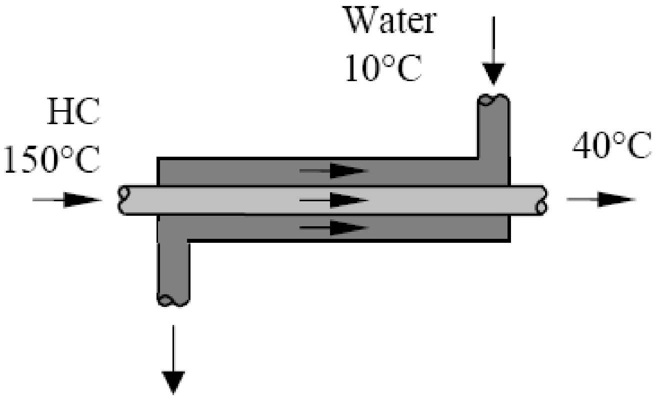
$$\dot{Q} = [\dot{m}c_p(T_{out} - T_{in})]_{\mathrm{HC}} = (720/3600\ \mathrm{kg/s})(2.2\ \mathrm{kJ/kg.°C})(150°\mathrm{C} - 40°\mathrm{C}) = 48.4\ \mathrm{kW}$$

The outlet temperature of water is

$$\dot{Q} = [\dot{m}c_p(T_{out} - T_{in})]_{\mathrm{w}}$$

$$48.4\ \mathrm{kW} = (540/3600\ \mathrm{kg/s})(4.18\ \mathrm{kJ/kg.°C})(T_{\mathrm{w,out}} - 10°\mathrm{C})$$

$$T_{\mathrm{w,out}} = 87.2\ °\mathrm{C}$$



The logarithmic mean temperature difference is

$$\Delta T_1 = T_{h,in} - T_{c,out} = 150°\mathrm{C} - 87.2°\mathrm{C} = 62.8°\mathrm{C}$$

$$\Delta T_2 = T_{h,out} - T_{c,in} = 40°\mathrm{C} - 10°\mathrm{C} = 30°\mathrm{C}$$

and $$\Delta T_{lm} = \frac{\Delta T_1 - \Delta T_2}{\ln(\Delta T_1/\Delta T_2)} = \frac{62.8 - 30}{\ln(62.8/30)} = 44.4°\mathrm{C}$$

The overall heat transfer coefficient is determined from

$$\dot{Q} = UA\Delta T_{lm}$$

$$48.4\ \mathrm{kW} = U(\pi \times 0.025 \times 6.0)(44.4°\mathrm{C})$$

$$U = \mathbf{2.31\ kW/m^2 \cdot K}$$

THE EFFECTIVENESS-NTU METHOD

The heat transfer surface area of the heat exchanger can be determined from $\dot{Q} = UA_s\,\Delta T_{\mathrm{lm}}$

With the LMTD method, the task is to *select* a heat exchanger that will meet the prescribed heat transfer requirements. The procedure to be followed by the selection process is:

1. Select the type of heat exchanger suitable for the application.
2. Determine any unknown inlet or outlet temperature and the heat transfer rate using an energy balance.
3. Calculate the log mean temperature difference ΔT_{lm} and the correction factor *F*, if necessary.
4. Obtain (select or calculate) the value of the overall heat transfer coefficient *U*.
5. Calculate the heat transfer surface area A_s .

A second kind of problem encountered in heat exchanger analysis is the determination of the *heat transfer rate* and the *outlet temperatures* of the hot and cold fluids for prescribed fluid mass flow rates and inlet temperatures when the *type* and *size* of the heat exchanger are specified.

The effectiveness (ε) –NTU method:

$$\varepsilon = \frac{\dot{Q}}{\dot{Q}_{\max}} = \frac{\text{Actual heat transfer rate}}{\text{Maximum possible heat transfer rate}}$$

The heat capacity rates of the cold and the hot fluids:

$$C_c = \dot{m}_c C_{pc} \qquad C_h = \dot{m}_c C_{ph}$$

The *actual* heat transfer rate in a heat exchanger:

$$\dot{Q} = C_c(T_{c,\text{ out}} - T_{c,\text{ in}}) = C_h(T_{h,\text{ in}} - T_{h,\text{ out}})$$

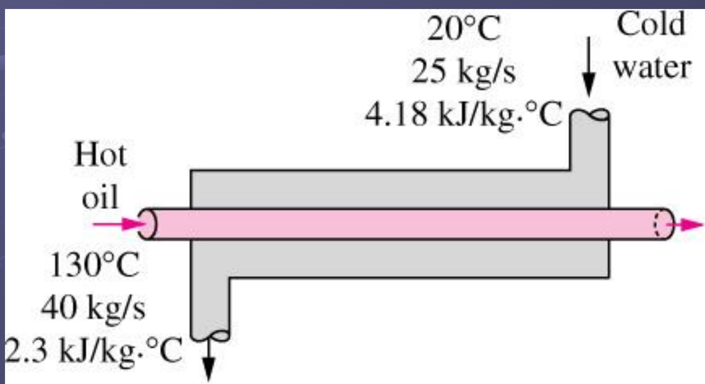
The *maximum temperature difference* in a heat exchanger is the difference between the *inlet* temperatures of the hot and cold fluids.

$$\Delta T_{\max} = T_{h,\text{ in}} - T_{c,\text{ in}}$$

The maximum possible heat transfer rate in a heat exchanger:

$$\dot{Q}_{\max} = C_{\min}(T_{h,\text{ in}} - T_{c,\text{ in}})$$

$C_{\min}$ is the smaller of $C_h = \dot{m}_h C_{ph}$ and $C_c = \dot{m}_c C_{pc}$



$C_c = \dot{m}_c c_{pc} = 104.5$ kW/°C

$C_h = \dot{m}_c c_{ph} = 92$ kW/°C

$C_{\min} = 92$ kW/°C

$\Delta T_{\max} = T_{h,\text{in}} - T_{c,\text{in}} = 110$°C

$\dot{Q}_{\max} = C_{\min}\,\Delta T_{\max} = 10{,}120$ kW

FIGURE 11–23

The determination of the maximum rate of heat transfer in a heat exchanger.

The determination of $\dot{Q}_{\max}$ requires the availability of the *inlet temperature* of the hot and cold fluids and their *mass flow rates*

The actual heat transfer rate: $\dot{Q} = \varepsilon \dot{Q}_{\max} = \varepsilon C_{\min}(T_{h,\,\text{in}} - T_{c,\,\text{in}})$

The effectiveness of a heat exchanger enables us to determine the heat transfer rate without knowing the *outlet temperatures* of the fluids.

For a parallel-flow heat exchanger:

$$\ln \frac{T_{h,\,\text{out}} - T_{c,\,\text{out}}}{T_{h,\,\text{in}} - T_{c,\,\text{in}}} = -\frac{UA_s}{C_c}\left(1 + \frac{C_c}{C_h}\right)$$

$$T_{h,\,\text{out}} = T_{h,\,\text{in}} - \frac{C_c}{C_h}(T_{c,\,\text{out}} - T_{c,\,\text{in}})$$

$$\ln \frac{T_{h,\,\text{in}} - T_{c,\,\text{in}} + T_{c,\,\text{in}} - T_{c,\,\text{out}} - \frac{C_c}{C_h}(T_{c,\,\text{out}} - T_{c,\,\text{in}})}{T_{h,\,\text{in}} - T_{c,\,\text{in}}} = -\frac{UA_s}{C_c}\left(1 + \frac{C_c}{C_h}\right)$$

$$\ln\left[1 - \left(1 + \frac{C_c}{C_h}\right)\frac{T_{c,\,\text{out}} - T_{c,\,\text{in}}}{T_{h,\,\text{in}} - T_{c,\,\text{in}}}\right] = -\frac{UA_s}{C_c}\left(1 + \frac{C_c}{C_h}\right)$$

$$\varepsilon = \frac{\dot{Q}}{\dot{Q}_{\max}} = \frac{C_c(T_{c,\,\text{out}} - T_{c,\,\text{in}})}{C_{\min}(T_{h,\,\text{in}} - T_{c,\,\text{in}})} \longrightarrow \frac{T_{c,\,\text{out}} - T_{c,\,\text{in}}}{T_{h,\,\text{in}} - T_{c,\,\text{in}}} = \varepsilon \frac{C_{\min}}{C_c}$$

The relation for the effectiveness of a *parallel-flow* heat exchanger:

$$\varepsilon_{\text{parallel flow}} = \frac{1 - \exp\left[-\frac{UA_s}{C_c}\left(1 + \frac{C_c}{C_h}\right)\right]}{\left(1 + \frac{C_c}{C_h}\right)\frac{C_{\min}}{C_c}}$$

$$\varepsilon_{\text{parallel flow}} = \frac{1 - \exp\left[-\frac{UA_s}{C_{\min}}\left(1 + \frac{C_{\min}}{C_{\max}}\right)\right]}{1 + \frac{C_{\min}}{C_{\max}}}$$

The **number of transfer units (NTU):**

$$\text{NTU} = \frac{UA_s}{C_{\min}} = \frac{UA_s}{(\dot{m}C_p)_{\min}}$$

The **capacity ratio** c:

$$c = \frac{C_{\min}}{C_{\max}}$$

TABLE 11–4

Effectiveness relations for heat exchangers: $\text{NTU} = UA_s/C_{\min}$ and $c = C_{\min}/C_{\max} = (\dot{m}c_p)_{\min}/(\dot{m}c_p)_{\max}$

Heat exchanger type	Effectiveness relation
1 *Double pipe:*	
Parallel-flow	$\varepsilon = \dfrac{1 - \exp[-\text{NTU}(1 + c)]}{1 + c}$
Counter-flow	$\varepsilon = \dfrac{1 - \exp[-\text{NTU}(1 - c)]}{1 - c\exp[-\text{NTU}(1 - c)]}$
2 *Shell-and-tube:* One-shell pass 2, 4, . . . tube passes	$\varepsilon = 2\left\{1 + c + \sqrt{1 + c^2}\,\dfrac{1 + \exp[-\text{NTU}\sqrt{1 + c^2}]}{1 - \exp[-\text{NTU}\sqrt{1 + c^2}]}\right\}^{-1}$
3 *Cross-flow (single-pass)*	
Both fluids unmixed	$\varepsilon = 1 - \exp\left\{\dfrac{\text{NTU}^{0.22}}{c}\left[\exp(-c\,\text{NTU}^{0.78}) - 1\right]\right\}$
$C_{\max}$ mixed, $C_{\min}$ unmixed	$\varepsilon = \dfrac{1}{c}(1 - \exp\{1 - c[1 - \exp(-\text{NTU})]\})$
$C_{\min}$ mixed, $C_{\max}$ unmixed	$\varepsilon = 1 - \exp\left\{-\dfrac{1}{c}[1 - \exp(-c\,\text{NTU})]\right\}$
4 *All heat exchangers with $c = 0$*	$\varepsilon = 1 - \exp(-\text{NTU})$

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$$\varepsilon = \text{function}\,(UA_s/C_{\min},\ C_{\min}/C_{\max}) = \text{function}\,(\text{NTU},\ c)$$

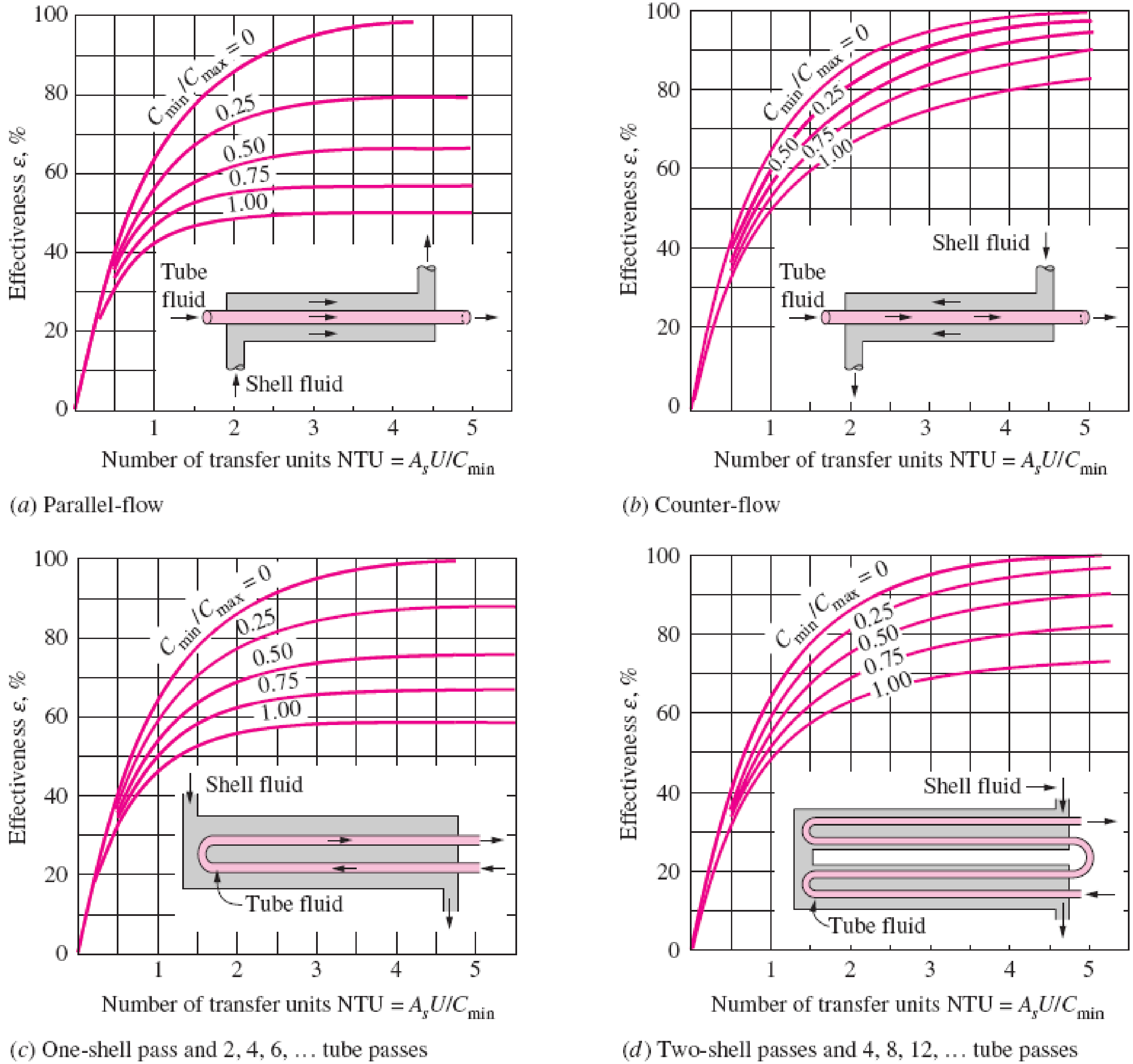
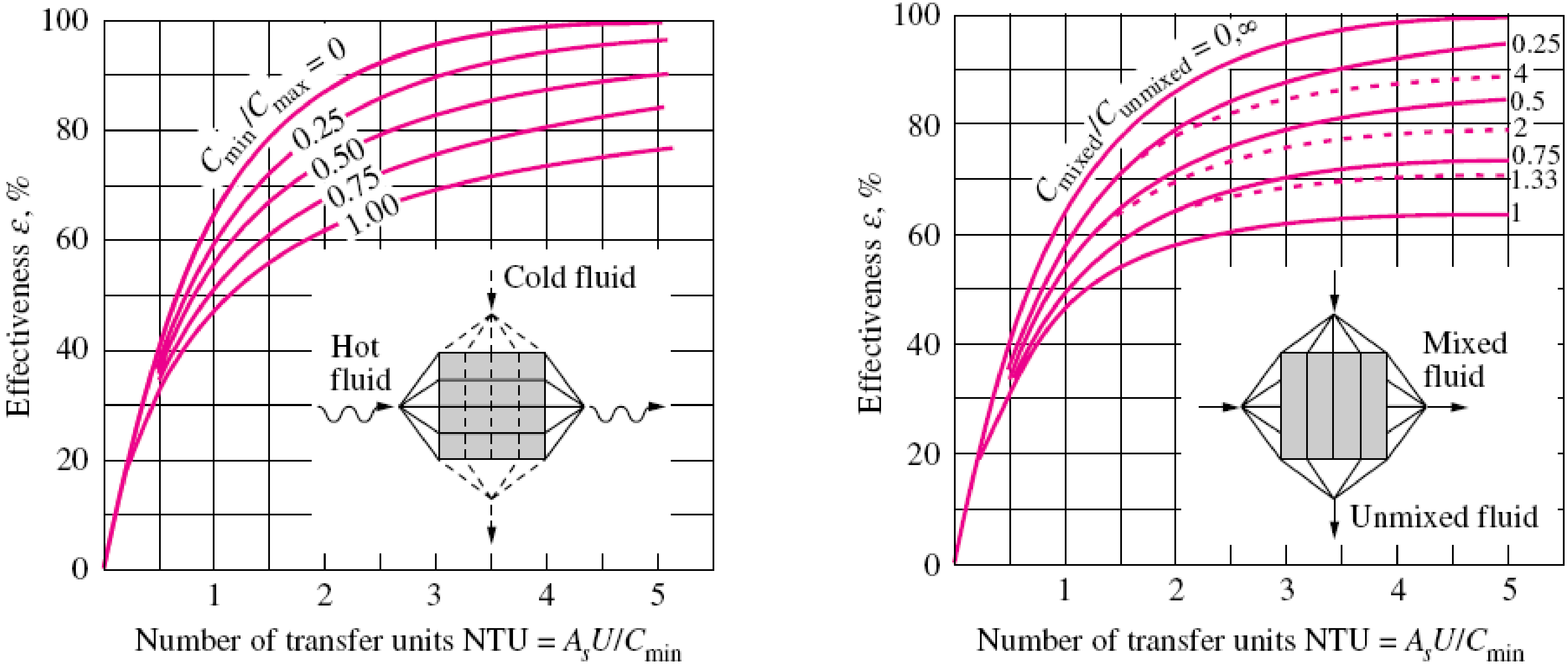


FIGURE 11–26

Effectiveness for heat exchangers.

(From Kays and London)



(*e*) Cross-flow with both fluids unmixed

(*f*) Cross-flow with one fluid mixed and the other unmixed

FIGURE 11–26

Effectiveness for heat exchangers.

(From Kays and London)

We make the following observations from the effectiveness relations and charts already given:

1. The value of the effectiveness ranges from 0 to 1. It increases rapidly with NTU for small values (up to about NTU = 1.5) but rather slowly for larger values. Therefore, the use of a heat exchanger with a large NTU (usually larger than 3) and thus a large size cannot be justified economically, since a large increase in NTU in this case corresponds to a small increase in effectiveness. Thus, a heat exchanger with a very high effectiveness may be highly desirable from a heat transfer point of view but rather undesirable from an economical point of view.
2. For a given NTU and capacity ratio $c = C_{min}/C_{max}$, the *counter-flow* heat exchanger has the *highest* effectiveness, followed closely by the cross-flow heat exchangers with both fluids unmixed. As you might expect, the lowest effectiveness values are encountered in parallel-flow heat exchangers (Fig. 23–27).
3. The effectiveness of a heat exchanger is independent of the capacity ratio c for NTU values of less than about 0.3.

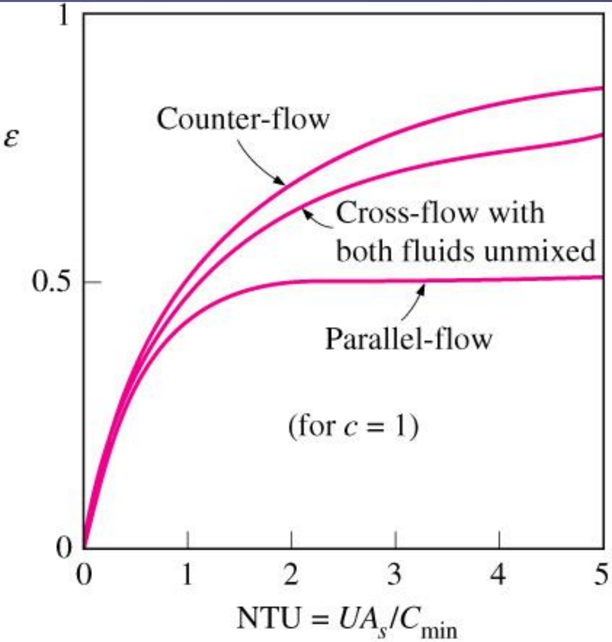


FIGURE 11–27
For a specified NTU and capacity ratio c, the counter-flow heat exchanger has the highest effectiveness and the parallel-flow the lowest.

4. The value of the capacity ratio c ranges between 0 and 1. For a given NTU, the effectiveness becomes a *maximum* for $c = 0$ and a *minimum* for $c = 1$. The case $c = C_{min}/C_{max} \to 0$ corresponds to $C_{max} \to \infty$, which is realized during a phase-change process in a *condenser* or *boiler.* All effectiveness relations in this case reduce to

$$\varepsilon = \varepsilon_{max} = 1 - \exp(-NTU)$$

regardless of the type of heat exchanger (Fig. 23–28). Note that the temperature of the condensing or boiling fluid remains constant in this case. The effectiveness is the *lowest* in the other limiting case of $c = C_{min}/C_{max} = 1$, which is realized when the heat capacity rates of the two fluids are equal.

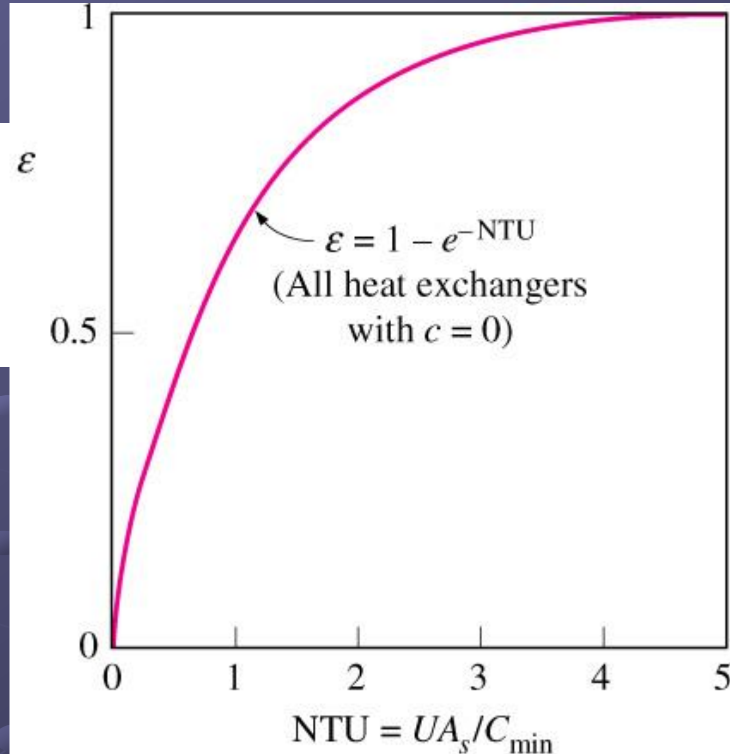


FIGURE 11–28
The effectiveness relation reduces to $\varepsilon = \varepsilon_{max} = 1 - \exp(-NTU)$ for all heat exchangers when the capacity ratio $c = 0$.

TABLE 11–5

NTU relations for heat exchangers: $NTU = UA_s/C_{min}$ and $c = C_{min}/C_{max} = (\dot{m}c_p)_{min}/(\dot{m}c_p)_{max}$

Heat exchanger type	NTU relation
1 *Double-pipe:* Parallel-flow	$NTU = -\dfrac{\ln[1 - \varepsilon(1 + c)]}{1 + c}$
Counter-flow	$NTU = \dfrac{1}{c - 1}\ln\left(\dfrac{\varepsilon - 1}{\varepsilon c - 1}\right)$
2 *Shell and tube:* One-shell pass 2, 4, . . . tube passes	$NTU = -\dfrac{1}{\sqrt{1 + c^2}}\ln\left(\dfrac{2/\varepsilon - 1 - c - \sqrt{1 + c^2}}{2/\varepsilon - 1 - c + \sqrt{1 + c^2}}\right)$
3 *Cross-flow (single-pass):* C_{max} mixed, C_{min} unmixed	$NTU = -\ln\left[1 + \dfrac{\ln(1 - \varepsilon c)}{c}\right]$
C_{min} mixed, C_{max} unmixed	$NTU = -\dfrac{\ln[c\ln(1 - \varepsilon) + 1]}{c}$
4 *All heat exchangers with $c = 0$*	$NTU = -\ln(1 - \varepsilon)$

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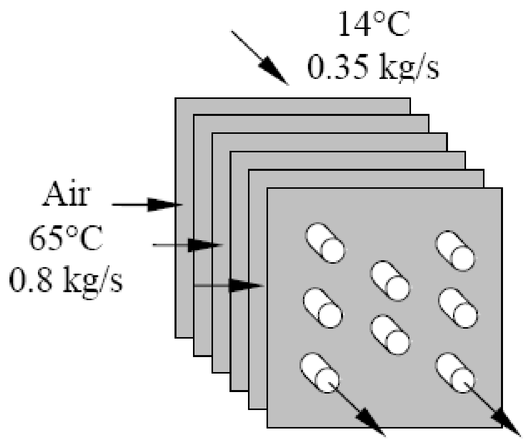
11–90 Cold water (c_p = 4.18 kJ/kg · °C) enters a cross-flow heat exchanger at 14°C at a rate of 0.35 kg/s where it is heated by hot air (c_p = 1.0 kJ/kg · °C) that enters the heat exchanger at 65°C at a rate of 0.8 kg/s and leaves at 25°C. Determine the maximum outlet temperature of the cold water and the effectiveness of this heat exchanger.

Properties The specific heats of water and air are given to be 4.18 and 1.0 kJ/kg.°C.



Analysis The heat capacity rates of the hot and cold fluids are

$$C_h = \dot{m}_h c_{ph} = (0.8 \text{ kg/s})(1.0 \text{ kJ/kg.°C}) = 0.8 \text{ kW/°C}$$

$$C_c = \dot{m}_c c_{pc} = (0.35 \text{ kg/s})(4.18 \text{ kJ/kg.°C}) = 1.463 \text{ kW/°C}$$

Therefore

$$C_{\min} = C_h = 0.8 \text{ kW/°C}$$

which is the smaller of the two heat capacity rates. Then the maximum heat transfer rate becomes

$$\dot{Q}_{\max} = C_{\min}(T_{h,in} - T_{c,in}) = (0.8 \text{ kW/°C})(65\text{°C} - 14\text{°C}) = 40.80 \text{ kW}$$

The maximum outlet temperature of the cold fluid is determined to be

$$\dot{Q}_{\max} = C_c(T_{c,out,\max} - T_{c,in}) \longrightarrow T_{c,out,\max} = T_{c,in} + \frac{\dot{Q}_{\max}}{C_c} = 14\text{°C} + \frac{40.80 \text{ kW}}{1.463 \text{ kW/°C}} = \mathbf{41.9°C}$$

The actual rate of heat transfer and the effectiveness of the heat exchanger are

$$\dot{Q} = C_h(T_{h,in} - T_{h,out}) = (0.8 \text{ kW/°C})(65\text{°C} - 25\text{°C}) = 32 \text{ kW}$$

$$\varepsilon = \frac{\dot{Q}}{\dot{Q}_{\max}} = \frac{32 \text{ kW}}{40.8 \text{ kW}} = \mathbf{0.784}$$

SELECTION OF HEAT EXCHANGERS

The rate of heat transfer in a heat exchanger: $\dot{Q}_{\max} = \dot{m} c_p (T_{\text{in}} - T_{\text{out}})$

Heat Transfer Rate: A heat exchanger should be capable of transferring heat at the specified rate in order to achieve the desired temperature change of the fluid at the specified mass flow rate.

Cost: Budgetary limitations usually play an important role in the selection of heat exchangers, except for some specialized cases where "money is no object."

Pumping Power:

$$\text{Operating cost} = (\text{Pumping power, kW}) \times (\text{Hours of operation, h}) \times (\text{Price of electricity, \$/kWh})$$

Minimizing the pressure drop and the mass flow rate of the fluids will *minimize* the operating cost of the heat exchanger, but it will *maximize* the size of the heat exchanger and thus the initial cost.

Size and Weight: Normally, the *smaller* and the *lighter* the heat exchanger, the better it is.

Type: The type of heat exchanger to be selected depends primarily on the type of *fluids* involved, the *size* and *weight* limitations, and the presence of any *phase change* processes.

Materials: The materials used in the construction of the heat exchanger may be an important consideration in the selection of heat exchangers.

Other Considerations: *Toxic or expensive fluids*

Concluding Points

- Types of Heat Exchangers
- Parallel, Counter and Cross Flows
- Fouling Factor
- Analysis of Heat Exchangers
- The Log Mean Temperature Difference (LMTD) Method
- Counter-Flow, Multipass and Cross-Flow Heat Exchangers
- Correction Factor
- The Effectiveness—NTU Method
- Selection of Heat Exchangers

HEAT AND MASS TRANSFER

Fundamentals of Thermal Radiation

OUTLINE

- Introduction
- Thermal Radiation
- Blackbody Radiation
- Radiation Intensity
- Radiative Properties
- Atmospheric and Solar Radiation
- Conclusions

INTRODUCTION

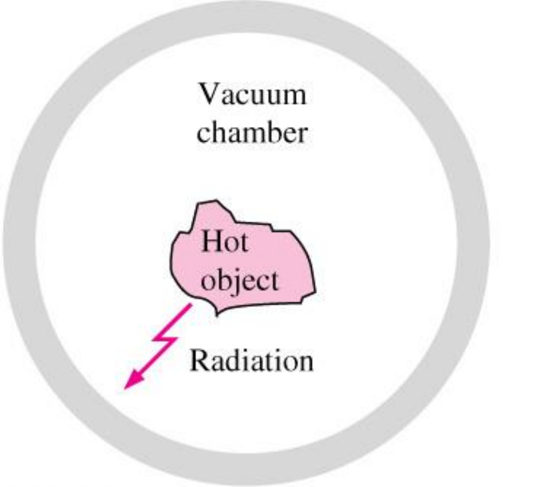


FIGURE 12–1
A hot object in a vacuum chamber loses heat by radiation only.

$$\lambda = \frac{c}{\upsilon}$$

λ: frequency

υ : wavelength

c: speed of propagation

c_0: speed of light = 2.9979×10^8 m/s

n: index of refraction

$$c = c_0/n$$

$$h = 6.6256 \times 10^{-34}\ \text{J} \cdot \text{s}$$

$$e = h\upsilon = \frac{hc}{\lambda}$$

h: Planck's constant

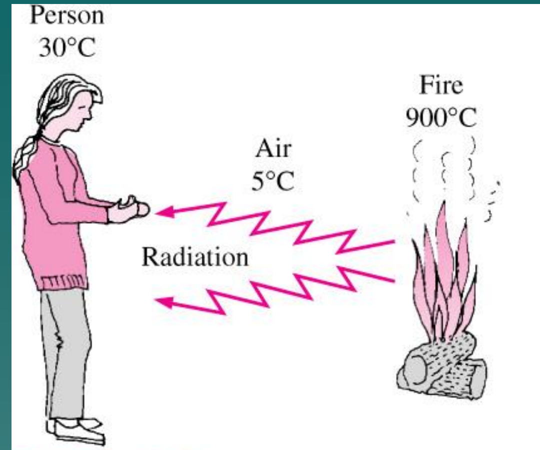


FIGURE 12–2
Unlike conduction and convection, heat transfer by radiation can occur between two bodies, even when they are separated by a medium colder than both.

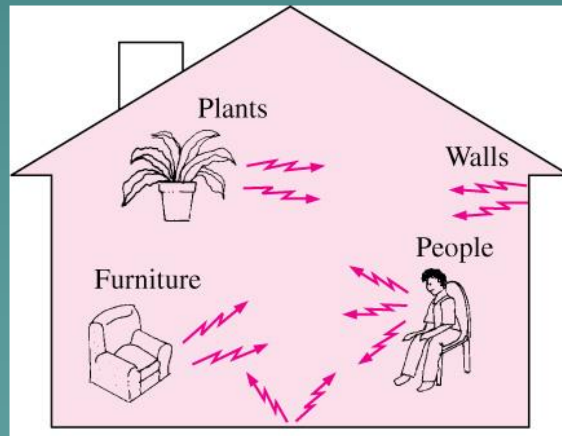


FIGURE 12–4
Everything around us constantly emits thermal radiation.

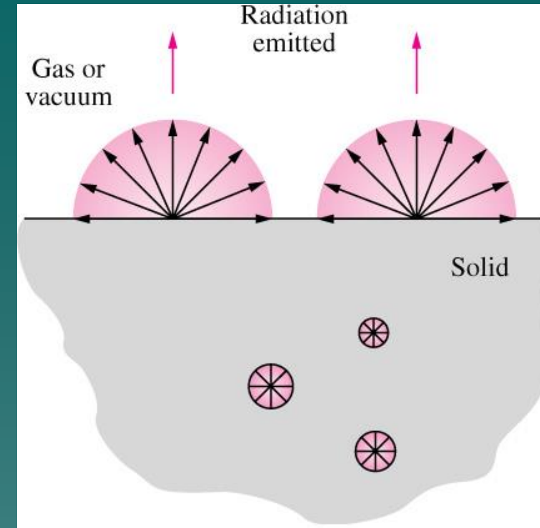


FIGURE 12–6
Radiation in opaque solids is considered a surface phenomenon since the radiation emitted only by the molecules at the surface can escape the solid.

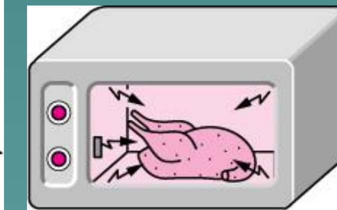


FIGURE 12–5
Food is heated or cooked in a microwave oven by absorbing the electromagnetic radiation energy generated by the magnetron of the oven.

THERMAL RADIATION

Thermal Radiation: electromagnetic radiation, pertinent to heat transfer (emitted as a result of energy transitions of molecules, atoms, and electrons of a substance).

Light: the *visible* portion of the electromagnetic spectrum (between 0.40 and 0.76 μm).

Solar Radiation: the electromagnetic radiation emitted by the sun (mainly in the wavelength band 0.3–3 μm).

The radiation emitted by bodies at room temperature falls into the **infrared** region of the spectrum (from 0.76 to 100 μm).

UV radiation: the low-wavelength end of the thermal radiation spectrum (0.01 and 0.40 μm).

TABLE 12–1

The wavelength ranges of different colors

Color	Wavelength band
Violet	0.40–0.44 μm
Blue	0.44–0.49 μm
Green	0.49–0.54 μm
Yellow	0.54–0.60 μm
Orange	0.60–0.67 μm
Red	0.63–0.76 μm

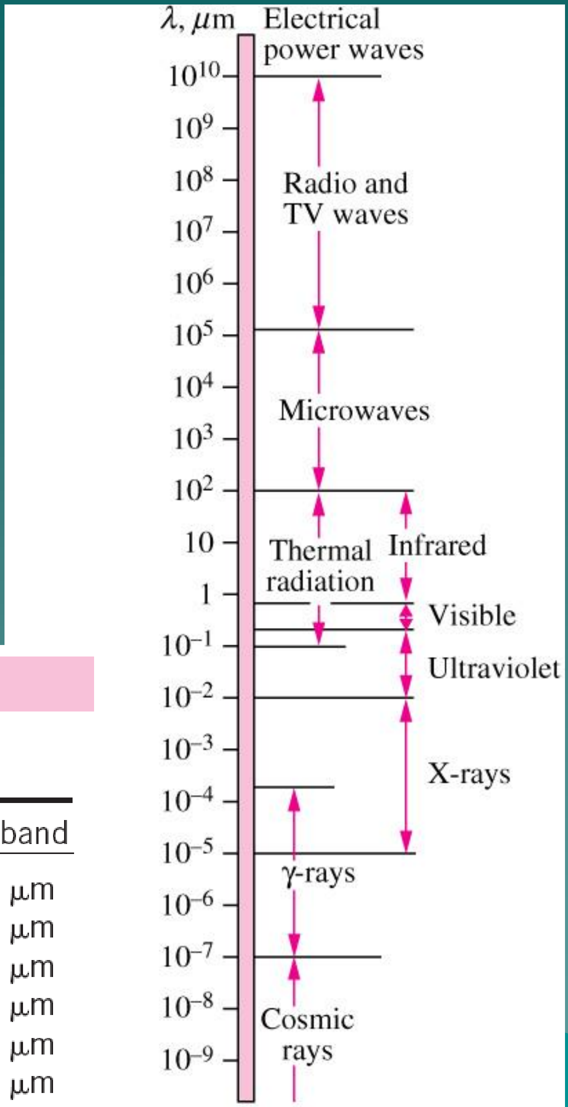


FIGURE 12–3
The electromagnetic wave spectrum.

Microwave ovens use electro-magnetic radiation in the **microwave** region of the spectrum generated by microwave tubes (magnetrons).

BLACKBODY RADIATION

A blackbody: *perfect emitter and absorber of radiation.*

The radiation energy emitted by a blackbody per unit time and per unit surface area (blackbody emissive power):

$$E_b(T) = \sigma T^4 \qquad (\mathrm{W/m^2})$$

Stefan–Boltzmann constant:

$$\sigma = 5.67 \times 10^{-8}\ \mathrm{W/m^2 \cdot K^4}$$

T : the absolute temperature of the surface in K.

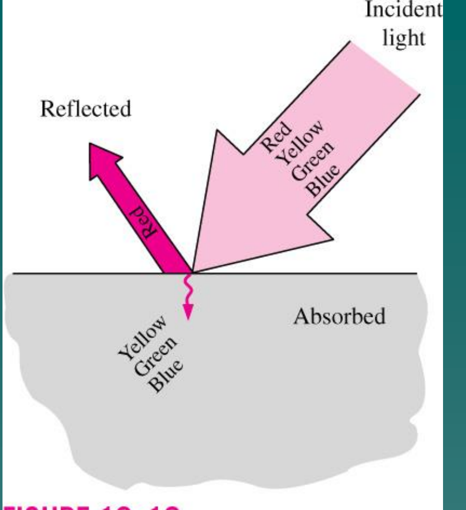


FIGURE 12–10
A surface that reflects red while absorbing the remaining parts of the incident light appears red to the eye.

Spectral Blackbody Emissive Power: the amount of radiation energy emitted by a blackbody at an absolute temperature per unit time, per unit surface area, and per unit wavelength about the wavelength.

Planck's law:

$$E_{b\lambda}(\lambda, T) = \frac{C_1}{\lambda^5[\exp(C_2/\lambda T) - 1]} \qquad (\mathrm{W/m^2 \cdot \mu m})$$

$$C_1 = 2\pi h c_0^2 = 3.742 \times 10^8\ \mathrm{W \cdot \mu m^4/m^2}$$
$$C_2 = hc_0/k = 1.439 \times 10^4\ \mathrm{\mu m \cdot K}$$

Boltzmann's constant:

$$k = 1.38065 \times 10^{-23}\ \mathrm{J/K}$$

T : the absolute temperature of the surface and c = speed of light

The total blackbody emissive power (E_b):

$$E_b(T) = \int_0^\infty E_{b\lambda}(\lambda, T)\, d\lambda = \sigma T^4 \qquad (\mathrm{W/m^2})$$

h = Planck's constant 6.6256×10^{-34} Js

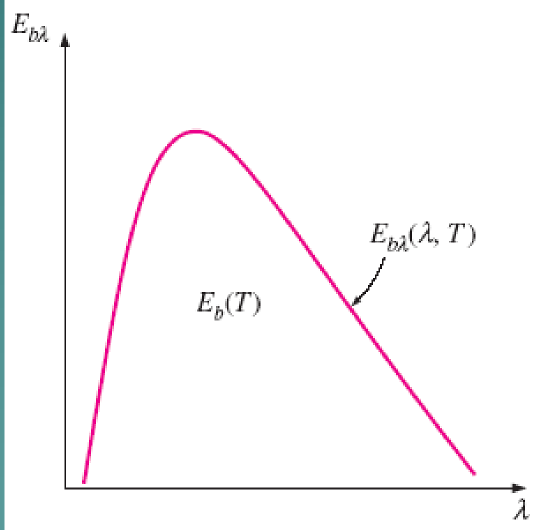


FIGURE 21–11
On an $E_{b\lambda}$–λ chart, the area under a curve for a given temperature represents the total radiation energy emitted by a blackbody at that temperature.

Emissivity (ε)

Emissivity of a Surface : the ratio of the radiation emitted by the surface at a given temperature to the radiation emitted by a blackbody at the same temperature. $0 \leq \varepsilon \leq 1$

For a blackbody: ε = 1

Real surface:

$\varepsilon_\theta \neq$ constant

$\varepsilon_\lambda \neq$ constant

Diffuse surface:

$\varepsilon_\theta =$ constant

Gray surface:

$\varepsilon_\lambda =$ constant

Diffuse, gray surface:

$\varepsilon = \varepsilon_\lambda = \varepsilon_\theta =$ constant

FIGURE 12–25
The effect of diffuse and gray approximations on the emissivity of a surface.

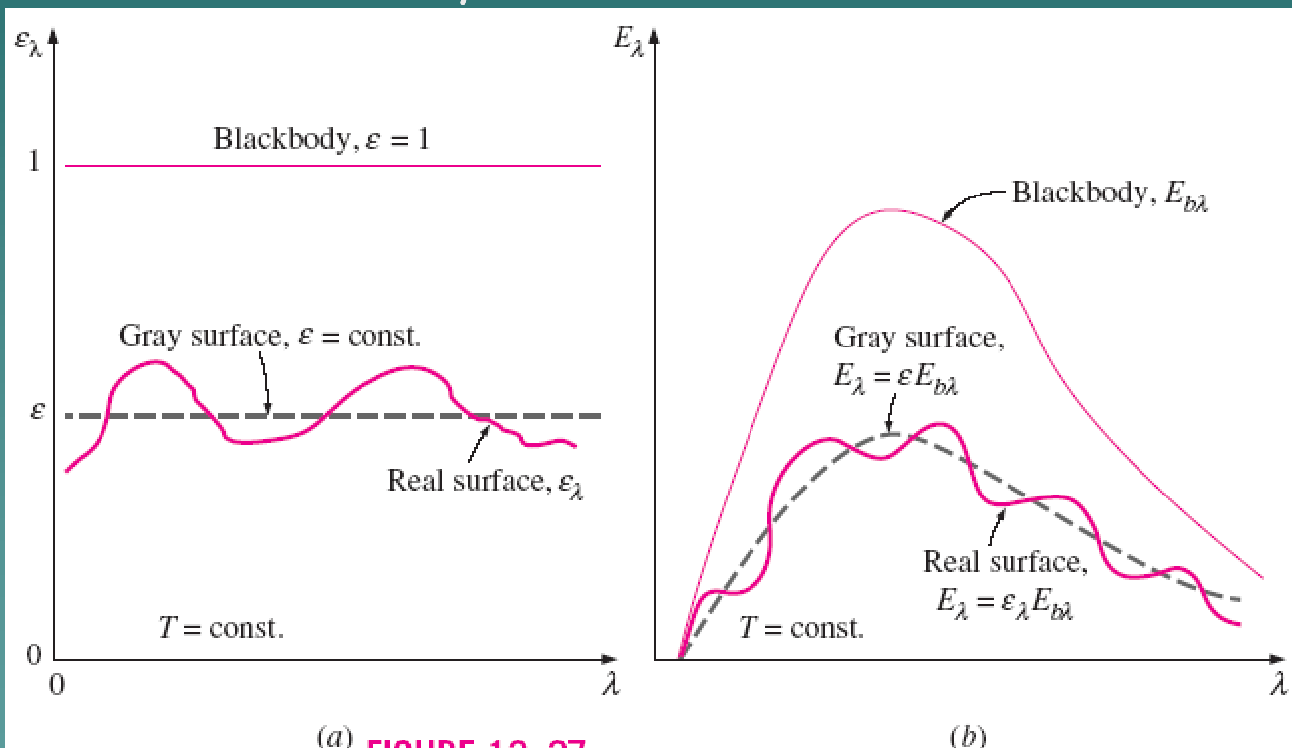


FIGURE 12–27
Comparison of the emissivity (*a*) and emissive power (*b*) of a real surface with those of a gray surface and a blackbody at the same temperature.

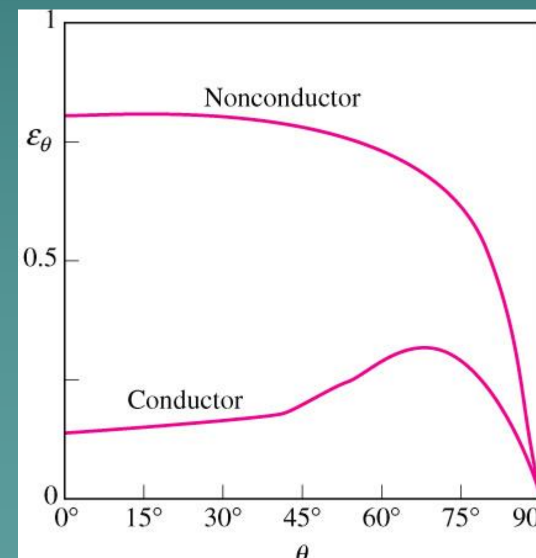
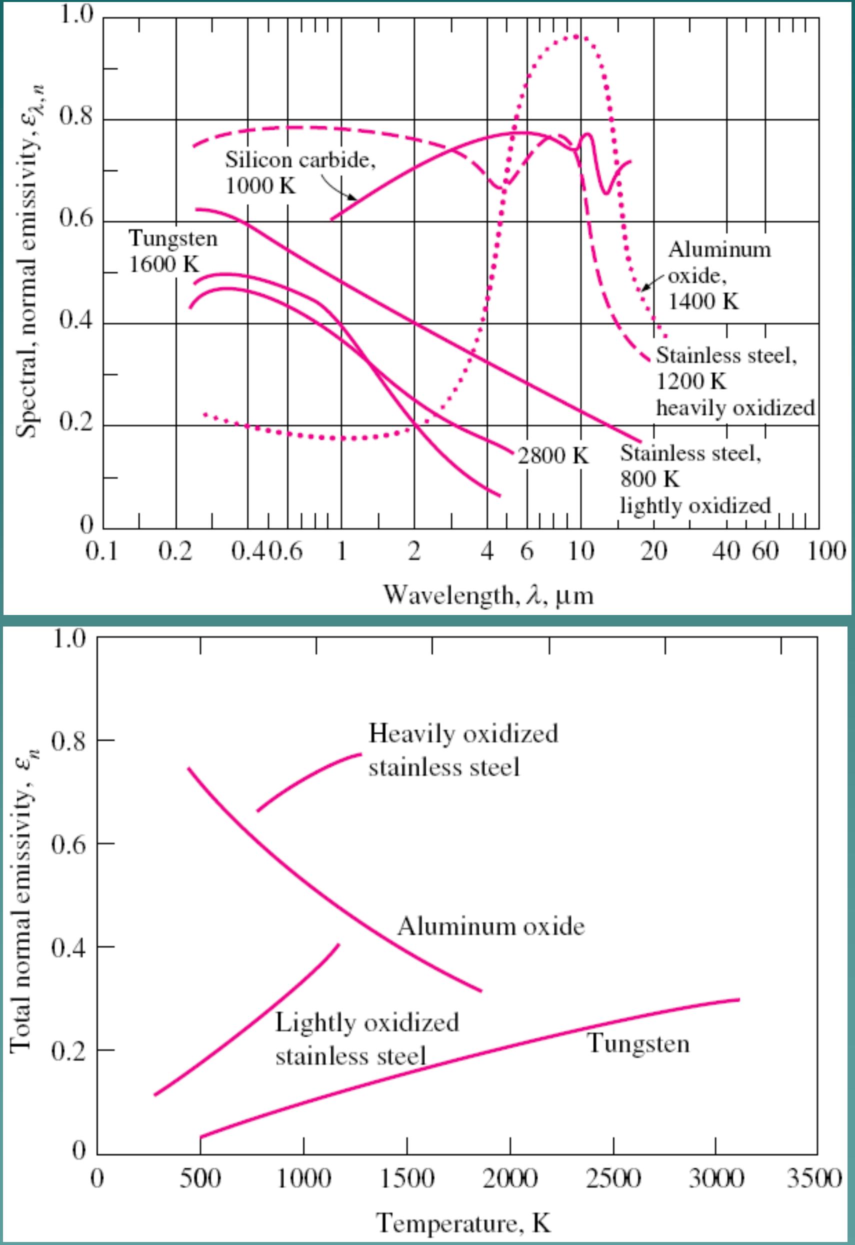


FIGURE 12–26
Typical variations of emissivity with direction for electrical conductors and nonconductors.

FIGURE 12–28

The variation of normal emissivity with (*a*) wavelength and (*b*) temperature for various materials.



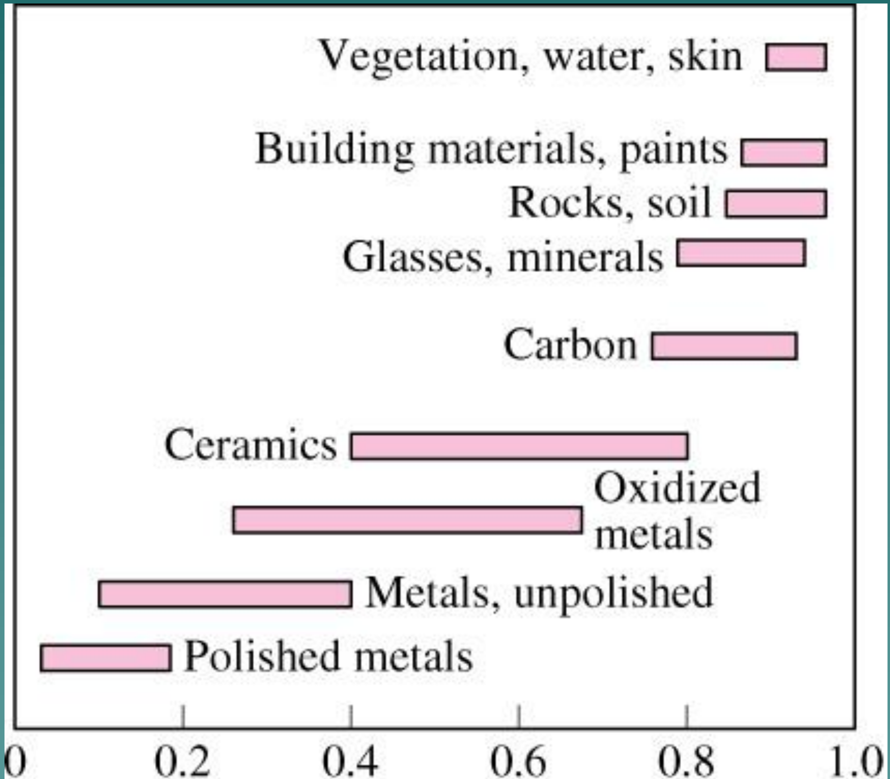


FIGURE 12–29

Typical ranges of emissivity for various materials.

Absorptivity, Reflectivity, and Transmissivity

Irradiation (G): radiation flux incident on a surface

$$\text{Absorptivity:} \quad \alpha = \frac{\text{Absorbed radiation}}{\text{Incident radiation}} = \frac{G_{abs}}{G}, \quad 0 \le \alpha \le 1$$

$$\text{Reflectivity:} \quad \rho = \frac{\text{Reflected radiation}}{\text{Incident radiation}} = \frac{G_{ref}}{G}, \quad 0 \le \rho \le 1$$

$$\text{Transmissivity:} \quad \tau = \frac{\text{Transmitted radiation}}{\text{Incident radiation}} = \frac{G_{tr}}{G}, \quad 0 \le \tau \le 1$$

$$G_{abs} + G_{ref} + G_{tr} = G \qquad \alpha + \rho + \tau = 1$$

For opaque surfaces: $\tau = 0$

$$\alpha + \rho = 1$$

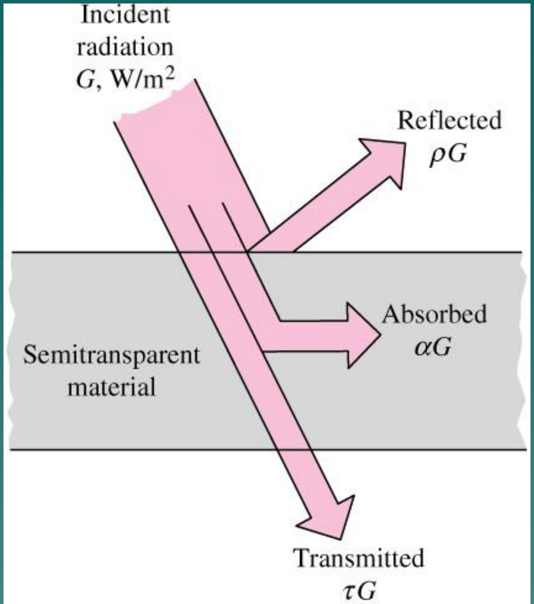


FIGURE 12–31
The absorption, reflection, and transmission of incident radiation by a semitransparent material.

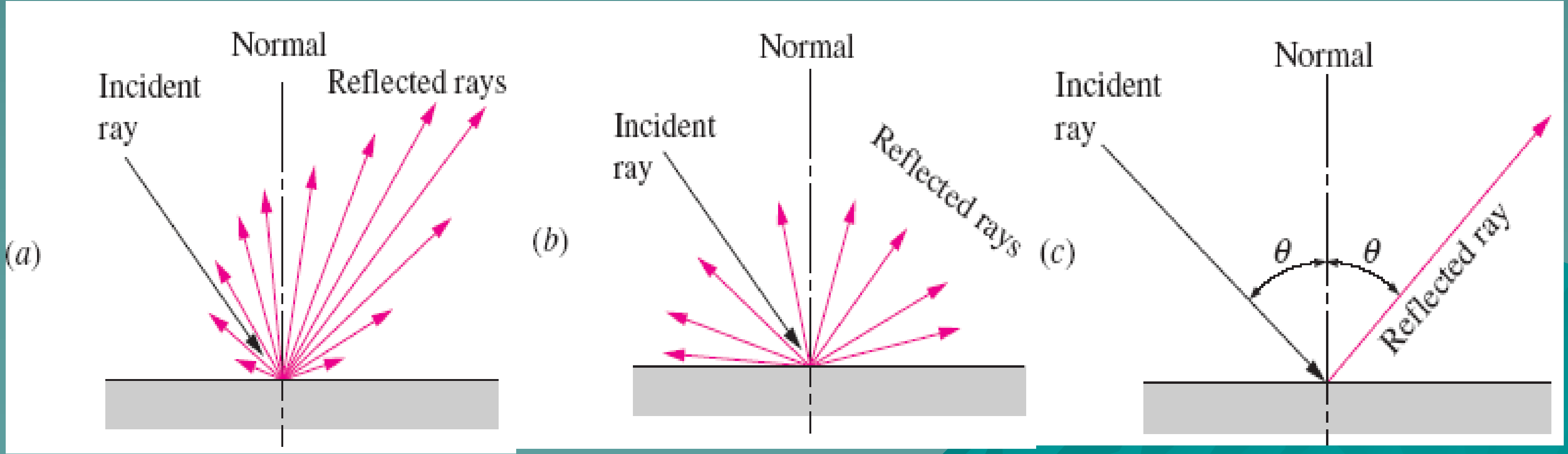


FIGURE 12–32
Different types of reflection from a surface: (*a*) actual or irregular, (*b*) diffuse, and (*c*) specular or mirrorlike.

Kirchhoff's Law

$$G = E_b(T) = \sigma T^4$$

The radiation absorbed by the small body per unit of its surface area:

$$G_{abs} = \alpha G = \alpha \sigma T^4$$

The radiation emitted by the small body:

$$E_{emit} = \varepsilon \sigma T^4$$

Considering that the small body is in thermal equilibrium with the enclosure:

$$A_s \varepsilon \sigma T^4 = A_s \alpha \sigma T^4$$

Kirchhoff's law: $\varepsilon(T) = \alpha(T)$

The total hemispherical emissivity of a surface at temperature T is equal to its total hemispherical absorptivity for radiation coming from a blackbody at the same temperature.

The *spectral* form of Kirchhoff's law: $\varepsilon_\lambda(T) = \alpha_\lambda(T)$

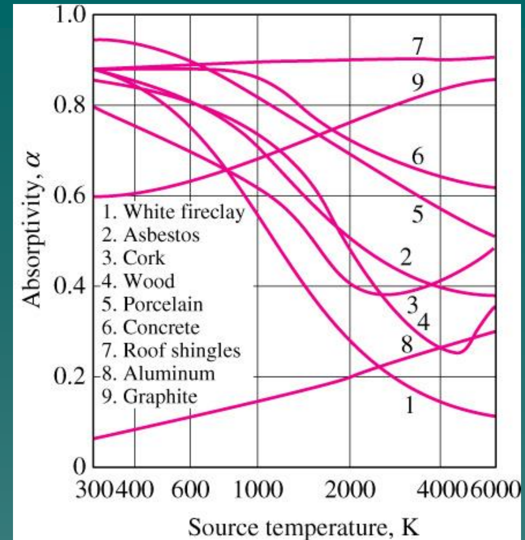


FIGURE 12–33
Variation of absorptivity with the temperature of the source of irradiation for various common materials at room temperature.

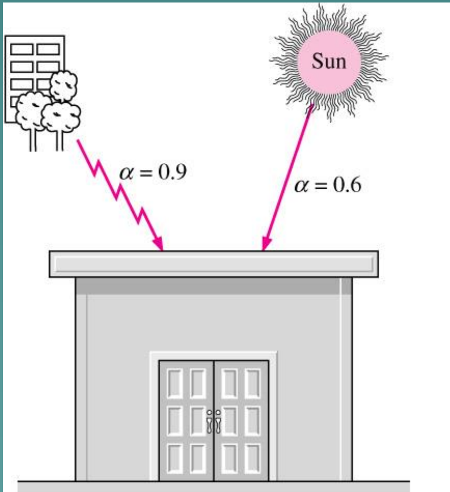


FIGURE 12–34
The absorptivity of a material may be quite different for radiation originating from sources at different temperatures.

Glass at thicknesses encountered in practice transmits over 90 percent of radiation in the visible range and is practically opaque (nontransparent) to radiation in the longer-wavelength infrared regions of the electromagnetic spectrum (roughly $\lambda > 3\ \mu m$). Therefore, glass has a transparent window in the wavelength range $0.3\ \mu m < \lambda < 3\ \mu m$ in which over 90 percent of solar radiation is emitted. On the other hand, the entire radiation emitted by surfaces at room temperature falls in the infrared region. Consequently, glass allows the solar radiation to enter but does not allow the infrared radiation from the interior surfaces to escape. This causes a rise in the interior temperature as a result of the energy buildup in the space. This heating effect, which is due to the nongray characteristic of glass (or clear plastics), is known as the **greenhouse effect**.

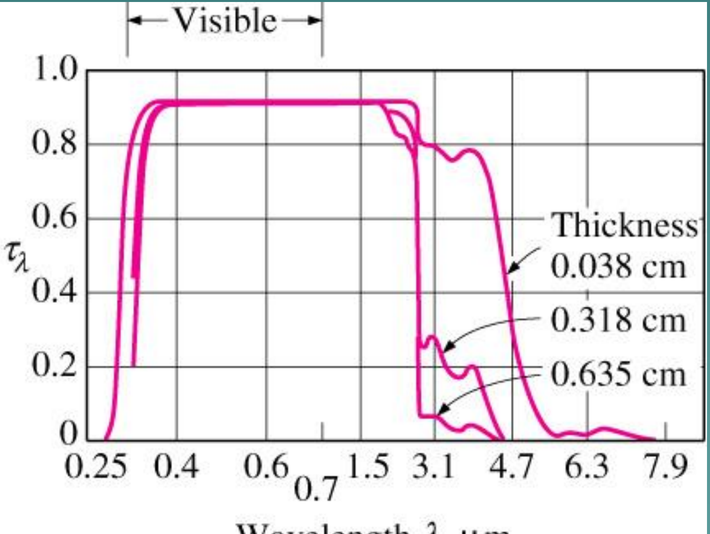


FIGURE 12–36

The spectral transmissivity of low-iron glass at room temperature for different thicknesses.

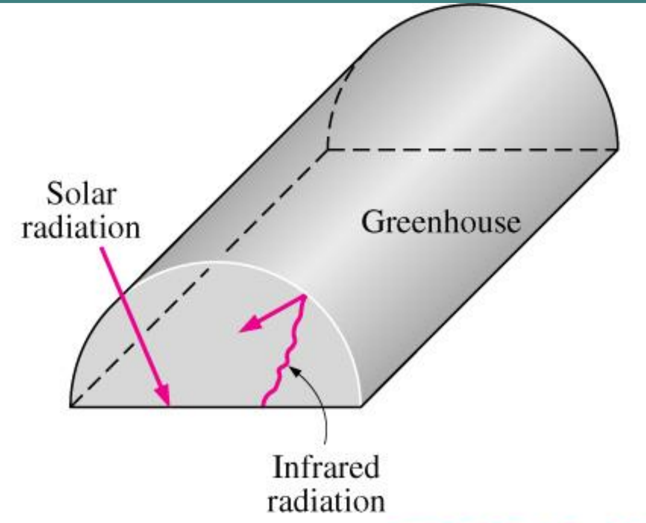


FIGURE 12–37

A greenhouse traps energy by allowing the solar radiation to come in but not allowing the infrared radiation to go out.

Greenhouse effect

ATMOSPHERIC AND SOLAR RADIATION

The *total solar energy* incident on the unit area of a *horizontal surface* on the ground:

$$G_{\text{solar}} = G_D \cos\theta + G_d \qquad (\text{W/m}^2)$$

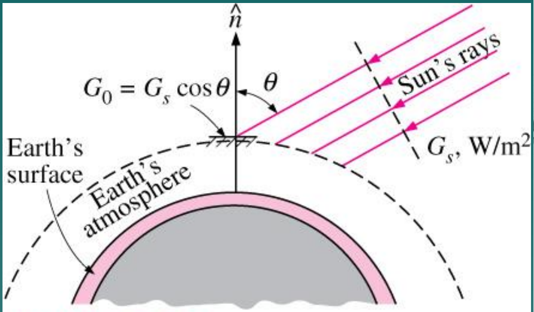


FIGURE 12–38
Solar radiation reaching the earth's atmosphere and the total solar irradiance.

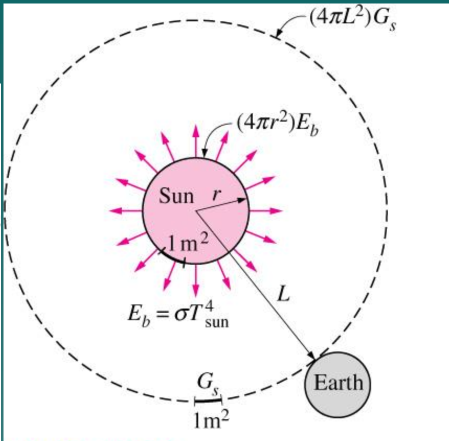


FIGURE 12–39
The total solar energy passing through concentric spheres remains constant, but the energy falling per unit area decreases with increasing radius.

The radiation emission from the atmosphere to the earth's surface:

$$G_{\text{sky}} = \sigma T^4_{\text{sky}} \qquad (\text{W/m}^2)$$

T_{sky}: the effective sky temperature

T_{sky} ranges from about 230 K for cold, clear-sky conditions to about 285 K for warm, cloudy-sky conditions.

The sky radiation absorbed by a surface:

$$E_{\text{sky, absorbed}} = \alpha G_{\text{sky}} = \alpha\sigma T^4_{\text{sky}} = \varepsilon\sigma T^4_{\text{sky}} \qquad (\text{W/m}^2)$$

The net rate of radiation heat transfer to a surface exposed to solar and atmospheric radiation:

$$\begin{aligned}
\dot{q}_{\text{net, rad}} &= \sum E_{\text{absorbed}} - \sum E_{\text{emitted}} \\
&= E_{\text{solar, absorbed}} + E_{\text{sky, absorbed}} - E_{\text{emitted}} \\
&= \alpha_s G_{\text{solar}} + \varepsilon\sigma T^4_{\text{sky}} - \varepsilon\sigma T_s^4 \\
&= \alpha_s G_{\text{solar}} + \varepsilon\sigma (T^4_{\text{sky}} - T_s^4) \qquad (\text{W/m}^2)
\end{aligned}$$

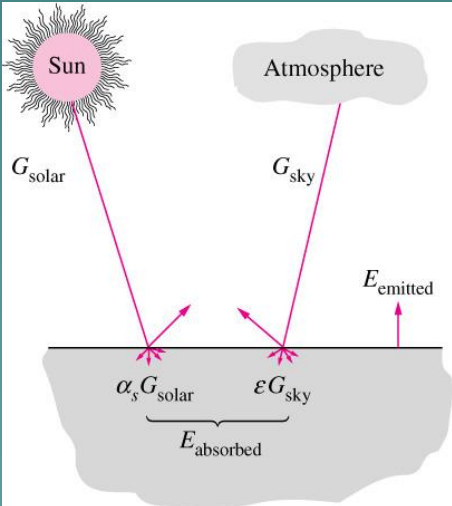


FIGURE 12–43
Radiation interactions of a surface exposed to solar and atmospheric radiation.

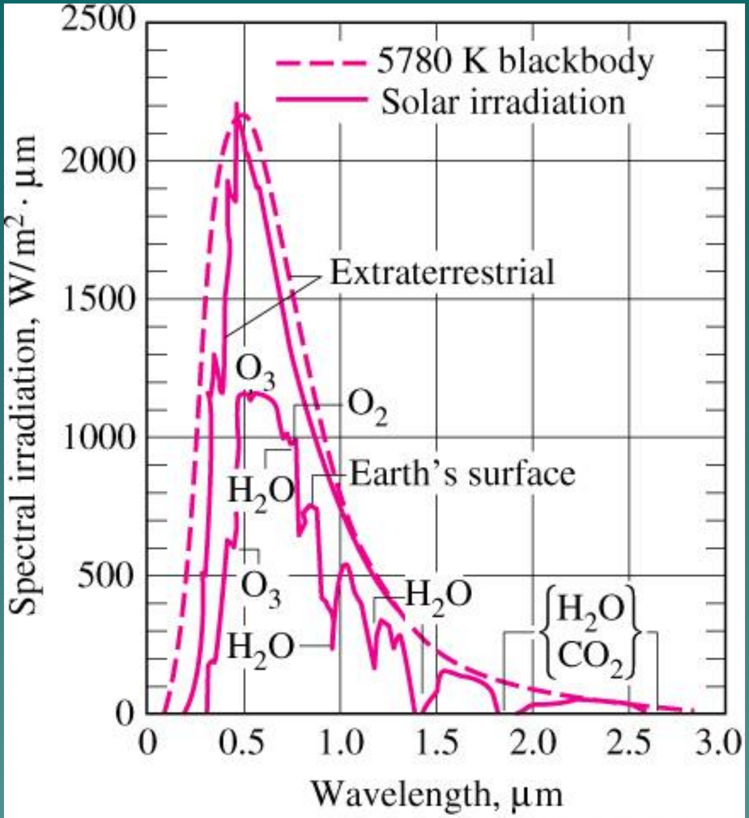


FIGURE 12–40

Spectral distribution of solar radiation just outside the atmosphere, at the surface of the earth on a typical day, and comparison with blackbody radiation at 5780 K.

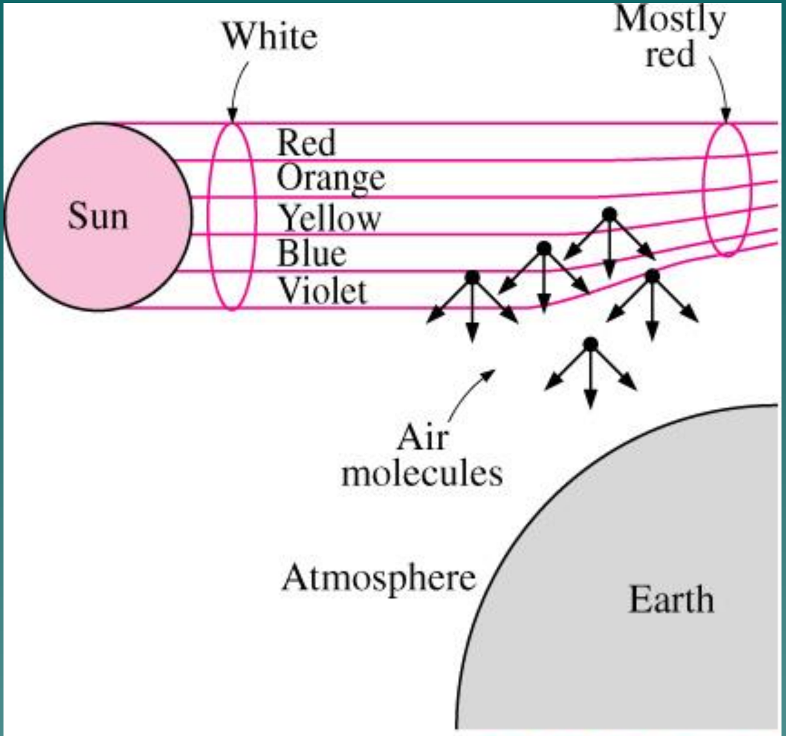


FIGURE 12–41

Air molecules scatter blue light much more than they do red light. At sunset, light travels through a thicker layer of atmosphere, which removes much of the blue from the natural light, allowing the red to dominate.

TABLE 12–3

Comparison of the solar absorptivity α_s of some surfaces with their emissivity ε at room temperature

Surface	α_s	ε
Aluminum		
Polished	0.09	0.03
Anodized	0.14	0.84
Foil	0.15	0.05
Copper		
Polished	0.18	0.03
Tarnished	0.65	0.75
Stainless steel		
Polished	0.37	0.60
Dull	0.50	0.21
Plated metals		
Black nickel oxide	0.92	0.08
Black chrome	0.87	0.09
Concrete	0.60	0.88
White marble	0.46	0.95
Red brick	0.63	0.93
Asphalt	0.90	0.90
Black paint	0.97	0.97
White paint	0.14	0.93
Snow	0.28	0.97
Human skin		
(Caucasian)	0.62	0.97

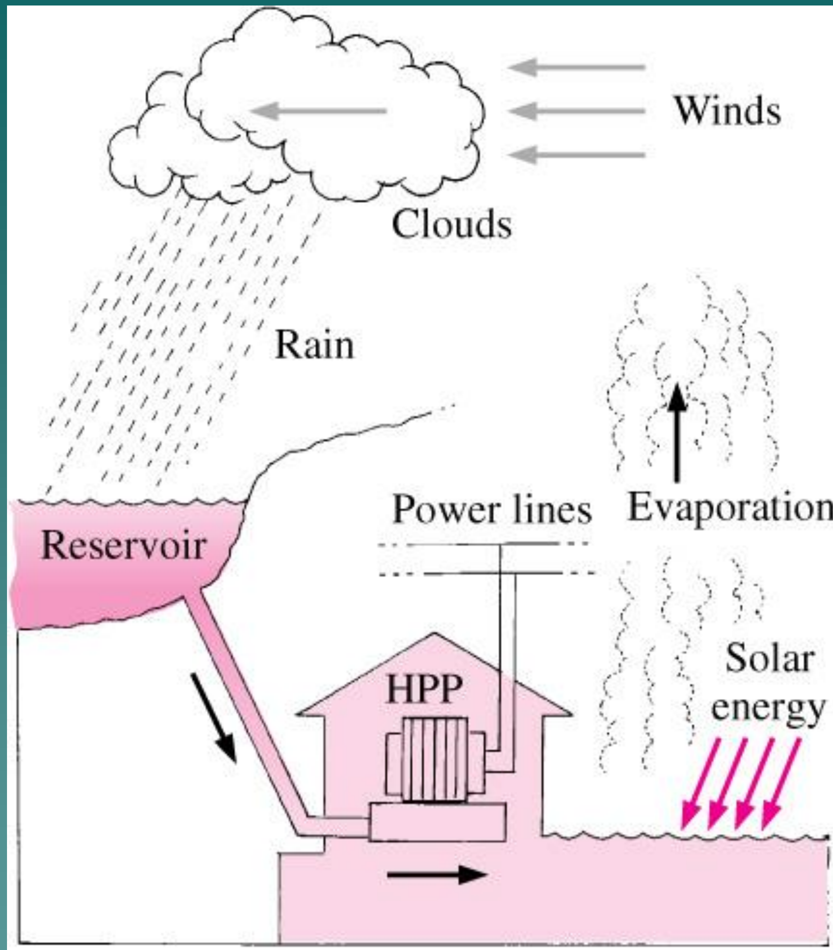
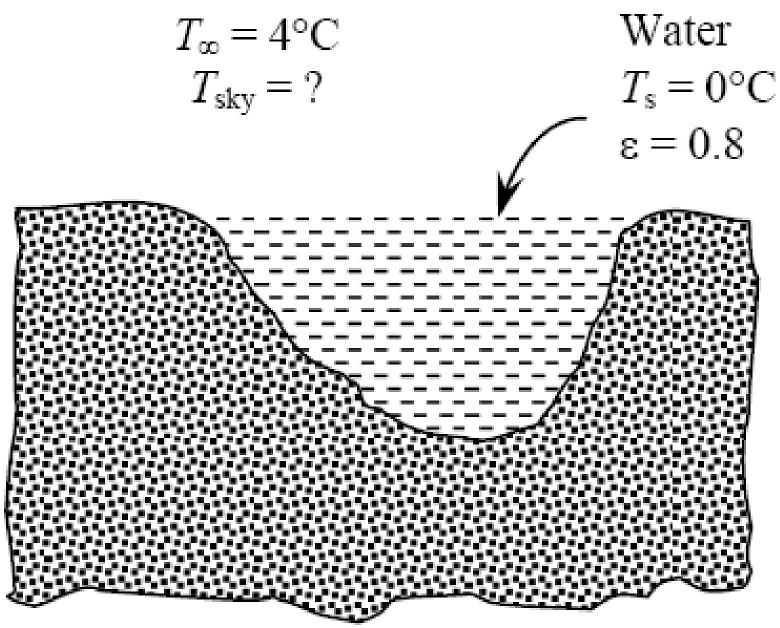


FIGURE 12–44

The cycle that water undergoes in a hydroelectric power plant.

12–62 The air temperature on a clear night is observed to remain at about 4°C. Yet water is reported to have frozen that night due to radiation effect. Taking the convection heat transfer coefficient to be 18 W/m² · °C, determine the value of the maximum effective sky temperature that night.

12-62 Water is observed to have frozen one night while the air temperature is above freezing temperature. The effective sky temperature is to be determined.



Properties The emissivity of water is $\varepsilon = 0.95$ (Table A-18).

Analysis Assuming the water temperature to be 0°C, the value of the effective sky temperature is determined from an energy balance on water to be

$$h(T_{air} - T_{surface}) = \varepsilon\sigma(T_s^4 - T_{sky}^4)$$

and

$$(18\ \mathrm{W/m^2 \cdot °C})(4°C - 0°C) = 0.95(5.67\times10^{-8}\ \mathrm{W/m^2 \cdot K^4})\left[(273\ \mathrm{K})^4 - T_{sky}^{\ 4}\right]$$

$$\longrightarrow T_{sky} = \mathbf{254.8\ K}$$

Therefore, the effective sky temperature must have been below 255 K.

Concluding Points:

- The electromagnetic spectrum?
- Thermal radiation?
- A blackbody and the blackbody emissive power?
- Stefan-Boltzman law?
- Planck's law?
- Emissivity?
- Absorptivity, reflectivity, and transmissivity?
- Kirshhoff's law?
- Effective sky temperature?

HEAT AND MASS TRANSFER

Radiation Heat Transfer

Objectives

- Define view factor, and understand its importance in radiation heat transfer calculations,
- Develop view factor relations, and calculate the unknown view factors in an enclosure by using these relations,
- Calculate radiation heat transfer between black surfaces,
- Determine radiation heat transfer between diffuse and gray surfaces in an enclosure using the concept of radiosity,
- Obtain relations for net rate of radiation heat transfer between the surfaces of a two-zone enclosure, including two large parallel plates, two long concentric cylinders, and two concentric spheres,
- Quantify the effect of radiation shields on the reduction of radiation heat transfer between two surfaces, and become aware of the importance of radiation effect in temperature measurements.

The View Factor

- Radiation heat transfer between surfaces depends on the *orientation* of the surfaces relative to each other as well as their radiation properties and temperatures.
- *View factor is defined* to account for the effects of orientation on radiation heat transfer between two surfaces.
- *View factor* is a purely geometric quantity and is independent of the surface properties and temperature.
- *Diffuse view factor* — view factor based on the assumption that the surfaces are diffuse emitters and diffuse reflectors.
- *Specular view factor* — view factor based on the assumption that the surfaces are specular reflectors.
- Here we consider radiation exchange between diffuse surfaces only, and thus the term *view factor* simply means *diffuse view factor*.

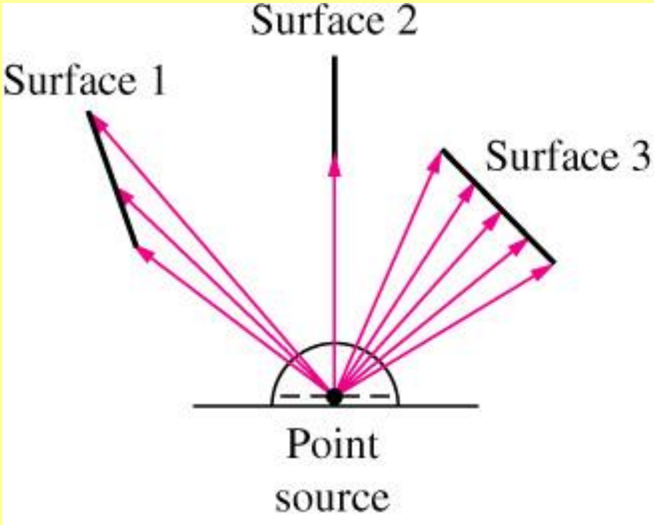


FIGURE 13–1
Radiation heat exchange between surfaces depends on the *orientation* of the surfaces relative to each other, and this dependence on orientation is accounted for by the *view factor*.

- The view factor from a surface i to a surface j is denoted by $F_{i \rightarrow j}$ or just F_{ij}, and is defined as
- $F_{i \rightarrow j}$ = *the fraction of the radiation leaving surface i that strikes surface j directly.*
- The view factor F_{12} represents the fraction of radiation leaving surface 1 that strikes surface 2 directly, and F_{21} represents the fraction of radiation leaving surface 2 that strikes surface 1 directly.
- Note that the radiation that strikes a surface does not need to be absorbed by that surface.
- Also, radiation that strikes a surface after being reflected by other surfaces is not considered in the evaluation of view factors.

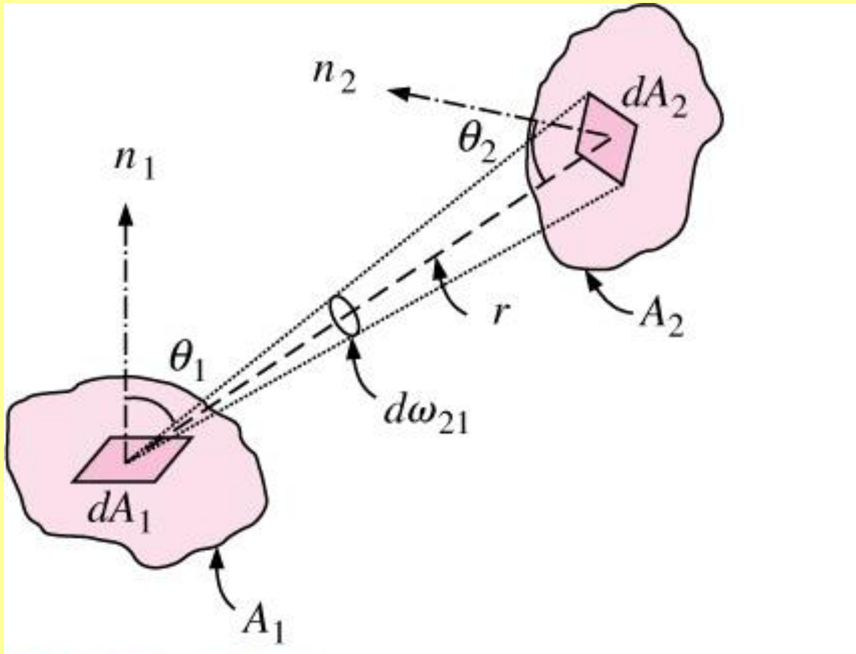


FIGURE 13–2

Geometry for the determination of the view factor between two surfaces.

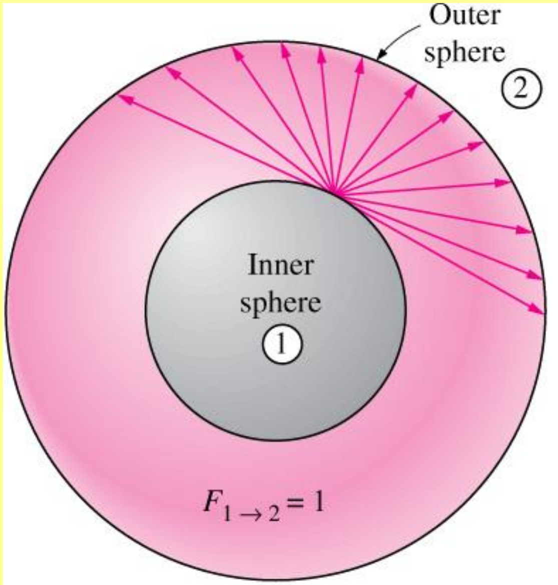


FIGURE 13–4

In a geometry that consists of two concentric spheres, the view factor $F_{1\to2} = 1$ since the entire radiation leaving the surface of the smaller sphere is intercepted by the larger sphere.

- *When j=i:*

  $F_{i\to i}$=*the fraction of radiation leaving surface i that strikes itself directly*.

  – $F_{i\to i}$=0: for plane or convex surfaces and

  – $F_{i\to i}\neq0$: for concave surfaces

- The value of the view factor ranges between *zero* and *one*.

  – $F_{i\to j}$=0, the two surfaces do not have a direct view of each other,

  – $F_{i\to j}$=1, surface *j* completely surrounds surface.

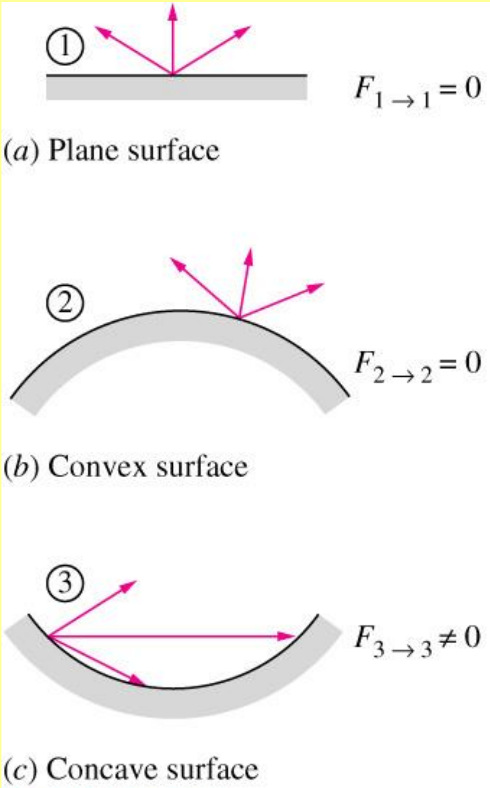


FIGURE 13–3

The view factor from a surface to itself is *zero* for *plane* or *convex* surfaces and *nonzero* for *concave* surfaces.

View Factors Tables for Selected Geometries (*analytical form*)

TABLE 13–1

View factor expressions for some common geometries of finite size (3-D)

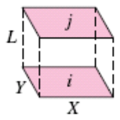
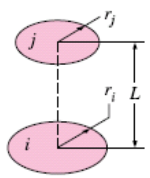
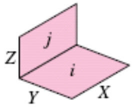
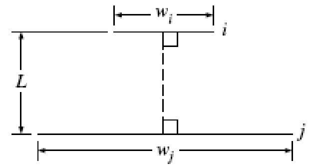
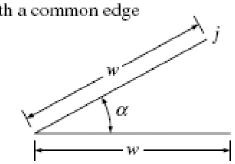
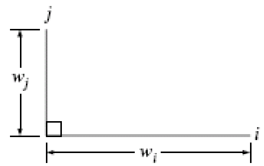
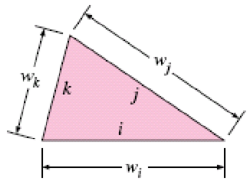
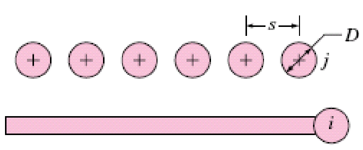
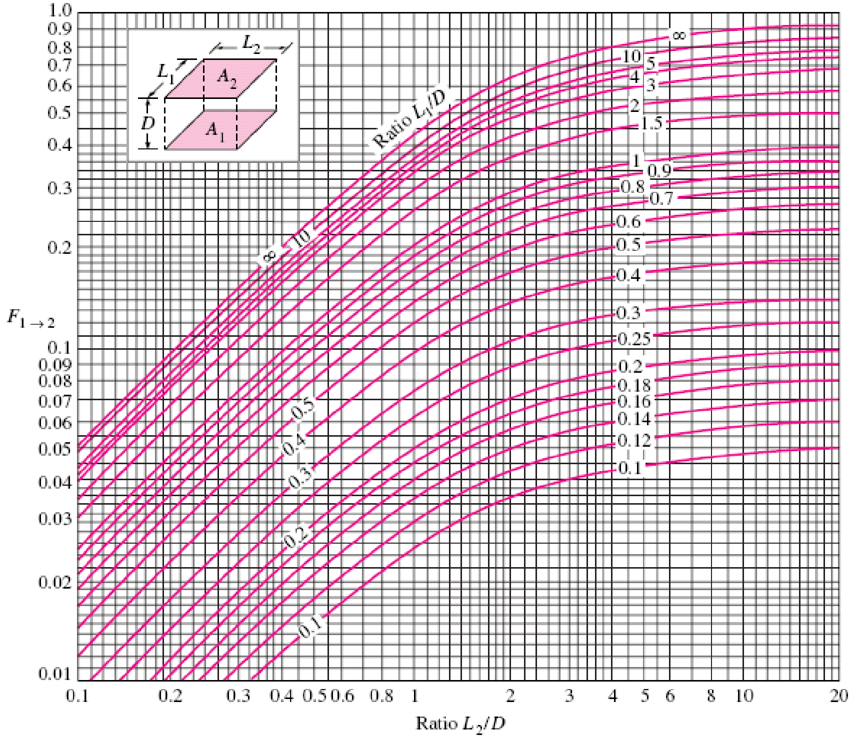
Geometry	Relation
Aligned parallel rectangles 	$\bar{X} = X/L$ and $\bar{Y} = Y/L$ $F_{i \to j} = \frac{2}{\pi \bar{X}\bar{Y}}\left\{\ln\left[\frac{(1+\bar{X}^2)(1+\bar{Y}^2)}{1+\bar{X}^2+\bar{Y}^2}\right]^{1/2} + \bar{X}(1+\bar{Y}^2)^{1/2}\tan^{-1}\frac{\bar{X}}{(1+\bar{Y}^2)^{1/2}} + \bar{Y}(1+\bar{X}^2)^{1/2}\tan^{-1}\frac{\bar{Y}}{(1+\bar{X}^2)^{1/2}} - \bar{X}\tan^{-1}\bar{X} - \bar{Y}\tan^{-1}\bar{Y}\right\}$
Coaxial parallel disks 	$R_i = r_i/L$ and $R_j = r_j/L$ $S = 1 + \frac{1+R_j^2}{R_i^2}$ $F_{i \to j} = \frac{1}{2}\left\{S - \left[S^2 - 4\left(\frac{r_j}{r_i}\right)^2\right]^{1/2}\right\}$
Perpendicular rectangles with a common edge 	$H = Z/X$ and $W = Y/X$ $F_{i \to j} = \frac{1}{\pi W}\left(W\tan^{-1}\frac{1}{W} + H\tan^{-1}\frac{1}{H} - (H^2+W^2)^{1/2}\tan^{-1}\frac{1}{(H^2+W^2)^{1/2}} + \frac{1}{4}\ln\left\{\frac{(1+W^2)(1+H^2)}{1+W^2+H^2}\left[\frac{W^2(1+W^2+H^2)}{(1+W^2)(W^2+H^2)}\right]^{W^2} \times \left[\frac{H^2(1+H^2+W^2)}{(1+H^2)(H^2+W^2)}\right]^{H^2}\right\}\right)$

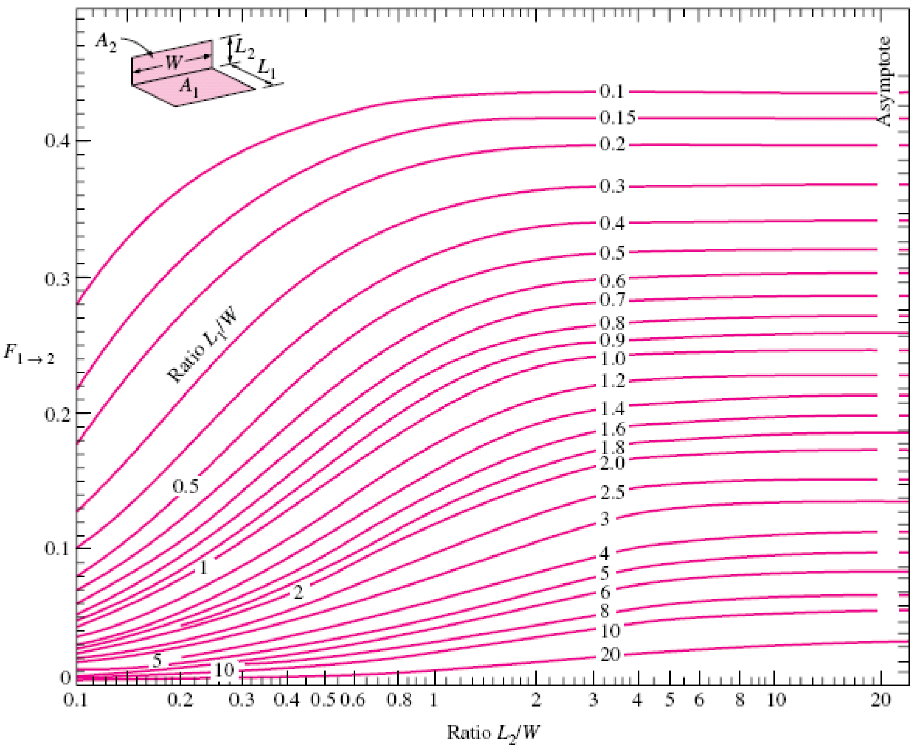
TABLE 13–2

View factor expressions for some infinitely long (2-D) geometries

Geometry	Relation
Parallel plates with midlines connected by perpendicular line 	$W_i = w_i/L$ and $W_j = w_j/L$ $F_{i \to j} = \frac{[(W_i + W_j)^2 + 4]^{1/2} - (W_j - W_i)^2 + 4]^{1/2}}{2W_i}$
Inclined plates of equal width and with a common edge 	$F_{i \to j} = 1 - \sin\frac{1}{2}\alpha$
Perpendicular plates with a common edge 	$F_{i \to j} = \frac{1}{2}\left\{1 + \frac{w_j}{w_i} - \left[1 + \left(\frac{w_j}{w_i}\right)^2\right]^{1/2}\right\}$
Three-sided enclosure 	$F_{i \to j} = \frac{w_i + w_j - w_k}{2w_i}$
Infinite plane and row of cylinders 	$F_{i \to j} = 1 - \left[1 - \left(\frac{D}{s}\right)^2\right]^{1/2} + \frac{D}{s}\tan^{-1}\left(\frac{s^2 - D^2}{D^2}\right)^{1/2}$

View Factors Figures for Selected Geometries (graphical form)





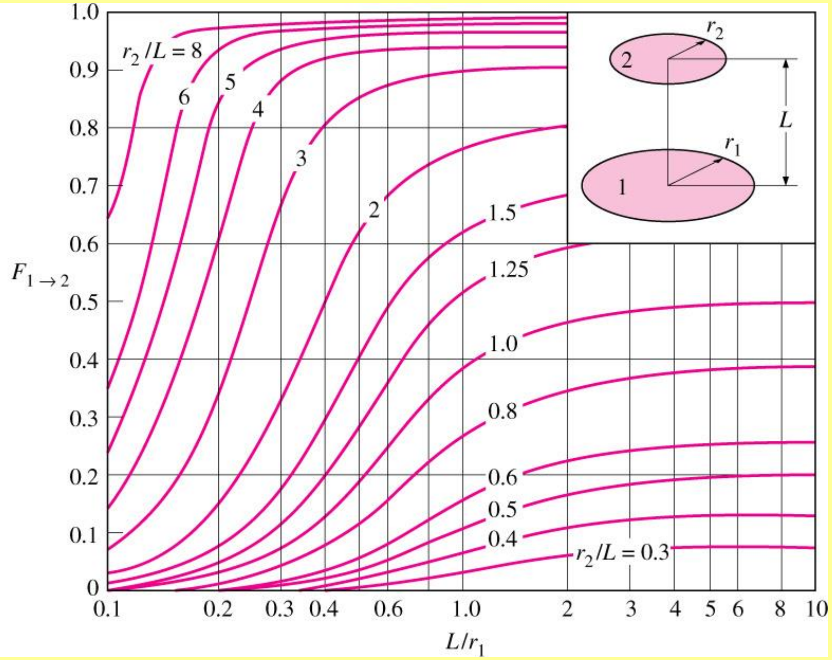


FIGURE 13–7
View factor between two coaxial parallel disks.

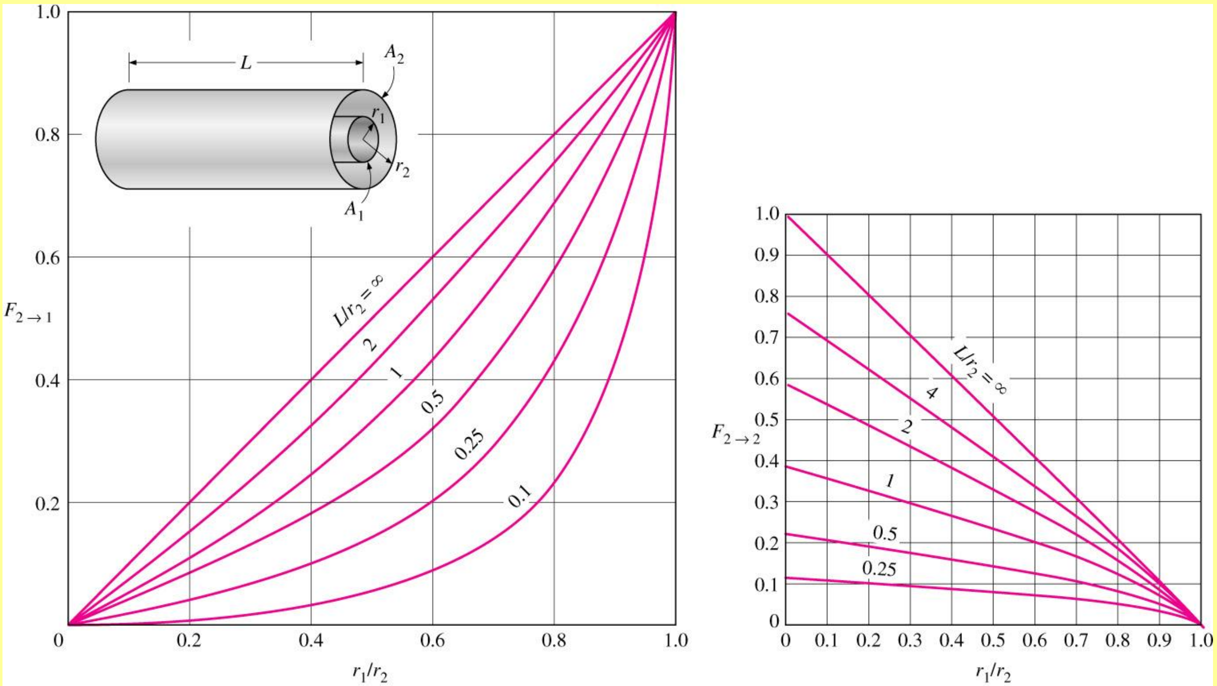


FIGURE 13–8
View factors for two concentric cylinders of finite length: (*a*) outer cylinder to inner cylinder; (*b*) outer cylinder to itself.

View Factor Relations

- Radiation analysis on an enclosure consisting of N surfaces requires the evaluation of N^2 view factors. However, it is neither practical nor necessary to evaluate all of the view factors directly.
- Once a sufficient number of view factors are available, the rest of them can be determined by utilizing some fundamental relations for view factors.
- Fundamental relations for view factors:
  - the reciprocity relation,
  - the summation rule,
  - the superposition rule,
  - the symmetry rule.

The Reciprocity Relation

$$A_i F_{i\to j} = A_j F_{j\to i}$$

$$F_{j\to i} = F_{i\to j} \quad \text{when} \quad A_i = A_j$$

$$F_{j\to i} \neq F_{i\to j} \quad \text{when} \quad A_i \neq A_j$$

The Summation Rule

$$\sum_{j=1}^{N} F_{i\to j} = 1$$

- The sum of the view factors from surface *i* of an enclosure to all surfaces of the enclosure, including to itself, must equal unity.
- For a three-surface enclosure,

$$\sum_{j=1}^{3} F_{1\to j} = F_{1\to 1} + F_{1\to 2} + F_{1\to 3} = 1$$

- The total number of view factors that need to be evaluated directly for an *N*-surface enclosure is

$$N^2 - [N + \tfrac{1}{2}N(N-1)] = \tfrac{1}{2}N(N-1)$$

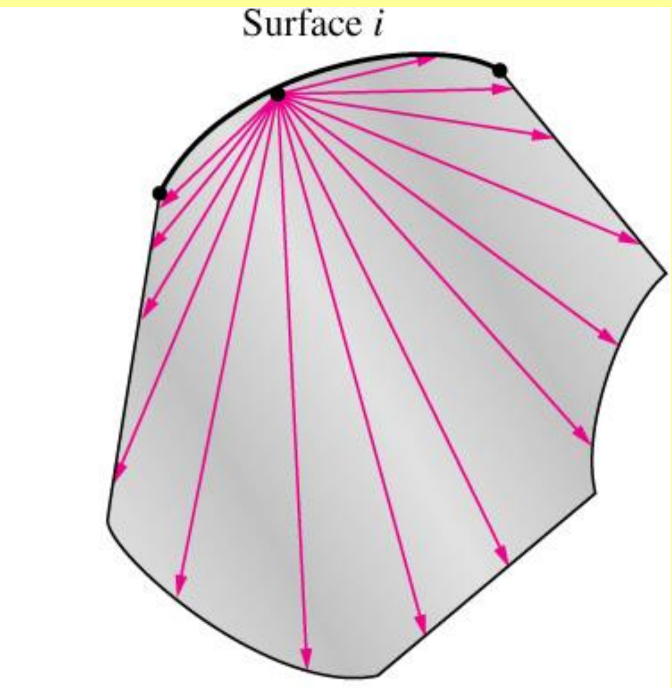


FIGURE 13–9

Radiation leaving any surface *i* of an enclosure must be intercepted completely by the surfaces of the enclosure. Therefore, the sum of the view factors from surface *i* to each one of the surfaces of the enclosure must be unity.

The Superposition Rule

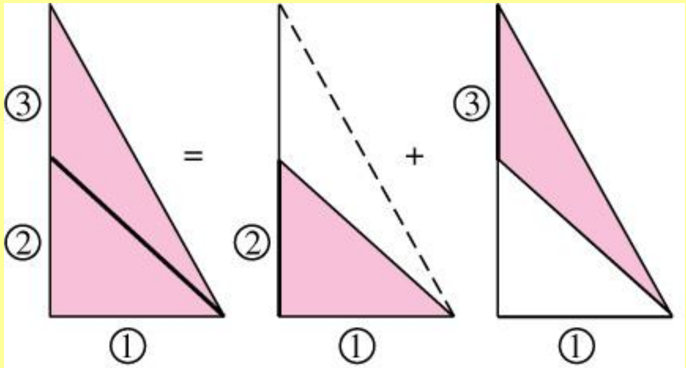
- *the view factor from a surface i to a surface j is equal to the sum of the view factors from surface i to the parts of surface j.*

$$F_{1\to(2,3)} = F_{1\to 2} + F_{1\to 3}$$

$$A_1F_{1\to(2,3)} = A_1F_{1\to 2} + A_1F_{1\to 3}$$

$$(A_2 + A_3)F_{(2,3)\to 1} = A_2F_{2\to 1} + A_3F_{3\to 1}$$

$$F_{(2,3)\to 1} = \frac{A_2F_{2\to 1} + A_3F_{3\to 1}}{A_2 + A_3}$$



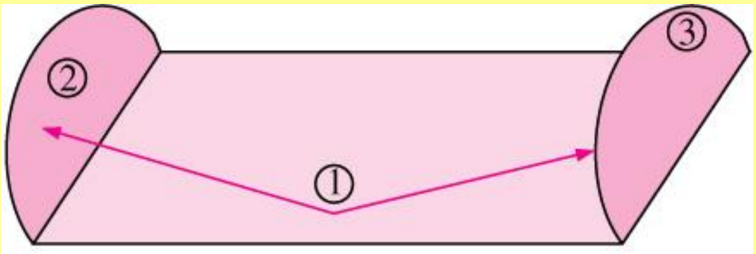
$$F_{1\to(2,3)} = F_{1\to 2} + F_{1\to 3}$$

FIGURE 13–11
The view factor from a surface to a composite surface is equal to the sum of the view factors from the surface to the parts of the composite surface.

The Symmetry Rule

$$F_{i\to j} = F_{i\to k}$$

$$F_{j\to i} = F_{k\to i}$$



$F_{1\to 2} = F_{1\to 3}$

(Also, $F_{2\to 1} = F_{3\to 1}$)

FIGURE 13–13
Two surfaces that are symmetric about a third surface will have the same view factor from the third surface.

View Factors between Infinitely Long Surfaces: The Crossed-Strings Method

- The view factor between two-dimensional surfaces can be determined by the simple *crossed-strings method.*

$$F_{i \to j} = \frac{\Sigma\,(\text{Crossed strings}) - \Sigma\,(\text{Uncrossed strings})}{2 \times (\text{String on surface } i)}$$

$$F_{1\to 2} = \frac{\left(L_5 + L_6\right) - \left(L_3 + L_4\right)}{2L_1}$$

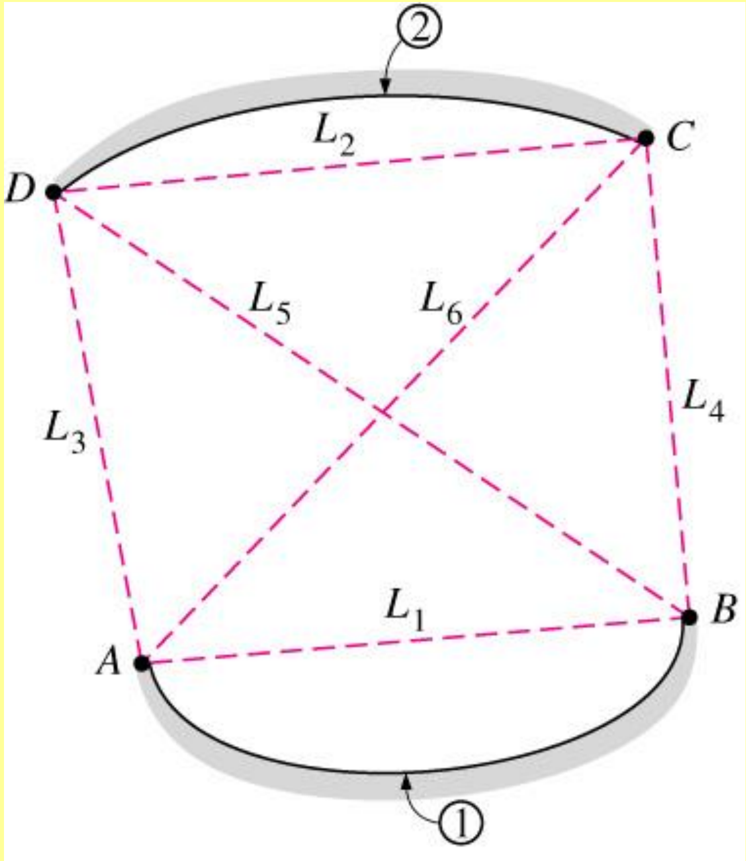
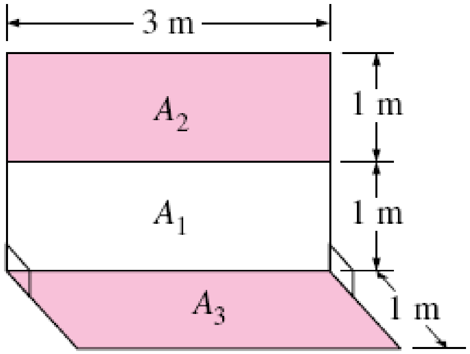


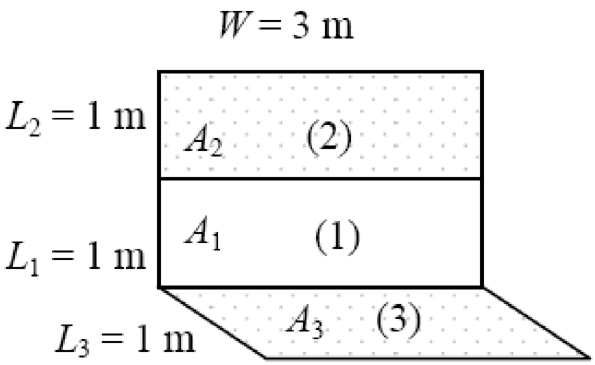
FIGURE 13–16
Determination of the view factor $F_{1\to 2}$ by the application of the crossed-strings method.

13–8 Determine the view factors F_{13} and F_{23} between the rectangular surfaces shown in Fig. P13–8.



Analysis From Fig. 13-6,

$$\left.\begin{array}{l}\dfrac{L_3}{W}=\dfrac{1}{3}=0.33\\[2ex]\dfrac{L_1}{W}=\dfrac{1}{3}=0.33\end{array}\right\}F_{31}=0.27$$



and

$$\left.\begin{array}{l}\dfrac{L_3}{W}=\dfrac{1}{3}=0.33\\[2ex]\dfrac{L_1+L_2}{W}=\dfrac{2}{3}=0.67\end{array}\right\}F_{3\to(1+2)}=0.32$$

We note that $A_1 = A_3$. Then the reciprocity and superposition rules gives

$$\mathrm{A}_1F_{13}=A_3F_{31}\longrightarrow F_{13}=F_{31}=\mathbf{0.27}$$

$$F_{3\to(1+2)}=F_{31}+F_{32}\longrightarrow\quad 0.32=0.27+F_{32}\longrightarrow F_{32}=0.05$$

Finally, $A_2=A_3\longrightarrow F_{23}=F_{32}=\mathbf{0.05}$

Radiation Heat Transfer: Black Surfaces

- Consider two black surfaces of arbitrary shape maintained at uniform temperatures T_1 and T_2.
- The *net* rate of radiation heat transfer from surface 1 to surface 2 can be expressed as

$$\dot{Q}_{1\rightarrow 2} = \left(\begin{array}{c}\textbf{Radiation leaving}\\ \textbf{the entire surface 1}\\ \textbf{that strikes surface 2}\end{array}\right) - \left(\begin{array}{c}\textbf{Radiation leaving}\\ \textbf{the entire surface 2}\\ \textbf{that strikes surface 1}\end{array}\right)$$

$$= A_1 E_{b1} F_{1\rightarrow 2} - A_2 E_{b2} F_{2\rightarrow 1} \qquad \text{(W)}$$

- Applying the reciprocity relation

$A_1F_{1\rightarrow 2}=A_2F_{2\rightarrow 1}$

$$\dot{Q}_{1\rightarrow 2} = A_1 F_{1\rightarrow 2}\sigma\left(T_1^4 - T_2^4\right) \qquad \text{(W)}$$

- For enclosure consisting of N black surfaces

$$\dot{Q}_i = \sum_{j=1}^{N} \dot{Q}_{i\rightarrow j} = \sum_{j=1}^{N} A_i F_{i\rightarrow j}\sigma\left(T_i^4 - T_j^4\right) \qquad \text{(W)}$$

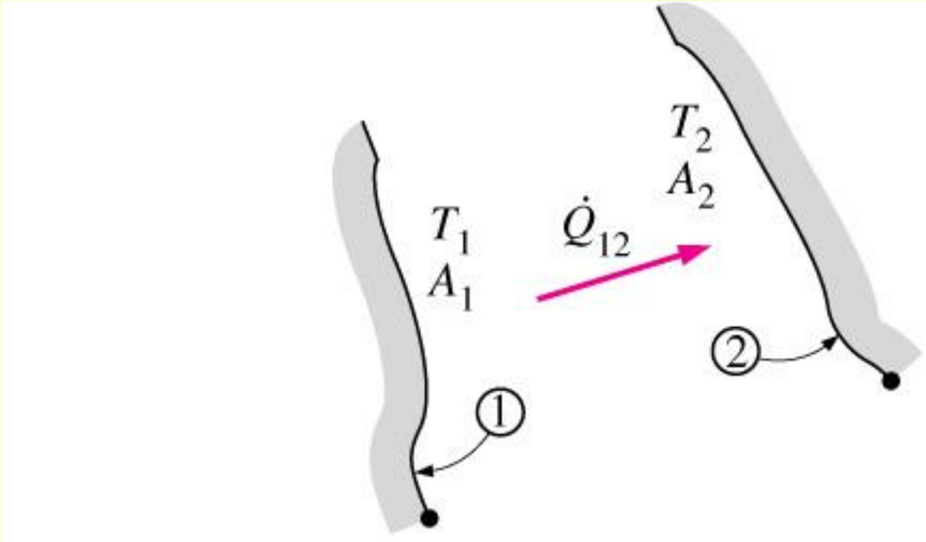


FIGURE 13–18
Two general black surfaces maintained at uniform temperatures T_1 and T_2.

Radiation Heat Transfer: Diffuse, Gray Surfaces

- To make a simple radiation analysis possible, it is common to assume the surfaces of an enclosure are:
  - opaque (nontransparent),
  - diffuse (diffuse emitters and diffuse reflectors),
  - gray (independent of wavelength),
  - isothermal, and
  - both the incoming and outgoing radiation are uniform over each surface.

Radiosity

Surfaces emit radiation as well as reflect it, and thus the radiation leaving a surface consists of emitted and reflected parts. The calculation of radiation heat transfer between surfaces involves the *total* radiation energy streaming away from a surface, with no regard for its origin. The *total radiation energy leaving a surface per unit time and per unit area* is the **radiosity** and is denoted by J (Fig. 13–20).

For a surface i that is *gray* and *opaque* ($\varepsilon_i = \alpha_i$ and $\alpha_i + \rho_i = 1$), the radiosity can be expressed as

$$\begin{aligned} J_i &= \begin{pmatrix} \text{Radiation emitted} \\ \text{by surface } i \end{pmatrix} + \begin{pmatrix} \text{Radiation reflected} \\ \text{by surface } i \end{pmatrix} \\ &= \varepsilon_i E_{bi} + \rho_i G_i \\ &= \varepsilon_i E_{bi} + (1 - \varepsilon_i) G_i \qquad (\text{W/m}^2) \end{aligned} \tag{13–21}$$

where $E_{bi} = \sigma T_i^4$ is the blackbody emissive power of surface i and G_i is irradiation (i.e., the radiation energy incident on surface i per unit time per unit area).

For a surface that can be approximated as a *blackbody* ($\varepsilon_i = 1$), the radiosity relation reduces to

$$J_i = E_{bi} = \sigma T_i^4 \qquad \text{(blackbody)} \tag{13–22}$$

That is, *the radiosity of a blackbody is equal to its emissive power.* This is expected, since a blackbody does not reflect any radiation, and thus radiation coming from a blackbody is due to emission only.

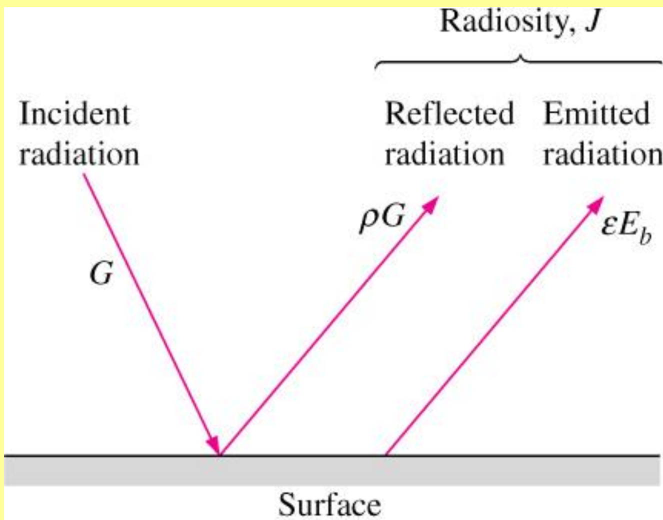


FIGURE 13–20

Radiosity represents the sum of the radiation energy emitted and reflected by a surface.

Net Radiation Heat Transfer to or from a Surface

$$\dot{Q}_i = \begin{pmatrix}\text{Radiation leaving} \\ \text{entire surface } i\end{pmatrix} - \begin{pmatrix}\text{Radiation incident} \\ \text{on entire surface } i\end{pmatrix}$$
$$= A_i(J_i - G_i) \qquad \text{(W)} \qquad \text{(13–23)}$$

Solving for G_i from Eq. 13–21 and substituting into Eq. 13–23 yields

$$\dot{Q}_i = A_i\left(J_i - \frac{J_i - \varepsilon_i E_{bi}}{1 - \varepsilon_i}\right) = \frac{A_i \varepsilon_i}{1 - \varepsilon_i}(E_{bi} - J_i) \qquad \text{(W)} \qquad \text{(13–24)}$$

In an electrical analogy to Ohm's law, this equation can be rearranged as

$$\dot{Q}_i = \frac{E_{bi} - J_i}{R_i} \qquad \text{(W)} \qquad \text{(13–25)}$$

where

$$R_i = \frac{1 - \varepsilon_i}{A_i \varepsilon_i} \qquad \text{(13–26)}$$

R_i is the **surface resistance** to radiation.

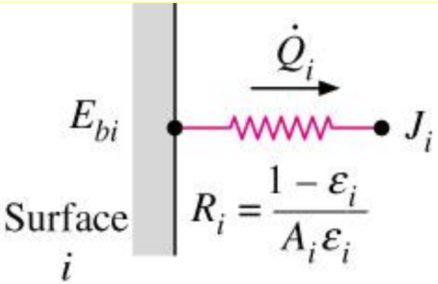


FIGURE 13–21
Electrical analogy of surface resistance to radiation.

When the convection effects on the front (heat transfer) side of such a surface is negligible and steady-state conditions are reached, the surface must lose as much radiation energy as it gains, and thus

$$\dot{Q}_i = 0$$

In such cases, the surface is said to *reradiate* all the radiation energy it receives, and such a surface is called a **reradiating surface**.

$$J_i = E_{bi} = \sigma T_i^4 \qquad (\text{W/m}^2)$$

Net Radiation Heat Transfer between any two surfaces

Consider two diffuse, gray, and opaque surfaces of arbitrary shape maintained at uniform temperatures, as shown in Fig. 13–22. Recognizing that the radiosity J represents the rate of radiation leaving a surface per unit surface area and that the view factor $F_{i \to j}$ represents the fraction of radiation leaving surface i that strikes surface j, the *net* rate of radiation heat transfer from surface i to surface j can be expressed as

$$\dot{Q}_{i \to j} = \begin{pmatrix} \text{Radiation leaving} \\ \text{the entire surface } i \\ \text{that strikes surface } j \end{pmatrix} - \begin{pmatrix} \text{Radiation leaving} \\ \text{the entire surface } j \\ \text{that strikes surface } i \end{pmatrix} \tag{13–28}$$

$$= A_i J_i F_{i \to j} - A_j J_j F_{j \to i} \qquad \text{(W)}$$

Applying the reciprocity relation $A_i F_{i \to j} = A_j F_{j \to i}$ yields

$$\dot{Q}_{i \to j} = A_i F_{i \to j} (J_i - J_j) \qquad \text{(W)} \tag{13–29}$$

Again in analogy to Ohm's law, this equation can be rearranged as

$$\dot{Q}_{i \to j} = \frac{J_i - J_j}{R_{i \to j}} \qquad \text{(W)} \tag{13–30}$$

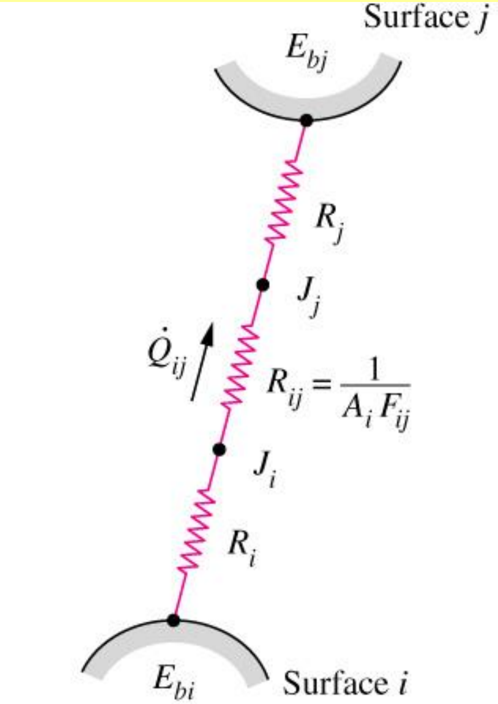


FIGURE 13–22
Electrical analogy of space resistance to radiation.

$$R_{i \to j} = \frac{1}{A_i F_{i \to j}}$$ **space resistance** to radiation

In an N-surface enclosure, the conservation of energy principle requires that the net heat transfer from surface i be equal to the sum of the net heat transfers from surface i to each of the N surfaces of the enclosure. That is,

$$\dot{Q}_i = \sum_{j=1}^{N} \dot{Q}_{i \to j} = \sum_{j=1}^{N} A_i F_{i \to j} (J_i - J_j) = \sum_{j=1}^{N} \frac{J_i - J_j}{R_{i \to j}} \qquad \text{(W)} \qquad \textbf{(13–32)}$$

The network representation of net radiation heat transfer from surface i to the remaining surfaces of an N-surface enclosure is given in Fig. 13–23. Note that $\dot{Q}_{i \to i}$ (the net rate of heat transfer from a surface to itself) is zero regardless of the shape of the surface. Combining Eqs. 13–25 and 13–32 gives

$$\frac{E_{bi} - J_i}{R_i} = \sum_{j=1}^{N} \frac{J_i - J_j}{R_{i \to j}} \qquad \text{(W)} \qquad \textbf{(13–33)}$$

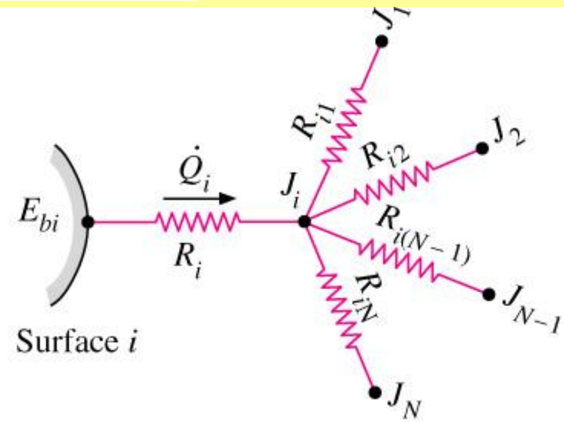


FIGURE 13–23

Network representation of net radiation heat transfer from surface i to the remaining surfaces of an N-surface enclosure.

Methods of Solving Radiation Problems

In the radiation analysis of an enclosure, either the temperature or the net rate of heat transfer must be given for each of the surfaces to obtain a unique solution for the unknown surface temperatures and heat transfer rates. There are two methods commonly used to solve radiation problems. In the first method, Eqs. 13–32 (for surfaces with specified heat transfer rates) and 13–33 (for surfaces with specified temperatures) are simplified and rearranged as

Surfaces with specified net heat transfer rate $\dot{Q}$

$$\dot{Q}_i = A_i \sum_{j=1}^{N} F_{i \rightarrow j}(J_i - J_j) \qquad \textbf{(13–34)}$$

Surfaces with specified temperature T_i

$$\sigma T_i^4 = J_i + \frac{1 - \varepsilon_i}{\varepsilon_i} \sum_{j=1}^{N} F_{i \rightarrow j}(J_i - J_j) \qquad \textbf{(13–35)}$$

Note that $\dot{Q}_i = 0$ for insulated (or reradiating) surfaces, and $\sigma T_i^4 = J_i$ for black surfaces since $\varepsilon_i = 1$ in that case. Also, the term corresponding to $j = i$ drops out from either relation since $J_i - J_j = J_i - J_i = 0$ in that case.

The equations above give N linear algebraic equations for the determination of the N unknown radiosities for an N-surface enclosure. Once the radiosities $J_1, J_2, \ldots, J_N$ are available, the unknown heat transfer rates can be determined from Eq. 13–34 while the unknown surface temperatures can be determined from Eq. 13–35. The temperatures of insulated or reradiating surfaces can be determined from $\sigma T_i^4 = J_i$. A positive value for $\dot{Q}_i$ indicates net radiation heat transfer *from* surface i to other surfaces in the enclosure while a negative value indicates net radiation heat transfer *to* the surface.

Radiation Heat Transfer in Two-Surface Enclosures

Consider an enclosure consisting of two opaque surfaces at specified temperatures T_1 and T_2, as shown in Fig. 13–24, and try to determine the net rate of radiation heat transfer between the two surfaces with the network method. Surfaces 1 and 2 have emissivities ε_1 and ε_2 and surface areas A_1 and A_2 and are maintained at uniform temperatures T_1 and T_2, respectively. There are only two surfaces in the enclosure, and thus we can write

$$\dot{Q}_{12} = \dot{Q}_1 = -\dot{Q}_2$$

That is, the net rate of radiation heat transfer from surface 1 to surface 2 must equal the net rate of radiation heat transfer *from* surface 1 and the net rate of radiation heat transfer *to* surface 2.

$$\dot{Q}_{12} = \frac{E_{b1} - E_{b2}}{R_1 + R_{12} + R_2} = \dot{Q}_1 = -\dot{Q}_2$$

$$\dot{Q}_{12} = \frac{\sigma(T_1^4 - T_2^4)}{\dfrac{1-\varepsilon_1}{A_1\varepsilon_1} + \dfrac{1}{A_1F_{12}} + \dfrac{1-\varepsilon_2}{A_2\varepsilon_2}}$$

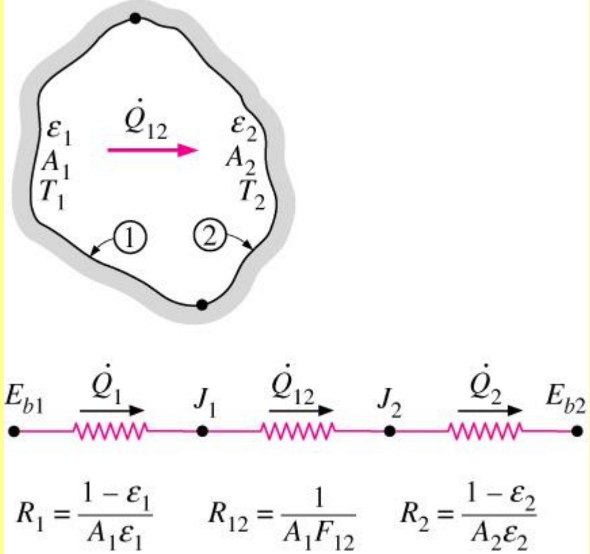
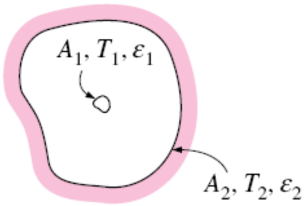
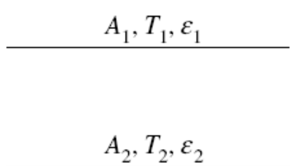
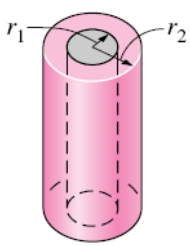
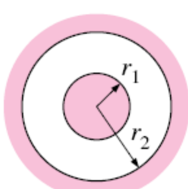


FIGURE 13–24
Schematic of a two-surface enclosure and the radiation network associated with it.

TABLE 13–3

Small object in a large cavity 	$\frac{A_1}{A_2} \approx 0$ $F_{12} = 1$	$\dot{Q}_{12} = A_1 \sigma \varepsilon_1 (T_1^4 - T_2^4)$	**(13–37)**
Infinitely large parallel plates 	$A_1 = A_2 = A$ $F_{12} = 1$	$\dot{Q}_{12} = \dfrac{A\sigma(T_1^4 - T_2^4)}{\dfrac{1}{\varepsilon_1} + \dfrac{1}{\varepsilon_2} - 1}$	**(13–38)**
Infinitely long concentric cylinders 	$\frac{A_1}{A_2} = \frac{r_1}{r_2}$ $F_{12} = 1$	$\dot{Q}_{12} = \dfrac{A_1\sigma(T_1^4 - T_2^4)}{\dfrac{1}{\varepsilon_1} + \dfrac{1 - \varepsilon_2}{\varepsilon_2}\left(\dfrac{r_1}{r_2}\right)}$	**(13–39)**
Concentric spheres 	$\frac{A_1}{A_2} = \left(\frac{r_1}{r_2}\right)^2$ $F_{12} = 1$	$\dot{Q}_{12} = \dfrac{A_1\sigma(T_1^4 - T_2^4)}{\dfrac{1}{\varepsilon_1} + \dfrac{1 - \varepsilon_2}{\varepsilon_2}\left(\dfrac{r_1}{r_2}\right)^2}$	**(13–40)**

Radiation Heat Transfer in Three-Surface Enclosures

$$\frac{E_{b1}-J_1}{R_1}+\frac{J_2-J_1}{R_{12}}+\frac{J_3-J_1}{R_{13}}=0$$

$$\frac{J_1-J_2}{R_{12}}+\frac{E_{b2}-J_2}{R_2}+\frac{J_3-J_2}{R_{23}}=0$$

$$\frac{J_1-J_3}{R_{13}}+\frac{J_2-J_3}{R_{23}}+\frac{E_{b3}-J_3}{R_3}=0$$

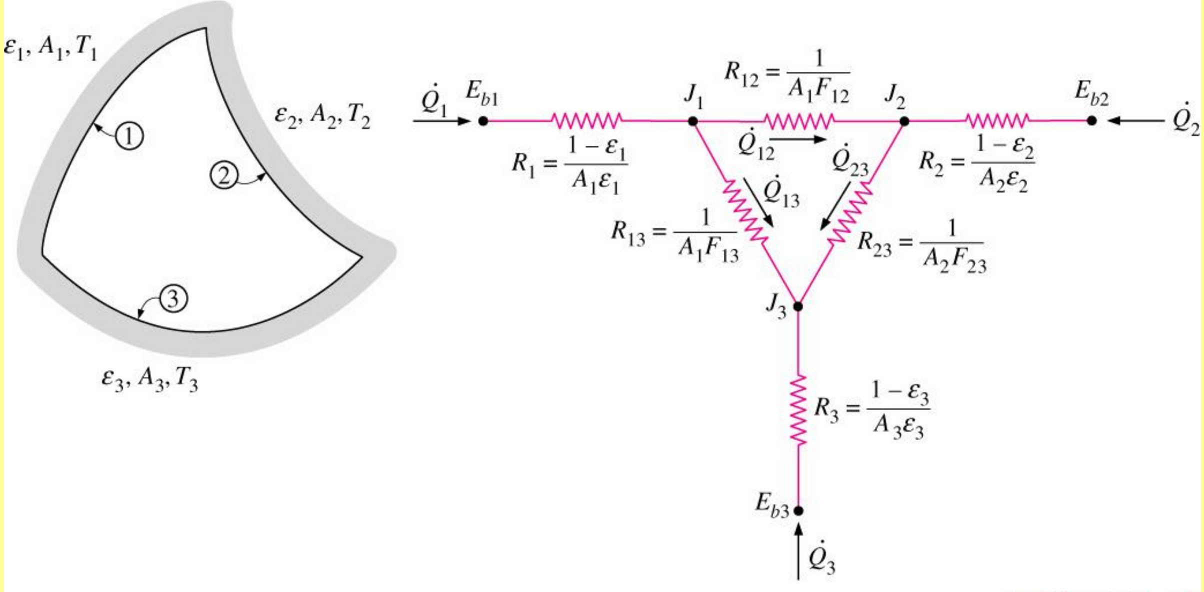


FIGURE 13–26

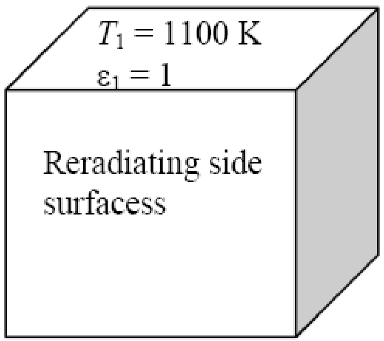
Schematic of a three-surface enclosure and the radiation network associated with it.

The set of equations above simplify further if one or more surfaces are "special" in some way. For example, $J_i = E_{bi} = \sigma T_i^4$ for a *black* or *reradiating* surface. Also, $\dot{Q}_i = 0$ for a reradiating surface. Finally, when the net rate of radiation heat transfer $\dot{Q}_i$ is specified at surface i instead of the temperature, the term $(E_{bi} - J_i)/R_i$ should be replaced by the specified $\dot{Q}_i$.

13–38 Consider a 4-m × 4-m × 4-m cubical furnace whose floor and ceiling are black and whose side surfaces are reradiating. The floor and the ceiling of the furnace are maintained at temperatures of 550 K and 1100 K, respectively. Determine the net rate of radiation heat transfer between the floor and the ceiling of the furnace.

Analysis We consider the ceiling to be surface 1, the floor to be surface 2 and the side surfaces to be surface 3. The furnace can be considered to be three-surface enclosure. We assume that steady-state conditions exist. Since the side surfaces are reradiating, there is no heat transfer through them, and the entire heat lost by the ceiling must be gained by the floor. The view factor from the ceiling to the floor of the furnace is $F_{12} = 0.2$. Then the rate of heat loss from the ceiling can be determined from

$$\dot{Q}_1 = \frac{E_{b1} - E_{b2}}{\left(\dfrac{1}{R_{12}} + \dfrac{1}{R_{13} + R_{23}}\right)^{-1}}$$



where

$$E_{b1} = \sigma T_1^{\,4} = (5.67 \times 10^{-8}\ \text{W/m}^2.\text{K}^4)(1100\ \text{K})^4 = 83{,}015\ \text{W/m}^2$$

$$E_{b2} = \sigma T_2^{\,4} = (5.67 \times 10^{-8}\ \text{W/m}^2.\text{K}^4)(550\ \text{K})^4 = 5188\ \text{W/m}^2$$

and

$$A_1 = A_2 = (4\ \text{m})^2 = 16\ \text{m}^2$$

$$R_{12} = \frac{1}{A_1 F_{12}} = \frac{1}{(16\ \text{m}^2)(0.2)} = 0.3125\ \text{m}^{-2}$$

$$R_{13} = R_{23} = \frac{1}{A_1 F_{13}} = \frac{1}{(16\ \text{m}^2)(0.8)} = 0.078125\ \text{m}^{-2}$$

Substituting,

$$\dot{Q}_{12} = \frac{(83{,}015 - 5188)\ \text{W/m}^2}{\left(\dfrac{1}{0.3125\ \text{m}^{-2}} + \dfrac{1}{2(0.078125\ \text{m}^{-2})}\right)^{-1}} = 7.47 \times 10^5\ \text{W} = \mathbf{747\ kW}$$

Radiation Shields and The radiation Effects

$$\dot{Q}_{12,\text{ no shield}} = \frac{A\sigma(T_1^4 - T_2^4)}{\dfrac{1}{\varepsilon_1} + \dfrac{1}{\varepsilon_2} - 1}$$

$$\dot{Q}_{12,\text{ one shield}} = \frac{E_{b1} - E_{b2}}{\dfrac{1-\varepsilon_1}{A_1\,\varepsilon_1} + \dfrac{1}{A_1\,F_{12}} + \dfrac{1-\varepsilon_{3,\,1}}{A_3\,\varepsilon_{3,\,1}} + \dfrac{1-\varepsilon_{3,\,2}}{A_3\,\varepsilon_{3,\,2}} + \dfrac{1}{A_3\,F_{32}} + \dfrac{1-\varepsilon_2}{A_2\,\varepsilon_2}}$$

$$\dot{Q}_{12,\text{ one shield}} = \frac{A\sigma(T_1^4 - T_2^4)}{\left(\dfrac{1}{\varepsilon_1} + \dfrac{1}{\varepsilon_2} - 1\right) + \left(\dfrac{1}{\varepsilon_{3,\,1}} + \dfrac{1}{\varepsilon_{3,\,2}} - 1\right)}$$

$$\dot{Q}_{12,\,N\text{ shields}} = \frac{A\sigma(T_1^4 - T_2^4)}{\left(\dfrac{1}{\varepsilon_1} + \dfrac{1}{\varepsilon_2} - 1\right) + \left(\dfrac{1}{\varepsilon_{3,\,1}} + \dfrac{1}{\varepsilon_{3,\,2}} - 1\right) + \cdots + \left(\dfrac{1}{\varepsilon_{N,\,1}} + \dfrac{1}{\varepsilon_{N,\,2}} - 1\right)}$$

$$\dot{Q}_{12,\,N\text{ shields}} = \frac{A\sigma(T_1^4 - T_2^4)}{(N+1)\left(\dfrac{1}{\varepsilon} + \dfrac{1}{\varepsilon} - 1\right)} = \frac{1}{N+1}\,\dot{Q}_{12,\text{ no shield}}$$

If the emissivities of all surfaces are equal

When all emissivitoes are equal, 1 shield reduces the rate of radiation heat transfer to one-half, 9 shields reduce it to one-tenth, and 19 shields reduce it to one-twentieth (or 5 percent) of what it was when there were no shields.

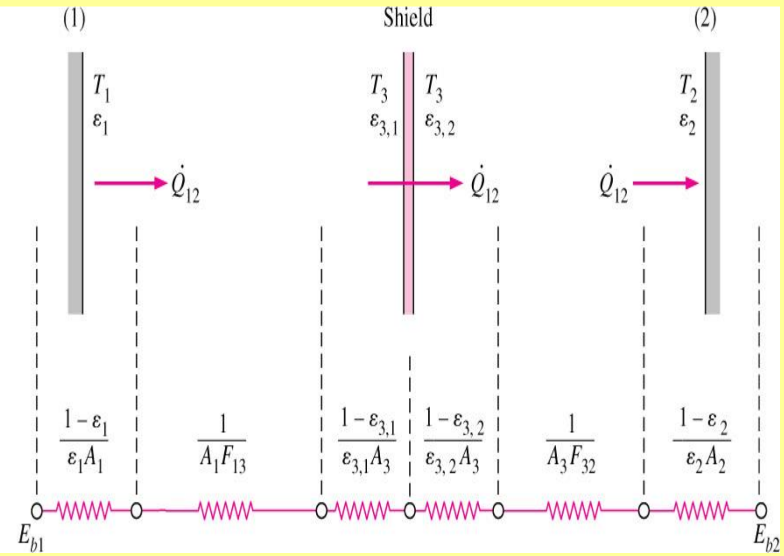


FIGURE 13–30
The radiation shield placed between two parallel plates and the radiation network associated with it.

Conclusions

- The View Factor
- View Factor Relations
- Radiation Heat Transfer: Black Surfaces
- Radiation Heat Transfer: Diffuse, Gray Surfaces
  - ✓ Radiosity
  - ✓ Net Radiation Heat Transfer to or from a Surface
  - ✓ Net Radiation Heat Transfer between Any Two Surfaces
- Methods of Solving Radiation Problems
- Radiation Heat Transfer in Two-Surface Enclosures
- Radiation Heat Transfer in Three-Surface Enclosures
- Radiation Shields and the Radiation Effect
- Radiation Effect on Temperature Measurements