ME302 FLUID MECHANICS II

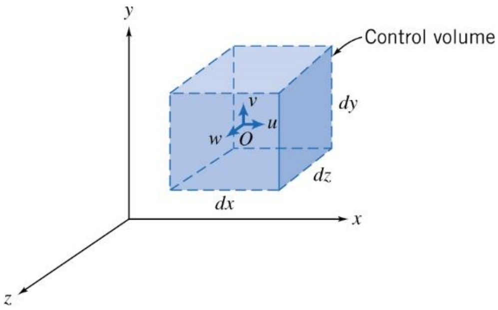
In ME301 Fluid Mechanics I course, we developed the basic equations in integral form for a control volume. The integral equations are particularly useful when we are interested in the gross behavior of a flow and its effect on various devices. However, the integral approach does not enable to us to obtain detailed point by point knowledge of the flow field.

To obtain this detailed knowledge, we must apply the equations of fluid motion in differential form.

CONSERVATION OF MASS (CONTINUITY EQUATION)

The application of the principle of conservation of mass to a fluid flow yields an equation which is referred as the continuity equation. We shall derive the differential equation for mass in rectangular and in cylindrical coordinates.

Rectangular Coordinate System



The differential form of the continuity equation may be obtained by applying the principle of conservation of mass to an infinitesimal control volume.

The sides of the control volume are *dx*, *dy*, and *dz*. The density at the center, O, of the control volume is ρ and the velocity is $\vec{V} = u\vec{i} + v\vec{j} + w\vec{k}$. The values of the mass fluxes at each of six faces of the control surface may be obtained by using a Taylor series expansion of the mass fluxes about point O. For example, at the right face,

$$\rho\Big)_{x+\frac{dx}{2}} = \rho + \left(\frac{\partial \rho}{\partial x}\right)\frac{dx}{2} + \left(\frac{\partial^2 \rho}{\partial x^2}\right)\frac{1}{2!}\left(\frac{dx}{2}\right)^2 + \cdots$$

Neglecting higher order terms, we can write

$$\rho\Big)_{x+\frac{dx}{2}} = \rho + \left(\frac{\partial \rho}{\partial x}\right)\frac{dx}{2}$$

and similarly,

$$u\Big)_{x+\frac{dx}{2}} = u + \left(\frac{\partial u}{\partial x}\right)\frac{dx}{2}$$

The corresponding terms at the left face are

$$\rho\Big)_{x-\frac{dx}{2}} = \rho + \left(\frac{\partial \rho}{\partial x}\right)\left(-\frac{dx}{2}\right) = \rho - \frac{\partial \rho}{\partial x}\frac{dx}{2}$$

$$u\Big)_{x-\frac{dx}{2}} = u + \left(\frac{\partial u}{\partial x}\right)\left(-\frac{dx}{2}\right) = u - \frac{\partial u}{\partial x}\frac{dx}{2}$$

A word statement of conservation of mass is

$$\begin{bmatrix}\text{Net rate of mass flux out}\\ \text{through the control surface}\end{bmatrix} + \begin{bmatrix}\text{Rate change of mass}\\ \text{inside the control volume}\end{bmatrix} = 0$$

$$\int_{CS} \rho\vec{V}\cdot d\vec{A} + \frac{\partial}{\partial t}\int_{C\forall} \rho d\forall = 0$$

To evaluate the first term in this equation, we must evaluate $\int_{CS} \rho\vec{V}\cdot d\vec{A}$. The mass flux through each of six faces are shown in Table below.

Table. Mass flux through the control surface of a rectangular differential control volume

Surface	Evaluation of $\int \rho \vec{V} \cdot d\vec{A}$
Left $(-x)$	$= -\left[\rho - \left(\frac{\partial \rho}{\partial x}\right)\frac{dx}{2}\right]\left[u - \left(\frac{\partial u}{\partial x}\right)\frac{dx}{2}\right] dy\,dz = -\rho u\,dy\,dz + \frac{1}{2}\left[u\left(\frac{\partial \rho}{\partial x}\right) + \rho\left(\frac{\partial u}{\partial x}\right)\right] dx\,dy\,dz$
Right $(+x)$	$= \left[\rho + \left(\frac{\partial \rho}{\partial x}\right)\frac{dx}{2}\right]\left[u + \left(\frac{\partial u}{\partial x}\right)\frac{dx}{2}\right] dy\,dz = \rho u\,dy\,dz + \frac{1}{2}\left[u\left(\frac{\partial \rho}{\partial x}\right) + \rho\left(\frac{\partial u}{\partial x}\right)\right] dx\,dy\,dz$
Bottom $(-y)$	$= -\left[\rho - \left(\frac{\partial \rho}{\partial y}\right)\frac{dy}{2}\right]\left[v - \left(\frac{\partial v}{\partial y}\right)\frac{dy}{2}\right] dx\,dz = -\rho v\,dx\,dz + \frac{1}{2}\left[v\left(\frac{\partial \rho}{\partial y}\right) + \rho\left(\frac{\partial v}{\partial y}\right)\right] dx\,dy\,dz$
Top $(+y)$	$= \left[\rho + \left(\frac{\partial \rho}{\partial y}\right)\frac{dy}{2}\right]\left[v + \left(\frac{\partial v}{\partial y}\right)\frac{dy}{2}\right] dx\,dz = \rho v\,dx\,dz + \frac{1}{2}\left[v\left(\frac{\partial \rho}{\partial y}\right) + \rho\left(\frac{\partial v}{\partial y}\right)\right] dx\,dy\,dz$
Back $(-z)$	$= -\left[\rho - \left(\frac{\partial \rho}{\partial z}\right)\frac{dz}{2}\right]\left[w - \left(\frac{\partial w}{\partial z}\right)\frac{dz}{2}\right] dx\,dy = -\rho w\,dx\,dy + \frac{1}{2}\left[w\left(\frac{\partial \rho}{\partial z}\right) + \rho\left(\frac{\partial w}{\partial z}\right)\right] dx\,dy\,dz$
Front $(+z)$	$= \left[\rho + \left(\frac{\partial \rho}{\partial z}\right)\frac{dz}{2}\right]\left[w + \left(\frac{\partial w}{\partial z}\right)\frac{dz}{2}\right] dx\,dy = \rho w\,dx\,dy + \frac{1}{2}\left[w\left(\frac{\partial \rho}{\partial z}\right) + \rho\left(\frac{\partial w}{\partial z}\right)\right] dx\,dy\,dz$

Adding the results for all six faces,

$$\int_{\mathrm{CS}} \rho \vec{V} \cdot d\vec{A} = \left[\left\{u\left(\frac{\partial \rho}{\partial x}\right) + \rho\left(\frac{\partial u}{\partial x}\right)\right\} + \left\{v\left(\frac{\partial \rho}{\partial y}\right) + \rho\left(\frac{\partial v}{\partial y}\right)\right\} + \left\{w\left(\frac{\partial \rho}{\partial z}\right) + \rho\left(\frac{\partial w}{\partial z}\right)\right\}\right] dx\,dy\,dz$$

or

$$\int_{\mathrm{CS}} \rho \vec{V} \cdot d\vec{A} = \left[\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z}\right] dx\,dy\,dz$$

The net rate of mass out through control surface is

$$\left[\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z}\right] dxdydz$$

The rate of change of mass inside the control volume is given by

$$\frac{\partial \rho}{\partial t} dxdydz$$

Therefore, the continuity equation in rectangular coordinate is

$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} + \frac{\partial \rho}{\partial t} = 0$$

Since, the vector operator, ∇, in rectangular coordinates is given by

$$\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$$

$$\therefore \quad \nabla \cdot \rho\vec{V} + \frac{\partial \rho}{\partial t} = 0$$

Two special cases, the continuity equation may be simplified.

1) **For an incompressible flow**, the density is constant, the continuity equation becomes,

$$\nabla \cdot \vec{V} = 0 \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

2) **For a steady flow**, the partial derivatives with respect to time are zero, that is $\frac{\partial}{\partial t} = 0$.

   Then,

$$\nabla \cdot \left(\rho\vec{V}\right) = 0 \qquad \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} = 0$$

Example: For a fluid flow in *xy* plane, the velocity component in the *y* direction is given by $v = y^2 - x^2 - 2y$.

a) Determine a possible velocity component in the *x* direction for steady flow of an incompressible fluid. How many possible *x* components are there?
b) Is the determined velocity component in the x direction also valid for unsteady flow of an incompressible fluid?

Basic equation: $\nabla \cdot \rho\vec{V} + \frac{\partial \rho}{\partial t} = 0$

a) For steady incompressible flow $\nabla \cdot \vec{V} = 0$

   For two-dimensional flow

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \text{or}$$

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y} = -2y + 2$$

Integrating the equation with respect to *x* yields,

$$u = -2xy + 2x + f(y)$$

Since, any function *f(y)* is allowable, the number of expressions for *u* to satisfy the differential continuity equation under given conditions is infinity.

b) Whether the flow is steady or not, the continuity equation for incompressible flow is $\nabla \cdot \vec{V} = 0$. Therefore, the velocity component in the x direction is also valid for unsteady flow of an incompressible fluid.

<u>Example:</u> A compressible flow field is described by $\rho\vec{V} = \left(ax\vec{\imath} - bxy\vec{\jmath}\right)e^{-kt}$
Determine the rate of change of the density at point *x = 3 m, y = 2 m* and *z = 2 m* for *t = 0.*

Basic equation: $\nabla \cdot \rho\vec{V} + \dfrac{\partial \rho}{\partial t} = 0$

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \rho\vec{V} = -\left(\frac{\partial}{\partial x}\vec{\imath} + \frac{\partial}{\partial y}\vec{\jmath} + \frac{\partial}{\partial z}\vec{k}\right)\cdot\left(ax\vec{\imath} - bxy\vec{\jmath}\right)e^{-kt}$$

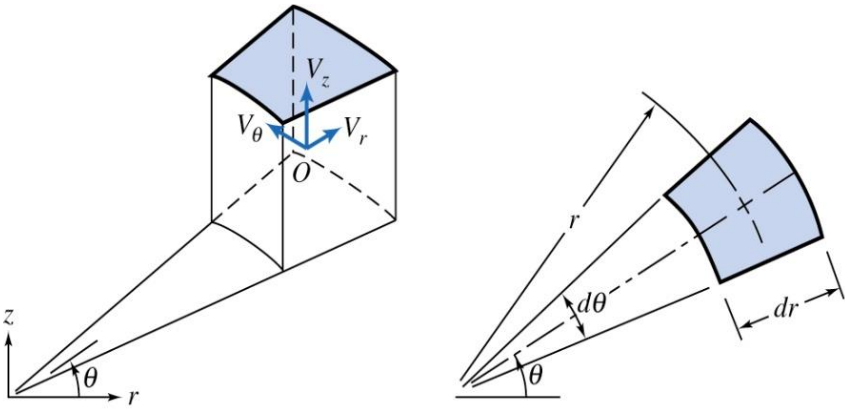
$$\frac{\partial \rho}{\partial t} = -\left(a - bx\right)e^{-kt} = \left(bx - a\right)e^{-kt}$$

for $(x, y, z, t) = (3, 2, 2, 0)$

$$\frac{\partial \rho}{\partial t} = \left(3b - a\right)e^{-k0} = \left(3b - a\right) \quad \left[\frac{\text{kg}}{\text{m}^3\text{s}}\right]$$

<u>Cylindrical Coordinate System</u>

In cylindrical coordinates, a suitable differential control volume is shown in the figure. The density at the center, O, is ρ and the velocity there is $\vec{V} = V_r\vec{e}_r + V_\theta\vec{e}_\theta + V_z\vec{e}_z$.



(a) Isometric view (b) Projection on $r\theta$ plane

Figure. Differential control volume in cylindrical coordinates

To evaluate $\int_{CS} \rho \vec{V} \cdot d\vec{A}$ we must consider the mass flux through each of the six faces of the control surface. The properties at each of the six faces of the control surface are obtained from Taylor series expansion about point O.

Table. Mass flux through the control surface of a cylindrical differential control volume

Surface	Evaluation of $\int \rho\vec{V}\cdot d\vec{A}$
Inside $(-r)$	$= -\left[\rho - \left(\frac{\partial\rho}{\partial r}\right)\frac{dr}{2}\right]\left[V_r - \left(\frac{\partial V_r}{\partial r}\right)\frac{dr}{2}\right]\left(r - \frac{dr}{2}\right)d\theta\,dz = -\rho V_r\,r d\theta\,dz + \rho V_r\frac{dr}{2}d\theta\,dz + \rho\left(\frac{\partial V_r}{\partial r}\right)r\frac{dr}{2}d\theta\,dz + V_r\left(\frac{\partial\rho}{\partial r}\right)r\frac{dr}{2}d\theta\,dz$
Outside $(+r)$	$= \left[\rho + \left(\frac{\partial\rho}{\partial r}\right)\frac{dr}{2}\right]\left[V_r + \left(\frac{\partial V_r}{\partial r}\right)\frac{dr}{2}\right]\left(r + \frac{dr}{2}\right)d\theta\,dz = \rho V_r\,r d\theta\,dz + \rho V_r\frac{dr}{2}d\theta\,dz + \rho\left(\frac{\partial V_r}{\partial r}\right)r\frac{dr}{2}d\theta\,dz + V_r\left(\frac{\partial\rho}{\partial r}\right)r\frac{dr}{2}d\theta\,dz$
Front $(-\theta)$	$= -\left[\rho - \left(\frac{\partial\rho}{\partial\theta}\right)\frac{d\theta}{2}\right]\left[V_\theta - \left(\frac{\partial V_\theta}{\partial\theta}\right)\frac{d\theta}{2}\right]dr\,dz = -\rho V_\theta\,dr\,dz + \rho\left(\frac{\partial V_\theta}{\partial\theta}\right)\frac{d\theta}{2}dr\,dz + V_\theta\left(\frac{\partial\rho}{\partial\theta}\right)\frac{d\theta}{2}dr\,dz$
Back $(+\theta)$	$= \left[\rho + \left(\frac{\partial\rho}{\partial\theta}\right)\frac{d\theta}{2}\right]\left[V_\theta + \left(\frac{\partial V_\theta}{\partial\theta}\right)\frac{d\theta}{2}\right]dr\,dz = \rho V_\theta\,dr\,dz + \rho\left(\frac{\partial V_\theta}{\partial\theta}\right)\frac{d\theta}{2}dr\,dz + V_\theta\left(\frac{\partial\rho}{\partial\theta}\right)\frac{d\theta}{2}dr\,dz$
Bottom $(-z)$	$= -\left[\rho - \left(\frac{\partial\rho}{\partial z}\right)\frac{dz}{2}\right]\left[V_z - \left(\frac{\partial V_z}{\partial z}\right)\frac{dz}{2}\right]r d\theta\,dr = -\rho V_z\,r d\theta\,dr + \rho\left(\frac{\partial V_z}{\partial z}\right)\frac{dz}{2}r d\theta\,dr + V_z\left(\frac{\partial\rho}{\partial z}\right)\frac{dz}{2}r d\theta\,dr$
Top $(+z)$	$= \left[\rho + \left(\frac{\partial\rho}{\partial z}\right)\frac{dz}{2}\right]\left[V_z + \left(\frac{\partial V_z}{\partial z}\right)\frac{dz}{2}\right]r d\theta\,dz = \rho V_z\,r d\theta\,dr + \rho\left(\frac{\partial V_z}{\partial z}\right)\frac{dz}{2}r d\theta\,dr + V_z\left(\frac{\partial\rho}{\partial z}\right)\frac{dz}{2}r d\theta\,dr$

Adding the results for all six faces,

$$\int_{CS}\rho\vec{V}\cdot d\vec{A} = \left[\rho V_r + r\left\{\rho\left(\frac{\partial V_r}{\partial r}\right) + V_r\left(\frac{\partial\rho}{\partial r}\right)\right\} + \left\{\rho\left(\frac{\partial V_\theta}{\partial\theta}\right) + V_\theta\left(\frac{\partial\rho}{\partial\theta}\right)\right\} + r\left\{\rho\left(\frac{\partial V_z}{\partial z}\right) + V_z\left(\frac{\partial\rho}{\partial z}\right)\right\}\right]dr\,d\theta\,dz$$

or

$$\int_{CS}\rho\vec{V}\cdot d\vec{A} = \left[\rho V_r + r\frac{\partial\rho V_r}{\partial r} + \frac{\partial\rho V_\theta}{\partial\theta} + r\frac{\partial\rho V_z}{\partial z}\right]dr\,d\theta\,dz$$

The net rate of mass flux out through the control surface is given by

$$\left[\rho V_r + r\frac{\partial\rho V_r}{\partial r} + \frac{\partial\rho V_\theta}{\partial\theta} + r\frac{\partial\rho V_z}{\partial z}\right]drd\theta dz$$

The rate of change of mass inside the control volume is given by $\frac{\partial\rho}{\partial t}rd\theta drdz$

In cylindrical coordinates the continuity equation becomes

$$\rho V_r + r\frac{\partial\rho V_r}{\partial r} + \frac{\partial\rho V_\theta}{\partial\theta} + r\frac{\partial\rho V_z}{\partial z} + r\frac{\partial\rho}{\partial t} = 0$$

Dividing by *r* gives,

$$\frac{\rho V_r}{r}+\frac{\partial \rho V_r}{\partial r}+\frac{1}{r}\frac{\partial \rho V_\theta}{\partial \theta}+\frac{\partial \rho V_z}{\partial z}+\frac{\partial \rho}{\partial t}=0$$

or

$$\frac{1}{r}\frac{\partial\left(r\rho V_r\right)}{\partial r}+\frac{1}{r}\frac{\partial\left(\rho V_\theta\right)}{\partial \theta}+\frac{\partial\left(\rho V_z\right)}{\partial z}+\frac{\partial \rho}{\partial t}=0$$

In cylindrical coordinates the vector operator ∇ is given by

$$\nabla=\frac{\partial}{\partial r}\vec{e}_r+\frac{1}{r}\frac{\partial}{\partial \theta}\vec{e}_\theta+\frac{\partial}{\partial z}\vec{e}_z$$

Then the continuity equation can be written in vector notation as

$$\nabla\cdot\rho\vec{V}+\frac{\partial \rho}{\partial t}=0 \qquad \left[\text{Note: } \frac{\partial \vec{e}_r}{\partial r}=\vec{e}_\theta \text{ and } \frac{\partial \vec{e}_\theta}{\partial \theta}=-\vec{e}_r\right]$$

Two special cases, the continuity equation may be simplified.

1) **For an incompressible flow**, the density is constant, i.e.,

$$\nabla\cdot\vec{V}=0 \qquad \frac{1}{r}\frac{\partial\left(rV_r\right)}{\partial r}+\frac{1}{r}\frac{\partial V_\theta}{\partial \theta}+\frac{\partial V_z}{\partial z}=0$$

2) **For a steady flow**,

$$\nabla\cdot\left(\rho\vec{V}\right)=0 \qquad \frac{1}{r}\frac{\partial\left(r\rho V_r\right)}{\partial r}+\frac{1}{r}\frac{\partial\left(\rho V_\theta\right)}{\partial \theta}+\frac{\partial\left(\rho V_z\right)}{\partial z}=0$$

Example: Consider one-dimensional radial flow in the $r\theta$ plane, characterized by $v_r = f(r)$ and $v_\theta = 0$. Determine the conditions on *f(r)* required for incompressible flow.

For incompressible flow $\nabla\cdot\vec{V}=0$

$$\frac{1}{r}\frac{\partial\left(rV_r\right)}{\partial r}+\frac{1}{r}\frac{\partial V_\theta}{\partial \theta}=0 \quad \text{in the } r\theta \text{ plane.}$$

For the given velocity field, $\vec{V}=\vec{V}\left(r\right)=V_r\vec{e}_r+V_\theta\vec{e}_\theta=V_r\vec{e}_r$

$$\therefore \frac{1}{r}\frac{\partial(rV_r)}{\partial r}=0$$

Integrating with respect to r gives $rV_r=\text{constant}=C$

Thus, $V_r=f(r)=\dfrac{C}{r}$ for one-dimensional radial flow of an incompressible fluid.

STREAM FUNCTION FOR TWO-DIMENSIONAL

INCOMPRESSIBLE FLOW

For a two-dimensional flow in the *xy* plane of the cartesian coordinate systems, the continuity equation for an incompressible fluid reduces to

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

If a continuous function $\psi\left(x,y,t\right)$, called stream function, is defined such that

$$u\equiv\frac{\partial\psi}{\partial y} \qquad \text{and} \qquad v\equiv-\frac{\partial\psi}{\partial x}$$

then the continuity equation is satisfied exactly, since

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=\frac{\partial^2\psi}{\partial x\partial y}-\frac{\partial^2\psi}{\partial y\partial x}=0$$

Since streamlines are tangent to the direction of flow at every point in the flow field. Thus, if $d\vec{r}$ is an element of length along a streamline, the equation of streamline is given by

$$\vec{V}\times d\vec{r}=0=\left(u\vec{\imath}+v\vec{j}\right)\times\left(dx\vec{\imath}+dy\vec{j}\right)=\left(udy-vdx\right)\vec{k}$$

then

$$udy-vdx=0$$

Substituting for velocity components *u* and *v*, in terms of the stream function, ψ

$$\frac{\partial\psi}{\partial x}dx+\frac{\partial\psi}{\partial y}dy=0 \tag{A}$$

At a certain instant of time, t_0, the stream function may be expressed as $\psi=\psi\left(x,y,t_0\right)$. At this instant, the stream function

$$d\psi=\frac{\partial\psi}{\partial x}dx+\frac{\partial\psi}{\partial y}dy \tag{B}$$

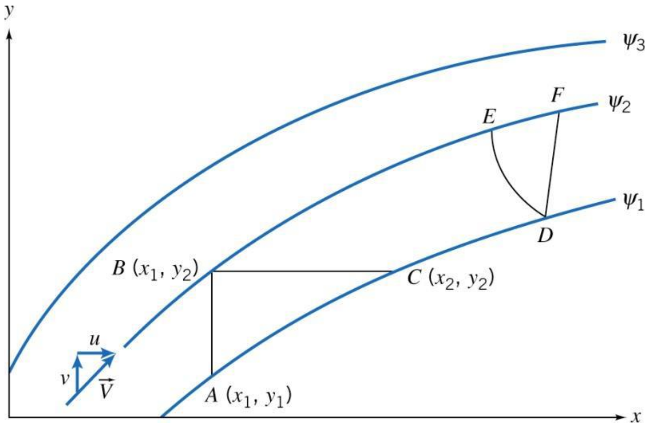
Comparing Equations (A) and (B), we see that along an instantaneous streamline

$$d\psi=0$$

and $\psi=\text{Constant}$ along a streamline.

In the flow field, $\psi_1-\psi_2$, depends only on the end points of integration, since the differential of ψ is exact.

Now, consider the **two-dimensional** flow of an **incompressible** fluid between two instantaneous streamlines, as shown in the Figure. The volumetric flow rate across areas AB, BC, DE, and DF must be equal, since there can be no flow across a streamline.



For a unit depth, the flow rate across AB is

$$Q = \int_{y_1}^{y_2} u dy = \int_{y_1}^{y_2} \frac{\partial \psi}{\partial y} dy$$

Along AB, x = *constant* and $d\psi = \frac{\partial \psi}{\partial y} dy$. Therefore,

$$Q = \int_{y_1}^{y_2} \frac{\partial \psi}{\partial y} dy = \int_{\psi_1}^{\psi_2} d\psi = \psi_2 - \psi_1$$

For a unit depth, the flow rate across BC is

$$Q = \int_{x_1}^{x_2} v dx = -\int_{x_1}^{x_2} \frac{\partial \psi}{\partial x} dx$$

Along BC, y = *constant* and $d\psi = \frac{\partial \psi}{\partial x} dx$. Therefore,

$$Q = -\int_{x_1}^{x_2} \frac{\partial \psi}{\partial x} dx = -\int_{\psi_2}^{\psi_1} d\psi = \psi_2 - \psi_1$$

Thus, the volumetric flow rate per unit depth between any two streamlines, can be expressed as the difference between constant values of ψ defining the two streamlines.

In $r\theta$ plane of the cylindrical coordinate system, the incompressible continuity equation reduces to

$$\frac{\partial r v_r}{\partial r} + \frac{\partial v_\theta}{\partial \theta} = 0$$

The stream function $\psi(r,\theta,t)$ then is defined such that

$$v_r \equiv \frac{1}{r}\frac{\partial \psi}{\partial \theta} \qquad \text{and} \qquad v_\theta \equiv -\frac{\partial \psi}{\partial r}$$

Example: Consider the stream function given by $\boldsymbol{\psi = xy}$. Find the corresponding velocity components and show that they satisfy the differential continuity equation. Then sketch a few streamlines and suggest any practical applications of the resulting flow field.

Given: $\psi = xy$

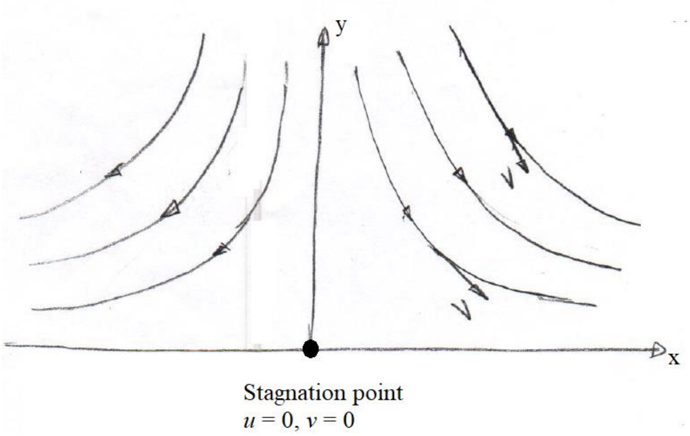
Find: u =?, v = ?, Do u and v satisfy continuity equation? Sketch few streamlines and suggest practical applications.

Assumptions: - Two-dimensional flow
- Incompressible flow
- Steady flow

$$u = \frac{\partial \psi}{\partial y} = x \qquad \text{and} \qquad v = -\frac{\partial \psi}{\partial x} = -y$$

The continuity equation for incompressible two-dimensional flow

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \Rightarrow \qquad 1 - 1 = 0$$



We could conclude that those streamlines might model the flow near the stagnation point on the nose of a blunt body. If we consider only the upper right quarter-plane, the streamlines might model flow in a 90° corner.

MOTION OF A FLUID ELEMENT (KINEMATICS)

Before formulating the effects of forces on fluid motion (dynamics), let us consider first the motion (kinematics) of a fluid in a flow field. When a fluid element moves in a flow field, it may under go **translation, linear deformation, rotation, and angular deformation** as a consequence of spatial variations in the velocity.

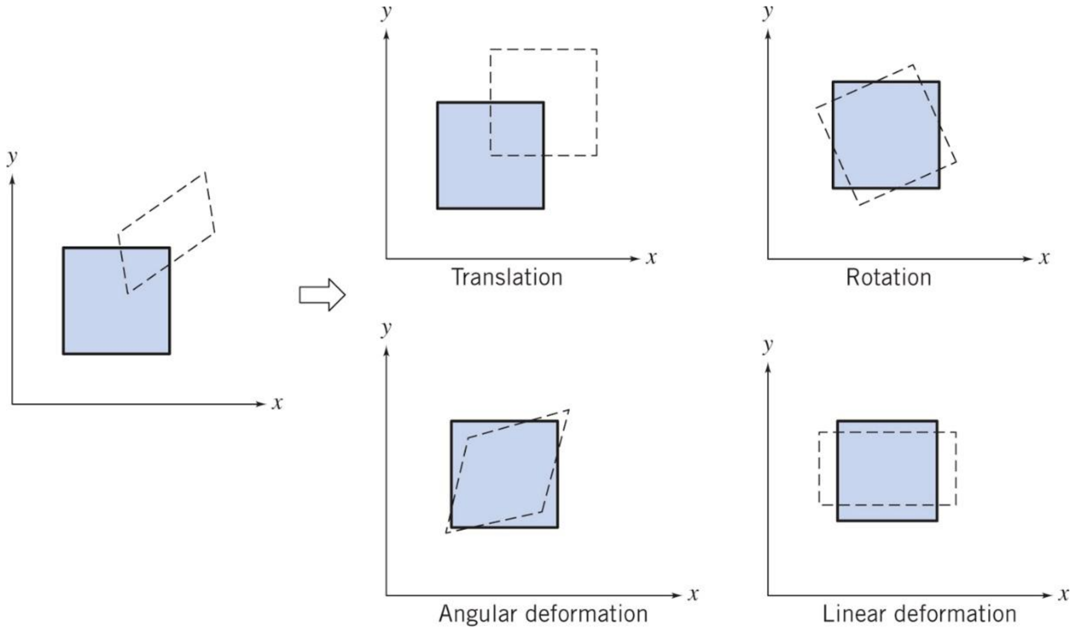


Figure. Pictorial representation of the components of fluid motion.

ACCELERATION OF A FLUID PARTICLE IN A VELOCITY FIELD

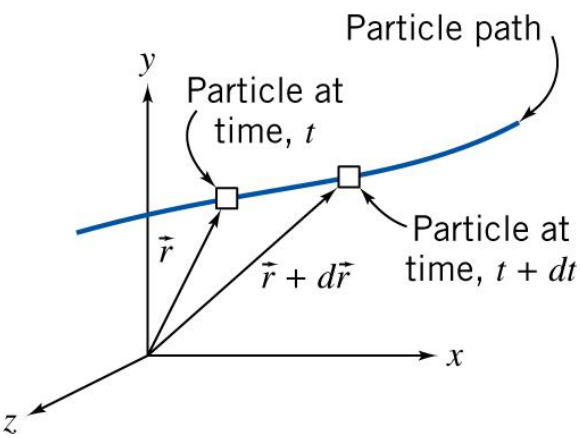


Figure. Motion of a particle in a flow field.

Consider a particle moving in a velocity field. At time t, the particle is at a position x, y, z and has a velocity $\vec{V}_p\Big]_t = \vec{V}(x, y, z, t)$.

At time *t+dt*, the particle has moved to a new position, with coordinates *x+dx, y+dy, z+dz*, and has a velocity given by $\vec{V}_p\Big]_{t+dt} = \vec{V}(x+dx, y+dy, z+dz, t+dt)$

The change in the velocity of the particle in moving from location $\vec{r}$ to $\vec{r} + d\vec{r}$, is given by

$$d\vec{V}_p = \frac{\partial \vec{V}}{\partial x} dx_p + \frac{\partial \vec{V}}{\partial y} dy_p + \frac{\partial \vec{V}}{\partial z} dz_p + \frac{\partial \vec{V}}{\partial t} dt$$

The total acceleration of the particle is given by

$$\vec{a}_p = \frac{d\vec{V}_p}{dt} = \frac{\partial \vec{V}}{\partial x}\frac{dx_p}{dt} + \frac{\partial \vec{V}}{\partial y}\frac{dy_p}{dt} + \frac{\partial \vec{V}}{\partial z}\frac{dz_p}{dt} + \frac{\partial \vec{V}}{\partial t}$$

since $\dfrac{dx_p}{dt} = u, \quad \dfrac{dy_p}{dt} = v \quad and \quad \dfrac{dz_p}{dt} = w$

then $$\vec{a}_p = \frac{d\vec{V}_p}{dt} = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z} + \frac{\partial \vec{V}}{\partial t}$$

Acceleration of a fluid particle in a velocity field requires a special derivative, it is given the symbol $\dfrac{D\vec{V}}{Dt}$.

Thus, $$\frac{D\vec{V}}{Dt} = \vec{a}_p = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z} + \frac{\partial \vec{V}}{\partial t}$$

It is called the **substantial**, the **material** or **particle derivative**.

The significance of the terms,

$$\vec{a}_p = \underset{\substack{\text{total}\\ \text{accelaration}\\ \text{of a particle}}}{\frac{D\vec{V}}{Dt}} = \underbrace{u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}}_{\substack{\text{convective}\\ \text{acceleration}}} + \underset{\substack{\text{local}\\ \text{acceleration}}}{\frac{\partial \vec{V}}{\partial t}}$$

The convective acceleration may be written as a single vector expression using the vector gradient operator, ∇.

$$u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z} = (\vec{V}\cdot\nabla)\vec{V}$$

Thus, $$\frac{D\vec{V}}{Dt} \equiv \vec{a}_p = (\vec{V}\cdot\nabla)\vec{V} + \frac{\partial \vec{V}}{\partial t}$$

It is possible to express above equation in terms of three scalar equations as

$$a_{x_p} = \frac{Du}{Dt} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}$$
$$a_{y_p} = \frac{Dv}{Dt} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t}$$
$$a_{z_p} = \frac{Dw}{Dt} = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial w}{\partial t}$$

The components of acceleration in cylindrical coordinates may be obtained by utilizing the appropriate expression for the vector operator ∇. Thus

$$a_{r_p} = V_r\frac{\partial V_r}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_r}{\partial \theta} - \frac{V_\theta^2}{r} + V_z\frac{\partial V_r}{\partial z} + \frac{\partial V_r}{\partial t}$$
$$a_{\theta_p} = V_r\frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{V_r V_\theta}{r} + V_z\frac{\partial V_\theta}{\partial z} + \frac{\partial V_\theta}{\partial t}$$
$$a_{z_p} = V_r\frac{\partial V_z}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_z}{\partial \theta} + V_z\frac{\partial V_z}{\partial z} + \frac{\partial V_z}{\partial t}$$

<u>Example:</u> The velocity field for a fluid flow is given by $\vec{V}(x,y,z,t) = x^2\vec{\imath} - 2xy\vec{\jmath} + 3zt\vec{k}$
Determine

a) the acceleration vector,
b) the acceleration of the fluid particle at point *P(1,2,3)* and at time *t = 1 sec*.

a) The components of the velocity vector are $u = x^2,\ v = -2xy$ and $w = 3zt$. The components of the acceleration vector

$$a_{x_p} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t} = x^2(2x) + (-2xy)(0) + (3zt)(0) + 0 = 2x^3$$
$$a_{y_p} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t} = x^2(-2y) + (-2xy)(-2x) + (3zt)(0) + 0 = 2x^2y$$
$$a_{zx_p} = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial w}{\partial t} = x^2(0) + (-2xy)(0) + (3zt)(3t) + 3z = 3z\left(1+3t^2\right)$$

Therefore,

$$\vec{a}_p = a_{x_p}\vec{\imath} + a_{y_p}\vec{\jmath} + a_{z_p}\vec{k} = 2x^3\vec{\imath} + 2x^2y\vec{\jmath} + 3z\left(1+3t^2\right)\vec{k}$$

b) The acceleration of the fluid particle at point *P(1,2,3)* and at time *t = 1 sec* is

$$\vec{a}_p = 2(1)^3\vec{\imath} + 2(1)^2(2)\vec{\jmath} + 3(3)\left(1+3(1)^2\right)\vec{k} = 2\vec{\imath} + 4\vec{\jmath} + 36\vec{k}$$

FLUID ROTATION

The **rotation**, $\vec{\omega}$, of a fluid particle is defined as the average angular velocity of any two mutually perpendicular line elements of the particle in each orthogonal plane. A particle may rotate about three coordinate axes. Thus, in general, $\vec{\omega} = \omega_x \vec{\imath} + \omega_y \vec{j} + \omega_z \vec{k}$

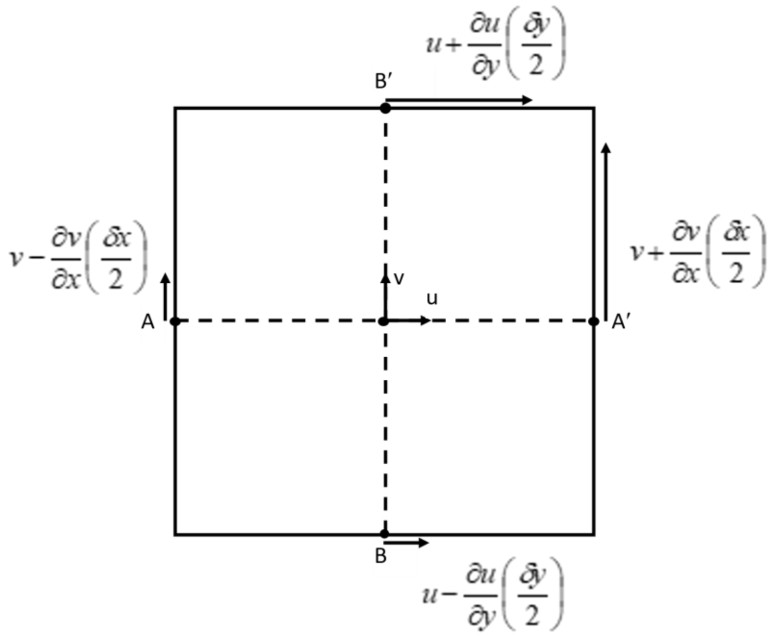


Figure. Rectangular fluid particle with two instantaneous perpendicular lines AA′ and BB′; velocities perpendicular to AA′and BB′ are also shown.

The figure shows a fluid particle with two lines AA′ and BB′. By definition

$$\omega_z \equiv \frac{1}{2}(\omega_{AA'} + \omega_{BB'})$$

where

$$\omega_{AA'} = \frac{v_{A'} - v_A}{\delta x} = \frac{\left[v + (\partial v / \partial x)(\delta x / 2)\right] - \left[v - (\partial v / \partial x)(\delta x / 2)\right]}{\delta x} = \frac{\partial v}{\partial x}$$

and

$$\omega_{BB'} = -\frac{u_{B'} - u_B}{\delta y} = -\frac{\left[u + (\partial u / \partial y)(\delta y / 2)\right] - \left[u + (\partial u / \partial y)(\delta y / 2)\right]}{\delta y} = -\frac{\partial u}{\partial y}$$

so

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

By considering the rotation of pairs perpendicular lines in the *yz* and *xz* planes, one can show that

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \text{and } \omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

then

$$\vec{\omega} = \omega_x \vec{i} + \omega_y \vec{j} + \omega_z \vec{k} = \frac{1}{2}\left[\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\vec{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\vec{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k}\right]$$

We recognize the term in the square brackets as

$$\text{curl}\,\vec{V} = \nabla \times \vec{V}$$

Then, in vector notation, we can write

$$\vec{\omega} = \frac{1}{2}\nabla \times \vec{V}$$

The factor of ½ can be eliminated in above equation by defining a quantity called the **vorticity**, $\vec{\zeta}$, to be twice the rotation,

$$\vec{\zeta} \equiv 2\vec{\omega} = \nabla \times \vec{V}$$

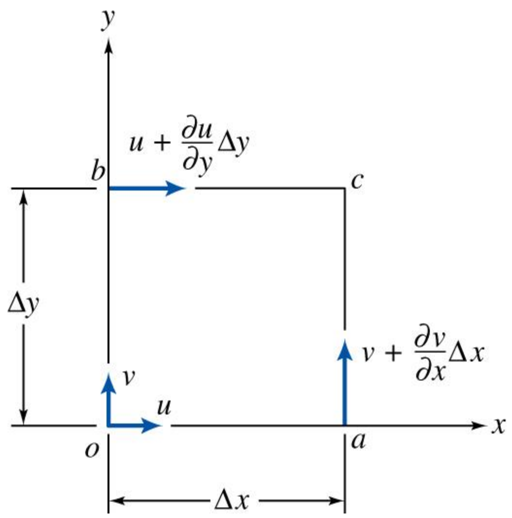
The vorticity is the measure of the rotation of a fluid element as it moves in the flow field. In cylindrical coordinates the vorticity is

$$\nabla \times \vec{V} == \left(\frac{1}{r}\frac{\partial V_z}{\partial \theta} - \frac{\partial V_\theta}{\partial z}\right)\vec{e}_r + \left(\frac{\partial V_r}{\partial z} - \frac{\partial V_z}{\partial r}\right)\vec{e}_\theta + \left(\frac{1}{r}\frac{\partial r V_\theta}{\partial r} - \frac{\partial V_r}{\partial \theta}\right)\vec{k}$$

The **circulation**, Γ, is defined as the line integral of the tangential velocity component about a closed curve fixed in the flow,

$$\Gamma = \oint_C \vec{V} \cdot d\vec{s}$$

where, $d\vec{s}$ is an elemental vector, of the length ds, tangent to curve; a positive sense corresponds to a counterclockwise path of integration around the curve.



For Oa line

$$\Gamma_{0a} = \int_{Oa} \left(u\vec{i} + v\vec{j}\right) \cdot dx\vec{i} = \int_{Oa} udx = u\Delta x$$

For the closed curve ***Oacb***,

$$\Delta\Gamma = u\Delta x + \left(v + \frac{\partial v}{\partial x}\right)\Delta y - \left(u + \frac{\partial u}{\partial y}\right)\Delta x - v\Delta y$$

$$\Delta\Gamma = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\Delta x \Delta y$$

$$\Delta\Gamma = 2\omega_z \Delta x \Delta y$$

$$\Gamma = \oint_C \vec{V} \cdot d\vec{s} = \int_A 2\omega_z dA = \int_A (\nabla \times \vec{V})_z dA$$

Thus, the circulation around a closed contour is the total vorticity enclosed within it.

<u>Example:</u> Consider flow fields with purely tangential motion (circular streamlines): $V_r = 0$ and $V_\theta = f(r)$. Evaluate the rotation, vorticity, and circulation for rigid-body rotation, "a forced vortex". Show that it is possible to choose *f(r)* so that the flow is irrotational; to produce "a free vortex".

Basic Equation: $\vec{\zeta} \equiv 2\vec{\omega} = \nabla \times \vec{V}$

For motion in $r\theta$ plane, the only components of rotation and vorticity are in the z-direction

$$\zeta_z = 2\omega_z = \frac{1}{r}\frac{\partial r V_\theta}{\partial r} - \frac{\partial V_r}{\partial \theta}$$

Since $V_r = 0$ $\qquad \zeta_z = 2\omega_z = \frac{1}{r}\frac{\partial r V_\theta}{\partial r}$

a) For rigid body rotation, $V_\theta = \omega r$

Then, $\omega_z = \frac{1}{2}\frac{1}{r}\frac{\partial r V_\theta}{\partial r} = \frac{1}{2}\frac{1}{r}\frac{\partial\left(\omega r^2\right)}{\partial r} = \frac{2\omega r}{2r} = \omega$ and $\zeta_z = 2\omega$

The circulation is $\Gamma = \oint_C \vec{V} \cdot d\vec{s} = \int_A 2\omega_z dA$

since $\omega_z = \omega = \text{constant}$

$\therefore \qquad \Gamma = 2\omega \int_A dA = 2\omega A$

b) For irrotational flow $\frac{1}{r}\frac{\partial\left(rV_\theta\right)}{\partial r} = 0$. Integrating,

$rV_\theta = \text{contant} = c \qquad$ or $\qquad V_\theta = f(r) = \frac{c}{r}$

For this flow, the origin is a singular point where $V_\theta \to \infty$.

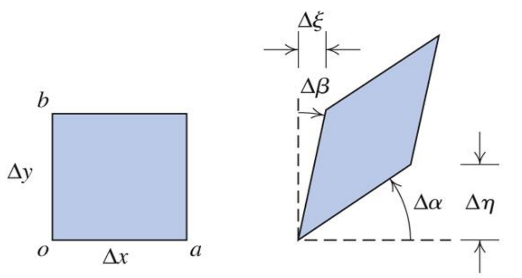
The circulation for any contour enclosing the origin is

$$\Gamma = \oint_C \vec{V} \cdot d\vec{s} = \int_0^{2\pi} \frac{c}{r} r\,d\theta = 2\pi c$$

The circulation around any contour not enclosing the singular point at the origin is zero.

FLUID DEFORMATION

Angular deformation of a fluid element involves changes in the angle between two mutually perpendicular lines in the fluid. The rate of angular momentum is given by



(a) Original particle (b) Particle after time Δt

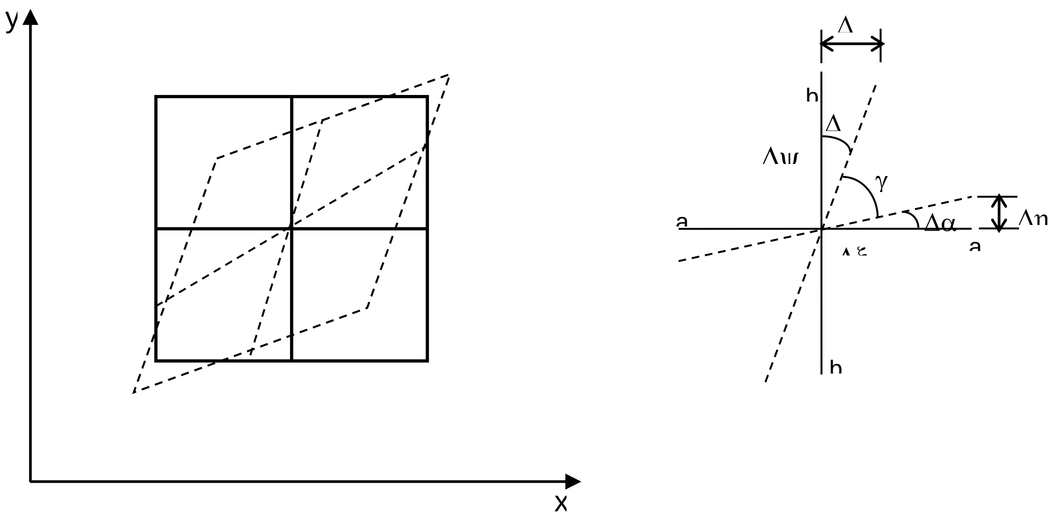


Figure. Angular deformation of a fluid element in a two dimensional flow field.

$$-\frac{d\gamma}{dt} = \frac{d\alpha}{dt} + \frac{d\beta}{dt}$$

Now,

$$\frac{d\alpha}{dt} = \lim_{\Delta t \to 0} \frac{\Delta\alpha}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta\eta / \Delta x}{\Delta t} = \lim_{\Delta t \to 0} \frac{(\partial v / \partial x)\Delta x \,\Delta t / \Delta x}{\Delta t} = \frac{\partial v}{\partial x}$$

and $$\frac{d\beta}{dt} = \lim_{\Delta t \to 0} \frac{\Delta\beta}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta\xi / \Delta y}{\Delta t} = \lim_{\Delta t \to 0} \frac{(\partial u / \partial y)\Delta y \, \Delta t / \Delta y}{\Delta t} = \frac{\partial u}{\partial y}$$

Consequently, the rate of the angular deformation in the *xy* plane is

$$\frac{d\alpha}{dt} + \frac{d\beta}{dt} = -\frac{d\gamma}{dt} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

The shear stress is related to the rate of angular deformation through the fluid viscosity.

MOMENTUM EQUATION

To derive the differential form of momentum equation, we shall apply Newton's second law to an infinitesimal fluid particle of mass *dm*.

Newton's second law for a finite system is given by

$$\vec{F} = \left.\frac{d\vec{P}}{dt}\right)_{system}$$

where the linear momentum, $\vec{P}$, of the system is given by

$$\vec{P}_{system} = \int_{mass\,(system)} \vec{V} dm$$

Then for an infinitesimal system of mass dm, Newton's second law is written

$$d\vec{F} = dm \left.\frac{d\vec{V}}{dt}\right)_{system} = dm \frac{d\vec{V}}{dt}$$

$$\therefore \qquad d\vec{F} = dm \left[u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z} + \frac{\partial \vec{V}}{\partial t} \right]$$

Forces Acting on a Fluid Particle

The forces acting on a fluid element may be classified as **body forces** and **surface forces**. Surface forces include both normal forces and tangential (shear) forces.

Stresses acting on a differential fluid element in the x-direction are shown in the figure.

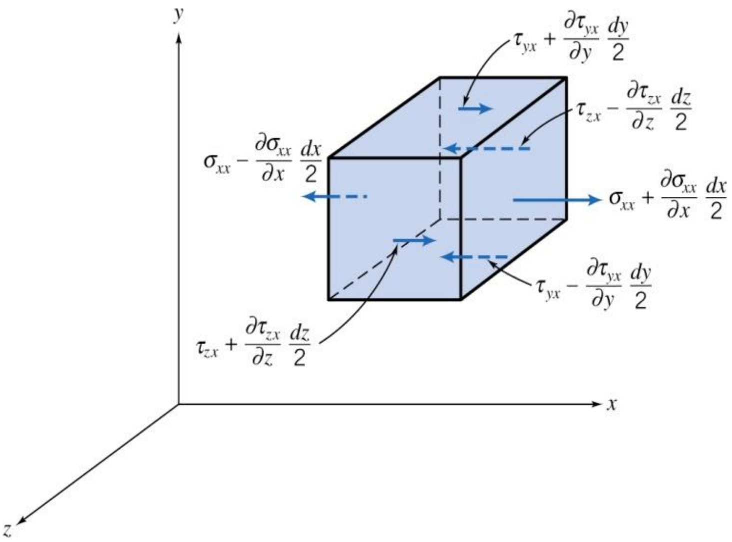


Figure. Stresses in the x direction on an element of fluid.

To obtain the net surface force in the *x* direction, dF_{S_x}, we must sum the forces in the *x* direction.

$$
\begin{aligned}
dF_{S_x} &= \left(\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x}\frac{dx}{2}\right)dydz - \left(\sigma_{xx} - \frac{\partial \sigma_{xx}}{\partial x}\frac{dx}{2}\right)dydz \\
&+ \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y}\frac{dy}{2}\right)dxdz - \left(\tau_{yx} - \frac{\partial \tau_{yx}}{\partial y}\frac{dy}{2}\right)dxdz \\
&+ \left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z}\frac{dz}{2}\right)dxdy - \left(\tau_{zx} - \frac{\partial \tau_{zx}}{\partial z}\frac{dz}{2}\right)dxdy
\end{aligned}
$$

By simplifying, we obtain

$$dF_{S_x} = \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right)dxdydz$$

When the force of gravity is the only body force acting, then the body force per unit mass in *x*-direction is given by $\rho g_x dxdydz$. Then the total net force in *x* direction can be expressed as

$$dF_x = dF_{B_x} + dF_{S_x} = \left(\rho g_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right)dxdydz$$

One can derive similar expressions for the force components in the *y* and *z* directions.

$$dF_y = dF_{B_y} + dF_{S_y} = \left(\rho g_y + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right)dxdydz$$

$$dF_z = dF_{B_z} + dF_{S_z} = \left(\rho g_z + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z}\right)dxdydz$$

Differential Momentum Equation

We have now formulated expressions for the components, dF_x, dF_y, and dF_z, of the force $d\vec{F}$, acting on the element of mass **dm**. If we substitute these expressions for the force components into *x*, *y*, and *z* components of equation, we obtain differential equations of motion.

$$\rho g_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)$$

$$\rho g_y + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right)$$

$$\rho g_z + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)$$

These three equations are the differential equations of motion for any fluid satisfying the continuum assumption. Before hte equations can be used to solve problems, suitable expressions for the stresses must be obtained in terms of the velocity and pressure fields.

Newtonian Fluid: Navier-Stokes Equations

For a Newtonian fluid the viscous stress is proportional to the rate of shearing strain (angular deformation rate). The stresses may be expressed in terms of velocity gradients and fluid properties in rectangular coordinates as follows:

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)$$

$$\tau_{zx} = \tau_{xz} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

$$\sigma_{xx} = -p - \frac{2}{3}\mu\nabla\cdot\vec{V} + 2\mu\frac{\partial u}{\partial x}$$

$$\sigma_{yy} = -p - \frac{2}{3}\mu\nabla\cdot\vec{V} + 2\mu\frac{\partial v}{\partial y}$$

$$\sigma_{zz} = -p - \frac{2}{3}\mu\nabla\cdot\vec{V} + 2\mu\frac{\partial w}{\partial z}$$

where *p* is the local thermodynamic pressure.

If these expressions are introduced into the differential equations of motion, we obtain

$$\rho\frac{Du}{Dt}=\rho g_x-\frac{\partial p}{\partial x}+\frac{\partial}{\partial x}\left[\mu\left(2\frac{\partial u}{\partial x}-\frac{2}{3}\nabla\cdot\vec{V}\right)\right]+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right]$$

$$\rho\frac{Dv}{Dt}=\rho g_y-\frac{\partial p}{\partial y}+\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]+\frac{\partial}{\partial y}\left[\mu\left(2\frac{\partial v}{\partial y}-\frac{2}{3}\nabla\cdot\vec{V}\right)\right]+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right]$$

$$\rho\frac{Dw}{Dt}=\rho g_z-\frac{\partial p}{\partial z}+\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)\right]+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right]+\frac{\partial}{\partial x}\left[\mu\left(2\frac{\partial w}{\partial z}-\frac{2}{3}\nabla\cdot\vec{V}\right)\right]$$

These equations of motion are called **the Navier-Stokes equations**. The equations are greatly simplified when applied **to incompressible flow with constant viscosity**. Under these conditions the equations reduce to

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=\rho g_x-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=\rho g_y-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=\rho g_z-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)$$

The Navier-Stokes equations in cylindrical coordinates, for constant density and viscosity, are given in the course textbook.

For the case of **frictionless flow (μ = 0)** the equations of motion reduce to Euler's equation,

$$\rho\frac{D\vec{V}}{Dt}=\rho\vec{g}-\nabla p$$

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=\rho g_x-\frac{\partial p}{\partial x}$$

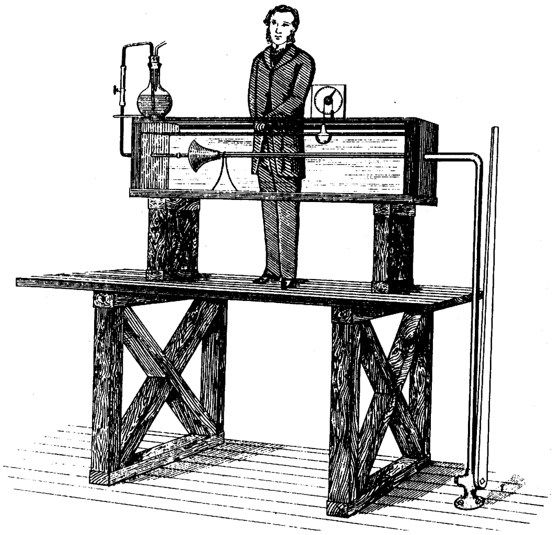
$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=\rho g_y-\frac{\partial p}{\partial y}$$

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=\rho g_z-\frac{\partial p}{\partial z}$$

INTERNAL INCOMPRESSIBLE FLOW

Flows completely bounded by solid surfaces are called internal flows. Internal flows may be laminar or turbulent. Some laminar flow cases may be solved analytically. In the case of turbulent flow, analytical solutions are not possible and we must rely heavily on semi-empirical theories or experimental data.

One can demonstrate the qualitative difference between the nature of laminar and turbulent flow by classical Reynolds experiment. The experimental set up consists of a constant diameter transparent pipe which is connected to a larger reservoir of water. A thin filament of dye, which is injected at the centerline of the pipe, allows visual observation of the flow.



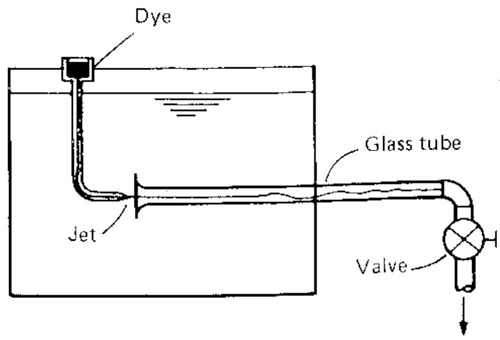


Figure. Set up for the Reynolds' experiment.

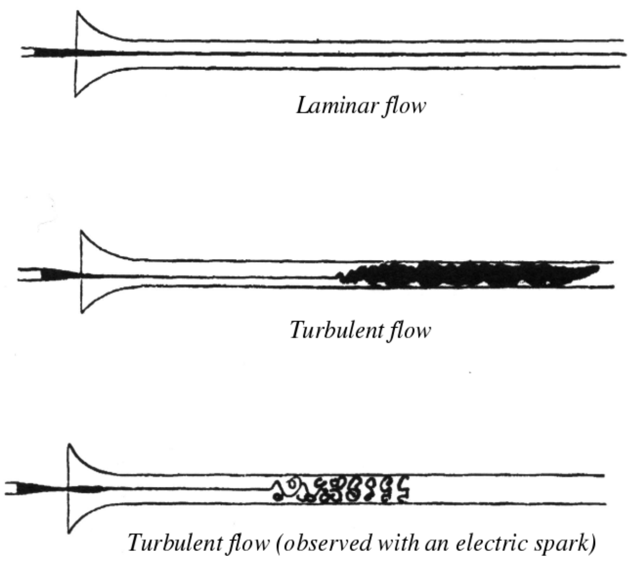
Conducting Reynolds' experiment, two points should be kept in mind.

a) The density and viscosity of the dye and water must be the same,
b) The water and dye levels in both reservoirs must be the same.

At the low flow rates, the dye injected into the flow remains in single filament; there is little dispersion of dye because the flow is laminar. A laminar flow is one in which the fluid flow in laminae, or layers.

As the flow rate through the tube is increased, the dye filament becomes wavy. This is known as transient flow.

At high flow rates, the dye filament becomes unstable and breaks up into a random motion. This behavior of turbulent flow is due to small, high-frequency velocity fluctuations superimposed on the mean motion of turbulent flow.



Under normal engineering applications, the transition from the laminar flow to turbulent flow in pipes occurs at Reynolds numbers of 2000 to 3000. However, in carefully controlled experiments, it is possible to obtain laminar flow up to a Reynolds number of 60000. Usually, the critical Reynolds number is taken to be 2300.

$$\mathrm{Re} = \frac{Vd}{\upsilon} = \frac{\rho Vd}{\mu}$$

Developing a Fully Developed Flow

Consider the flow of an incompressible fluid through a long pipe of constant diameter. At the entrance of the pipe has not been subjected to the action of viscosity, so that the velocity profile is constant.

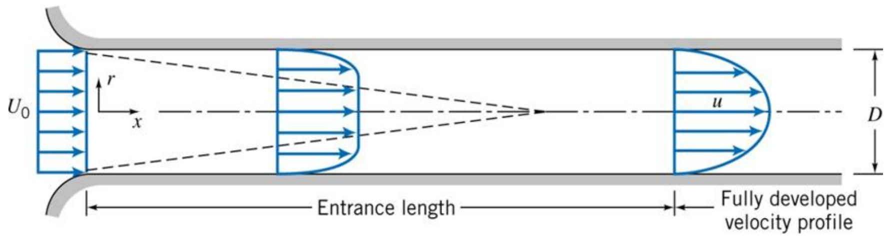


Figure. Development of viscous laminar flow in a pipe.

As soon as the fluid comes in contact with circumference of pipe, its velocity reduces to zero, and it satisfies **no-slip condition**. A boundary layer develops along the walls of the channel. The solid surface exerts a retarding force on the flow, thus the speed of the fluid in the neighborhood of the surface is reduced. Sufficiently far from the pipe entrance, the boundary layer reaches the pipe centerline and the flow becomes entirely viscous. After this point, the velocity profile will no longer change with the distance along the pipe. This region is known as **the fully developed region**.

For laminar flow, **the entrance length, L**, is a function of Reynolds number

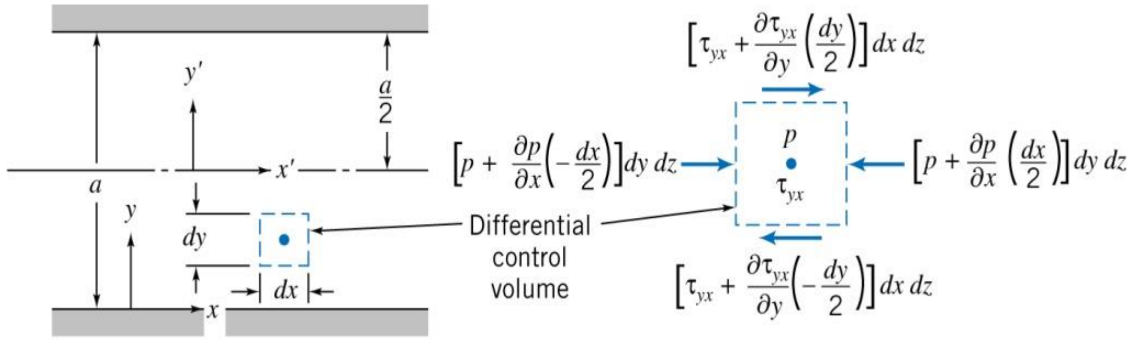
$$\frac{L}{D} \approx 0.06 \frac{\rho \bar{V} d}{\mu}$$

For turbulent flow, **the entrance length, L, is about 25 to 40 pipe diameters**.

FULLY DEVELOPED LAMINAR FLOW

FULLY DEVELOPED LAMINAR FLOW BETWEEN INFINITE PARALLEL PLATES

Let us consider the fully developed laminar flow between infinite parallel plates.



(a) Geometry of CV

(b) Forces acting on CV

Assumptions:
1. Steady flow
2. Fully developed flow ($\frac{\partial}{\partial x} = 0$)
3. Incompressible flow
4. Plates are infinite in the z direction ($w = 0,\ \frac{\partial}{\partial z} = 0$)
5. Body forces in x direction is negligible ($F_{B_x} = 0$)

Find:
a) Velocity profile
b) Shear stress distribution
c) Volume flow rate
d) Average velocity
e) Point of maximum velocity

a) Velocity profile:

Velocity distribution can be found by applying integral momentum equations or differential momentum equations.

For our analysis we select a differential control volume of size, and apply x component of momentum equation

$$\underbrace{F_{S_x}+F_{B_x}}_{0} = \underbrace{\frac{\partial}{\partial t}\int_{CV} u\rho d\forall}_{0} + \int_{CS} u\rho\vec{V}\cdot d\vec{A}$$

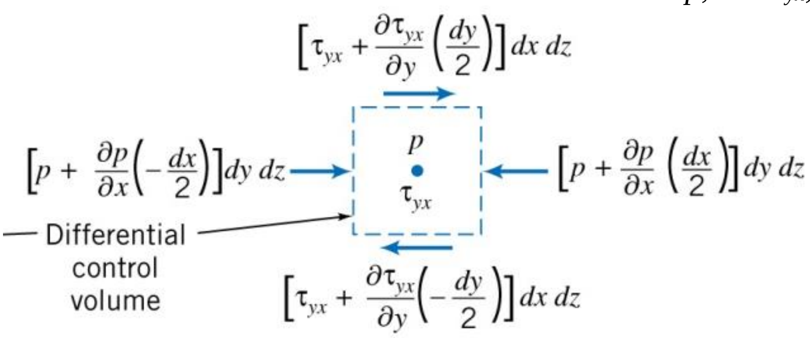
For fully developed flow, the net momentum flux through the control surface is zero. (The momentum flux through the right face of the control surface is equal in magnitude but opposite in sign to the momentum flux through the left face.)

$$\therefore \qquad F_{S_x}=0$$

There are two types of forces which act to the surface of control volume. Those are:

1. Pressure forces (normal forces)
2. Shear forces (tangential forces)

If the pressure and the shear stress at the center of fluid element are *p*, and τ_{yx}, respectively.



(*b*) Forces acting on CV

$$\left(p-\frac{\partial p}{\partial x}\frac{dx}{2}\right)dydz-\left(p+\frac{\partial p}{\partial x}\frac{dx}{2}\right)dydz+\left(\tau_{yx}+\frac{\partial\tau_{yx}}{\partial y}\frac{dy}{2}\right)dxdz-\left(\tau_{yx}-\frac{\partial\tau_{yx}}{\partial y}\frac{dy}{2}\right)dxdz=0$$

By arranging,

$$-\frac{\partial p}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}=0$$ For τ_{yx}, we used the total derivative, since τ_{yx} is only function of *y*.

[$u = u(y)$]

or

$$\frac{\partial\tau_{yx}}{\partial y}=\frac{\partial p}{\partial x}$$

The left hand side of this equation is the function of y, but, the right hand side of the equation is the function of x. Therefore, in order to be this equation valid, it should be equal to a constant.

$$\frac{\partial \tau_{yx}}{\partial y} = \frac{\partial p}{\partial x} = \text{constant}$$

Integrating this equation, we obtain

$$\tau_{yx} = \frac{\partial p}{\partial x} y + C_1$$

which indicates that the shear stress varies linearly with y. Since for Newtonian fluid

$$\tau_{yx} = \mu \frac{du}{dy}$$

then

$$\mu \frac{du}{dy} = \frac{\partial p}{\partial x} y + C_1$$

and

$$u = \frac{1}{2\mu} \frac{\partial p}{\partial x} y^2 + \frac{C_1}{\mu} y + C_2$$

To evaluate constants C_1 and C_2, we must apply the boundary conditions.

at $y = 0$ $\quad u = 0 \quad$ consequently $C_2 = 0$

at $y = a$ $\quad u = 0 \quad$ $\Rightarrow 0 = \frac{1}{2\mu} \frac{\partial p}{\partial x} a^2 + \frac{C_1}{\mu} a$

$$\therefore \; C_1 = -\frac{1}{2} \frac{\partial p}{\partial x} a$$

and hence

$$u = \frac{1}{2}\left(\frac{\partial p}{\partial x}\right) y^2 - \frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right) ay$$

or

$$u = \frac{a^2}{2\mu} \frac{\partial p}{\partial x}\left[\left(\frac{y}{a}\right)^2 - \left(\frac{y}{a}\right)\right]$$

b) Shear Stress Distribution:

The shear stress distribution is given by

$$\tau_{yx} = \mu \frac{du}{dy} = a\left(\frac{\partial p}{\partial x}\right)\left[\frac{y}{a} - \frac{1}{2}\right]$$

c) Volume Flow Rate:

The volume flow rate is given by $Q = \int_A \vec{V} \cdot d\vec{A}$

For a depth l in the z direction $Q = \int_0^a u l dy$

or $$\frac{Q}{l} = \int_0^a \frac{1}{2\mu}\frac{\partial p}{\partial x}\left(y^2 - ay\right)dy$$

Thus, the volume flow rate per unit depth l is given by
$$\frac{Q}{l} = -\frac{1}{12\mu}\left(\frac{\partial p}{\partial x}\right)a^3$$

Flow rate as a function of Pressure Drop

Since $\frac{\partial p}{\partial x}$ is constant, the pressure varies linearly with x,

$$\frac{\partial p}{\partial x} = \frac{p_2 - p_1}{L} = \frac{\Delta P}{L}$$

Substituting into the expression for volume flow rate gives

$$\frac{Q}{l} = -\frac{1}{12\mu}\left(-\frac{\Delta P}{L}\right)a^3 = \frac{a^3 \Delta P}{12\mu L}$$

d) Average Velocity:
The average velocity is given by

$$\bar{V} = \frac{Q}{A} = -\frac{1}{12\mu}\left(\frac{\partial p}{\partial x}\right)\frac{a^3 l}{la} = -\frac{1}{12\mu}\left(\frac{\partial p}{\partial x}\right)a^2$$

e) Point of Maximum Velocity:

To find the point of maximum velocity, we set $\frac{du}{dy}$ equal to zero and solve for corresponding y.

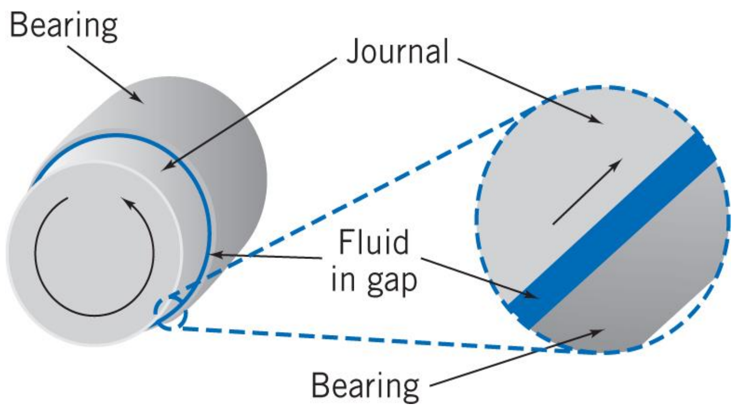
$$\frac{du}{dy} = \frac{a^2}{12\mu}\left(\frac{\partial p}{\partial x}\right)\left[\frac{2y}{a^2} - \frac{1}{a}\right]$$

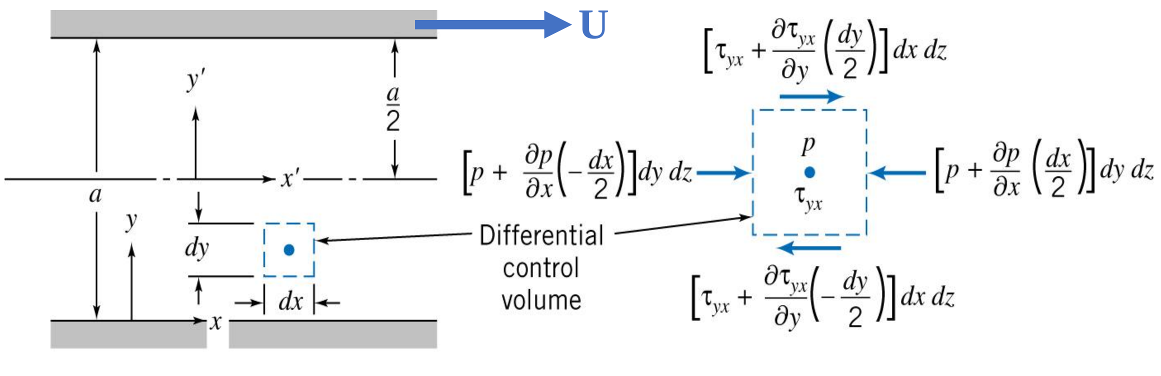
$$\frac{du}{dy} = 0 \quad \text{at} \quad y = \frac{a}{2}$$

$$\therefore \text{ at } \quad y = \frac{a}{2} \qquad u = u_{max} = -\frac{1}{8\mu}\left(\frac{\partial p}{\partial x}\right)a^2 = \frac{3}{2}\bar{V}$$

UPPER PLATE MOVING WITH CONSTANT SPEED, U

Second laminar flow case of practical importance is flow in a journal bearing. In such a bearing, an inner cylinder, the journal rotates inside a stationary member. It can be considered as flow between infinite parallel plates.





(a) Geometry of CV (b) Forces acting on CV

Assumptions:

1. Steady flow
2. Fully developed flow (i.e. $\frac{\partial}{\partial x} = 0$)
3. Laminar flow
4. Incompressible flow
5. Plates are infinite in the z direction ($w = 0,\ \frac{\partial}{\partial z} = 0$)
6. Body forces in x direction is negligible

Find:

a) Velocity distribution
b) Shear stress distribution
c) Volume flow rate
d) Average velocity
e) Point of maximum velocity

Boundary Conditions:

$u = 0$ at $y = 0$

$u = U$ at $y = a$

a) Velocity profile:

Since only the boundary conditions have changed, thus the velocity distribution is given by

$$u = \frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right)y^2 + \frac{C_1}{\mu}y + C_2$$

Integral constants C_1 and C_2 can be found by using boundary conditions

at $y = 0$ $\quad u = 0$ consequently $C_2 = 0$

at $y = a$ $\quad u = U$ $\quad \Rightarrow U = \frac{1}{2}\left(\frac{\partial p}{\partial x}\right)a^2 + \frac{C_1}{\mu}a$

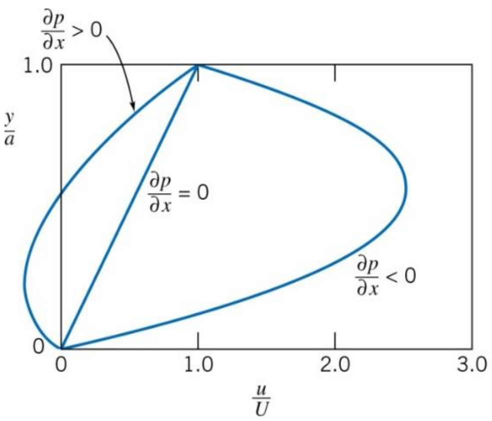
Thus, $C_1 = \frac{U\mu}{a} - \frac{1}{2}\left(\frac{\partial p}{\partial x}\right)a$

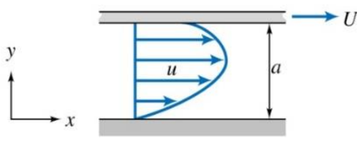
and

$$u = \frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right)y^2 + \frac{Uy}{a} - \frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right)ay$$

$$= \frac{Uy}{a} + \frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right)\left(y^2 - ay\right)$$

$$u = \frac{Uy}{a} + \frac{a^2}{2\mu}\frac{\partial p}{\partial x}\left[\left(\frac{y}{a}\right)^2 - \left(\frac{y}{a}\right)\right]$$





For the various values of $\left(\frac{\partial p}{\partial x}\right)$, the dimensionless velocity profile is plotted in the figure.

b) Shear Stress Distribution:
The shear stress distribution is given by

$$\tau_{yx} = \mu \frac{du}{dy} = \mu \frac{U}{a} + \frac{a^2}{2}\left(\frac{\partial p}{\partial x}\right)\left[\frac{2y}{a^2} - \frac{1}{a}\right] = \mu \frac{U}{a} + a\left(\frac{\partial p}{\partial x}\right)\left[\frac{y}{a} - \frac{1}{2}\right]$$

c) Volume Flow Rate:
The volume flow rate is given by $Q = \int_A \vec{V} \cdot d\vec{A}$. For a depth l in the z direction

$$Q = \int_0^a uldy \Rightarrow \frac{Q}{l} = \int_0^a \left[\frac{Uy}{a} + \frac{1}{2\mu}\frac{\partial p}{\partial x}\left(y^2 - ay\right)\right]dy$$

The volume flow rate per unit depth l is

$$\frac{Q}{l} = \frac{Ua}{2} - \frac{1}{12\mu}\left(\frac{\partial p}{\partial x}\right)a^3$$

d) Average Velocity:
The average velocity, $\bar{V}$, is given by

$$\bar{V} = \frac{Q}{A} = \frac{l\left[\frac{Ua}{2} - \frac{1}{12\mu}\left(\frac{\partial p}{\partial x}\right)a\right]^3}{la} = \frac{U}{2} - \frac{1}{12\mu}\left(\frac{\partial p}{\partial x}\right)a^2$$

e) Point of Maximum Velocity:
To find the point of maximum velocity, we set $\frac{du}{dy}$ equal to zero and solve for corresponding y.

$$\frac{du}{dy} = \frac{U}{a} + \frac{a^2}{2\mu}\left(\frac{\partial p}{\partial x}\right)\left[\frac{2y}{a^2} - \frac{1}{a}\right] = 0 \Rightarrow y = \frac{a}{2} - \frac{U/a}{\frac{1}{\mu}\left(\frac{\partial p}{\partial x}\right)}$$

<u>II. Method:</u> By using Differential Momentum Equation (Navier-Stokes equation), to find velocity distribution.

x - component of the momentum equation:

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = \rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

by utilizing assumptions, above equation simplifies to

$$\rho\left(\underbrace{\frac{\partial u}{\partial t}}_{0}+\underbrace{u\frac{\partial u}{\partial x}}_{0}+\underbrace{v}_{0}\frac{\partial u}{\partial y}+\underbrace{w}_{0}\frac{\partial u}{\partial z}\right)=\rho\underbrace{g_x}_{0}-\frac{\partial p}{\partial x}+\mu\left(\underbrace{\frac{\partial^2 u}{\partial x^2}}_{0}+\frac{\partial^2 u}{\partial y^2}+\underbrace{\frac{\partial^2 u}{\partial z^2}}_{0}\right)$$

$$0=-\frac{\partial p}{\partial x}+\mu\frac{\partial^2 u}{\partial y^2} \qquad \text{since } u=u(y)\Rightarrow\frac{\partial^2 u}{\partial y^2}=\frac{d^2 u}{dy^2}$$

or

$$\frac{d^2 u}{dy^2}=\frac{1}{\mu}\frac{\partial p}{\partial x}$$

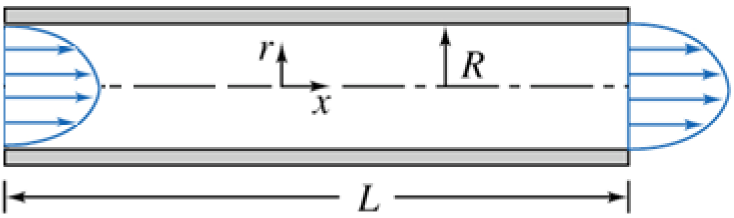
Integrating twice we find that:

$$u=\frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right)y^2+\frac{C_1}{\mu}y+C_2$$

Integral constants C_1 and C_2 can be found by using boundary conditions and so on …

FULLY DEVELOPED LAMINAR FLOW IN A PIPE

Let us consider fully developed laminar flow in a pipe. Here the flow is axisymmetric. Consequently, it is the most convenient to work in cylindrical coordinates. The control volume will be chosen a differential annulus.



Assumptions:

1. Fully developed flow ($\frac{\partial}{\partial x}=0$)
2. Steady flow
3. Laminar flow
4. Incompressible flow
5. There is no property change in θ - direction.
6. Radial velocity component is zero.
6. Neglect body forces

Find:

a) Velocity distribution
b) Shear stress distribution
c) Volume flow rate
d) Average velocity

e) Point of maximum velocity

Boundary conditions

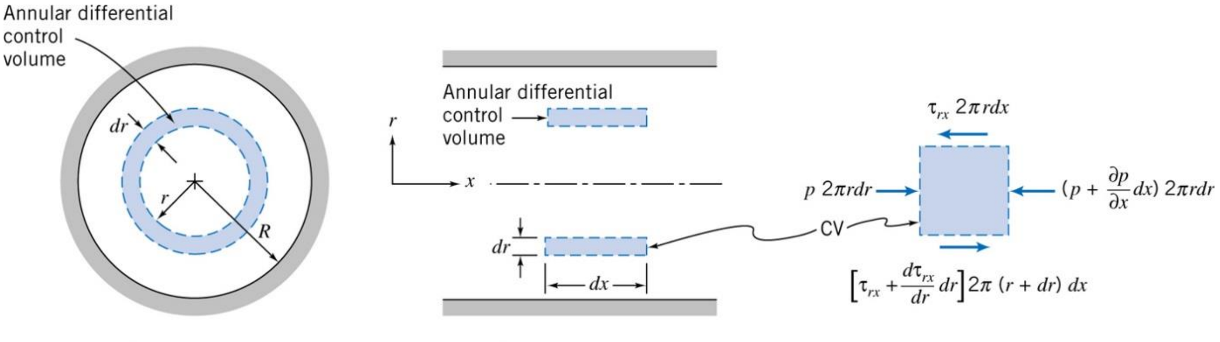
at $r = 0$ the velocity **must be finite** *(from physical consideration)*

at $r = R$ $u = 0$ *(no slip condition)*

a) Velocity profile:

Velocity distribution can be found by using the integral or differential form of the momentum equation. We will find the velocity distribution by using both methods.

If we apply the x - component of momentum equation for the control volume shown in the figure.



(*a*) End view of CV (*b*) Side view of CV (*c*) Forces on CV

For fully developed flow, the net momentum flux through the control surface is zero.

The normal (pressure) force and the tangential (shear) forces act to the control volume. The surface forces acting on the differential fluid element in x direction are

$$F_{S_x} + \underset{0}{F_{B_x}} = \underbrace{\frac{\partial}{\partial t}\int_{CV} u\rho d\forall}_{0} + \underbrace{\int_{CS} u\rho \vec{V}\cdot d\vec{A}}_{0}$$

For fully developed flow, the net momentum flux through the control surface is zero.

$$\therefore \qquad F_{S_x} = 0$$

The normal (pressure) force and the tangential (shear) forces act to the control volume. The surface forces acting on the differential fluid element in x direction are

$$+\left(p - \frac{\partial p}{\partial x}\frac{dx}{2}\right)2\pi r dr - \left(p + \frac{\partial p}{\partial x}\frac{dx}{2}\right)2\pi r dr + \left(\tau_{rx} + \frac{d\tau_{rx}}{dr}\frac{dr}{2}\right)2\pi\left(r + \frac{dr}{2}\right)dx$$
$$-\left(\tau_{rx} - \frac{d\tau_{rx}}{dr}\frac{dr}{2}\right)2\pi\left(r - \frac{dr}{2}\right)dx = 0$$

By simplifying

$$-\frac{\partial p}{\partial x}2\pi r dr dx + \tau_{rx}2\pi r dr dx + \frac{d\tau_{rx}}{dr}2\pi r dr dx = 0$$

Dividing this equation by $2\pi rdrdx$, and solving for $\frac{\partial p}{\partial x}$ gives

$$\frac{\partial p}{\partial x} = \frac{\tau_{rx}}{r} + \frac{d\tau_{rx}}{dr} = \frac{1}{r}\frac{d\left(r\tau_{rx}\right)}{dr}$$

The left hand side of the equation is only the function of *x*, but the right hand side of the equation is only the function of *r*. Then this equation holds only if each side of the equation is constant.

$$\frac{1}{r}\frac{d\left(r\tau_{rx}\right)}{dr} = \frac{\partial p}{\partial x} = \text{constant}$$

or

$$\frac{d\left(r\tau_{rx}\right)}{dr} = r\frac{\partial p}{\partial x}$$

Integrating this equation, we obtain

$$r\tau_{rx} = \frac{r^2}{2}\frac{\partial p}{\partial x} + C_1$$

or

$$\tau_{rx} = \frac{r}{2}\frac{\partial p}{\partial x} + \frac{C_1}{r}$$

Since $\tau_{rx} = \mu\frac{du}{dr}$

then

$$\mu\frac{du}{dr} = \frac{r}{2}\frac{\partial p}{\partial x} + \frac{C_1}{r}$$

and

$$u = \frac{r^2}{4\mu}\left(\frac{\partial p}{\partial x}\right) + \frac{C_1}{\mu}\ln r + C_2$$

By using the boundary conditions, integral constant C_1 and C_2 can be found.

<u>Boundary conditions</u>

From the first boundary condition (at $r = 0$ the velocity **must be finite)**

$$C_1 = 0$$

From the second boundary condition at ($r = R \quad u = 0$)

$$0=\frac{R^2}{4\mu}\left(\frac{\partial p}{\partial x}\right)+C_2$$

$$C_2=-\frac{R^2}{4\mu}\left(\frac{\partial p}{\partial x}\right)$$

and hence

$$u=\frac{r^2}{4\mu}\left(\frac{\partial p}{\partial x}\right)-\frac{R^2}{4\mu}\left(\frac{\partial p}{\partial x}\right)=\frac{1}{4\mu}\left(\frac{\partial p}{\partial x}\right)\left(r^2-R^2\right)$$

or

$$u=-\frac{R^2}{4\mu}\left(\frac{\partial p}{\partial x}\right)\left[1-\left(\frac{r}{R}\right)^2\right]$$

II. Method: By using the differential form of momentum equation in *x*-direction.

$$\rho\left(\underbrace{\frac{\partial u_z}{\partial t}}_{0}+\underbrace{u_r}_{0}\frac{\partial u_z}{\partial r}+\underbrace{\frac{u_\theta}{r}}_{0}\frac{\partial u_z}{\partial \theta}+u_z\underbrace{\frac{\partial u_z}{\partial z}}_{0}\right)=-\frac{\partial p}{\partial z}+\mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)+\underbrace{\frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2}}_{0}+\underbrace{\frac{\partial^2 u_z}{\partial z^2}}_{0}\right)$$

Note: By replacing $x\rightarrow z$ and $u_z\rightarrow u$, and simplifying the above differential equation

$$-\frac{\partial p}{\partial x}+\mu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)=0$$

or

$$\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)=\frac{r}{\mu}\frac{\partial p}{\partial x}$$

By integrating twice,

$$u=\frac{r^2}{4\mu}\left(\frac{\partial p}{\partial x}\right)+\frac{C_1}{\mu}\ln r+C_2$$

This equation is the same as the equation found by using integral momentum equation.

b) Shear Stress Distribution:

The shear stress is given by

$$\tau_{rx}=\mu\frac{du}{dr}=\frac{r}{2}\frac{\partial p}{\partial x}$$

c) Volume Flow Rate:

$$Q = \int_A \vec{V} \cdot d\vec{A} = \int_0^R u 2\pi r dr = \int_0^R \frac{1}{4\mu}\left(\frac{\partial p}{\partial x}\right)\left(r^2 - R^2\right) 2\pi r dr$$

$$Q = -\frac{\pi R^4}{8\mu}\left(\frac{\partial p}{\partial x}\right)$$

For fully developed flow $\frac{\partial p}{\partial x} = \text{constant} \Rightarrow \frac{\partial p}{\partial x} = \frac{p_2 - p_1}{L} = -\frac{\Delta p}{L}$

$$Q = -\frac{\pi R^4}{8\mu}\left[-\frac{\Delta p}{L}\right] = \frac{\pi \Delta p R^4}{8\mu L} = \frac{\pi \Delta p D^4}{128\mu L}$$ for laminar flow in a horizontal pipe.

d) Average Velocity:

The average velocity, $\bar{V}$, is given by

$$\bar{V} = \frac{Q}{A} = \frac{Q}{\pi R^2} = -\frac{R^2}{8\mu}\left(\frac{\partial p}{\partial x}\right)$$

e) Point of Maximum Velocity:

To find the maximum point of velocity, we set $\frac{du}{dr}$ equal to zero and solve for corresponding *r*,

$$\frac{du}{dr} = \frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right) r = 0$$

At *r = 0*, $u = u_{\max} = U = -\frac{R^2}{4\mu}\left(\frac{\partial p}{\partial x}\right) = 2\bar{V}$

FULLY DEVELOPED TURBULENT FLOW

In turbulent flows, there is no universally acceptable relation between shear stress and velocity gradients. Therefore, the analytical solutions of turbulent flow problems are impossible, we must rely on semi-empirical data.

INCOMPRESSIBLE INVISCID FLOW

MOMENTUM EQUATION FOR FRICTIONLESS FLOW: EULER'S EQUATIONS

The equations of motion for frictionless flow are called Euler's equations. These equations can be obtained from Navier-Stokes equations (by setting $\mu = 0$).

$$\rho g_x - \frac{\partial p}{\partial x} = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)$$
$$\rho g_y - \frac{\partial p}{\partial y} = \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right)$$
$$\rho g_z - \frac{\partial p}{\partial z} = \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)$$

We can also write the above equations as a single vector equation

$$\rho\frac{D\vec{V}}{Dt} = \rho\vec{g} - \nabla p = \rho\left(\frac{\partial \vec{V}}{\partial t} + u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}\right)$$

or

$$\rho\frac{D\vec{V}}{Dt} = \rho\vec{g} - \nabla p$$

If the z coordinate is directed vertically upward, then since, $\nabla z = \hat{k}$,

$$\rho\vec{g} = -\rho g\hat{k} = -\rho g\nabla z$$

In cylindrical coordinates, Euler equations in the component form, with gravity the only body force, are

$$g_r - \frac{1}{\rho}\frac{\partial p}{\partial r} = a_r = \frac{\partial V_r}{\partial t} + V_r\frac{\partial V_r}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_r}{\partial \theta} + V_z\frac{\partial V_r}{\partial z} - \frac{V_\theta^2}{r}$$
$$g_\theta - \frac{1}{\rho r}\frac{\partial p}{\partial \theta} = a_\theta = \frac{\partial V_\theta}{\partial t} + V_r\frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_\theta}{\partial \theta} + V_z\frac{\partial V_\theta}{\partial z} + \frac{V_r V_\theta}{r}$$
$$g_z - \frac{1}{\rho}\frac{\partial p}{\partial z} = a_z = \frac{\partial V_z}{\partial t} + V_r\frac{\partial V_z}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_z}{\partial \theta} + V_z\frac{\partial V_z}{\partial z}$$

If the *z axis* is directed vertically upward, then $g_r = g_\theta = 0$ and $g_z = -g$.

EULER'S EQUATION IN STREAMLINE COORDINATES

In this section, the Euler's equation will be first derived in the streamline coordinates, and then integrated along a streamline.

For this reason, consider an infinitesimal fluid element, which is moving along an instantaneous streamline, as shown in the figure. For simplicity, consider the flow in *yz* plane. Since velocity vector must be tangent to the streamline, the velocity field is given by $\vec{V} = \vec{V}(s,t)$.

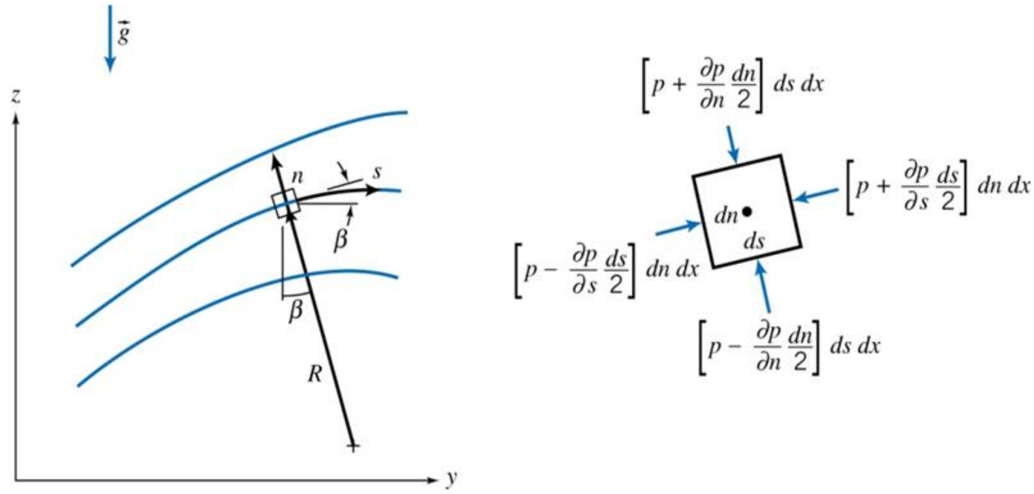


Figure. Fluid particle moving along a streamline.

If we apply Newton's second law of motion in the *streamwise (the s-) direction* to the fluid element of volume ***dsdndx***, then neglecting viscous forces we obtain

$$\left(p-\frac{\partial p}{\partial s}\frac{ds}{2}\right)dndx-\left(p+\frac{\partial p}{\partial s}\frac{ds}{2}\right)dndx-\rho g\sin\beta dsdndx=\rho a_s dsdndx$$

where "$\mathbf{a_s}$" is the acceleration of the fluid particle along the streamline. Simplifying the equation,

$$-\frac{\partial p}{\partial s}-\rho g\sin\beta=\rho a_s$$

since $\sin\beta=\dfrac{\partial z}{\partial s}$, we can write $-\dfrac{1}{\rho}\dfrac{\partial p}{\partial s}-g\dfrac{\partial z}{\partial s}=a_s$

Along any streamline *V=V(s,t)*, then the total acceleration in s-direction

$$a_s=\frac{DV}{Dt}=\frac{\partial V}{\partial s}+V\frac{\partial V}{\partial s}$$

Then, the Euler's equation becomes

$$-\frac{1}{\rho}\frac{\partial p}{\partial s}-g\frac{\partial z}{\partial s}=\frac{\partial V}{\partial t}+V\frac{\partial V}{\partial s}$$

For steady flow, and neglecting body forces, it reduces to

$$-\frac{1}{\rho}\frac{\partial p}{\partial s}=V\frac{\partial V}{\partial s}$$

which indicates that a decrease in velocity is accompanied by an increase in pressure and conversely.

If we apply Newton's second law in the *n-direction* to the fluid element. Neglecting viscous forces, we obtain

$$\left(p-\frac{\partial p}{\partial n}\frac{dn}{2}\right)dsdx-\left(p+\frac{\partial p}{\partial n}\frac{dn}{2}\right)dsdx-\rho g\cos\beta dsdndx=\rho a_s dndsdx$$

Simplifying the equation

$$-\frac{\partial p}{\partial n}-\rho g\cos\beta=\rho a_n$$

Since $\cos\beta=\frac{\partial z}{\partial n}$, we can write $-\frac{1}{\rho}\frac{\partial p}{\partial n}-g\frac{\partial z}{\partial n}=a_n$

The centripetal acceleration, an, for steady flow can be written $a_n=-\frac{V^2}{R}$ where **R is the radius** of the curvature of the streamline. Then, Euler's equation normal to the streamline is written for steady flow as

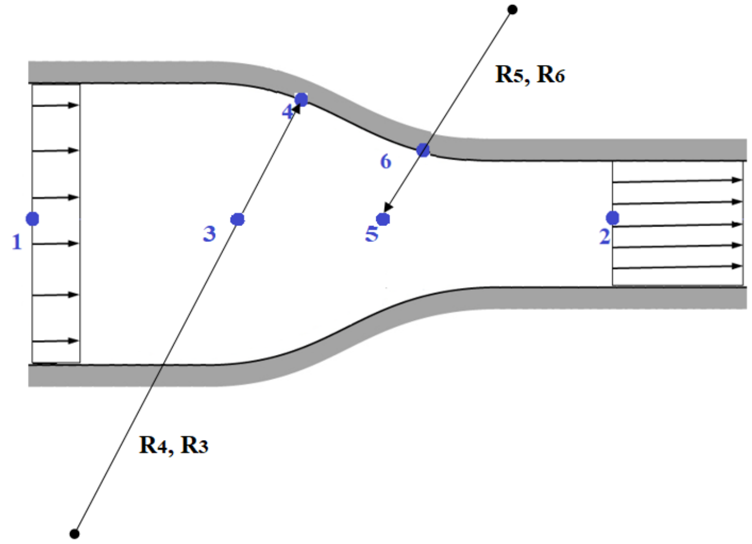
$$\frac{1}{\rho}\frac{\partial p}{\partial n}+g\frac{\partial z}{\partial n}=\frac{V^2}{R}$$

For steady flow in a horizontal plane, Euler's equation normal to streamline becomes

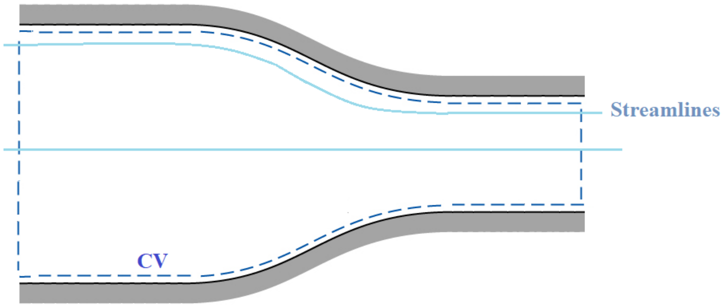
$$\frac{1}{\rho}\frac{\partial p}{\partial n}=\frac{V^2}{R}$$

It indicates that pressure increases in the direction outward from the center of curvature of the streamlines.

Example: An ideal fluid (zero viscosity and constant density) flowing through a planar converging nozzle that lies in a horizontal plane, shown in the figure. Compare the pressures at points 1 and 2, at 3 and 4, and at 5 and 6.



First, estimate the shape of the streamlines from the shape of the nozzle walls. Also, control volume is shown in the figure below.



From continuity equation steady, incompressible flow, we conclude that velocity increases from 1 t o3 to 5 to 2. Therefore, along the line 1-3-5-2,

$$\frac{\partial V}{\partial s} > 0$$

From $-\frac{1}{\rho}\frac{\partial p}{\partial s} = V\frac{\partial V}{\partial s}$ equation and the specification that the nozzle lies in a horizontal plane,

$$\frac{\partial p}{\partial s} = -\rho V\frac{\partial V}{\partial s}$$

As $\frac{\partial V}{\partial s} > 0$, we conclude $\frac{\partial p}{\partial s} < 0$ and pressure falls along line 1-3-5-2. Therefore,

$$p_1 > p_3 > p_5 > p_2$$

From $\frac{1}{\rho}\frac{\partial p}{\partial n} = \frac{V^2}{R}$ equation,

$$\frac{\partial p}{\partial n} = \rho \frac{V^2}{R}$$

Recall that n points toward the center of curvature. Both V^2 and R are positive, so the pressure increases outward from the center of the curvature, From the figure, we conclude that

$$p_4 > p_3 \qquad \text{and} \qquad p_5 > p_6$$

Although consideration of Euler's equations allowed us to comment on the relative magnitudes of the pressures, it did not permit us to calculate their values. The equations must be integrated before we calculate any numerical values for pressure.

BERNOULLI EQUATION INTEGRATION OF EULER'S EQUATION ALONG A STREAMLINE FOR STEADY FLOW

Consider the streamwise Euler equation in a streamline coordinates for steady flow. The equation is

$$V\frac{\partial V}{\partial s} + \frac{1}{\rho}\frac{\partial p}{\partial s} + g\frac{\partial z}{\partial s} = 0$$

Multiplying by ***ds*** we get

$$V\frac{\partial V}{\partial s}ds + \frac{1}{\rho}\frac{\partial p}{\partial s}ds + g\frac{\partial z}{\partial s}ds = 0$$

In general, the total differential of any parameter of the flow field (say pressure p) is given by

$$dp = \frac{\partial p}{\partial s}ds + \frac{\partial p}{\partial n}dn$$

because p is a function of both ***s* and *n***. If we restrict ourselves to remaining on the same streamline, $d\vec{s} = ds\vec{i}_s + dn\vec{i}_n$ $\qquad (d\vec{s} = dx\vec{i} + dy\vec{j} + dz\vec{k})$

$$dn = 0 \ \text{ and } \ dp = \frac{\partial p}{\partial s}ds$$

Similar relations hold for other properties.

With restriction of staying on the same streamline, Euler equation becomes

$$VdV + \frac{dp}{\rho} + gdz = 0$$

Integrating

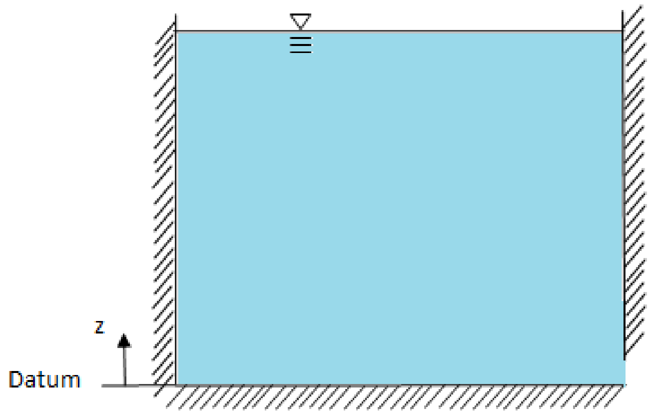
$$\frac{V^2}{2} + \int \frac{dp}{\rho} + gz = C \qquad \text{(a constant)}$$

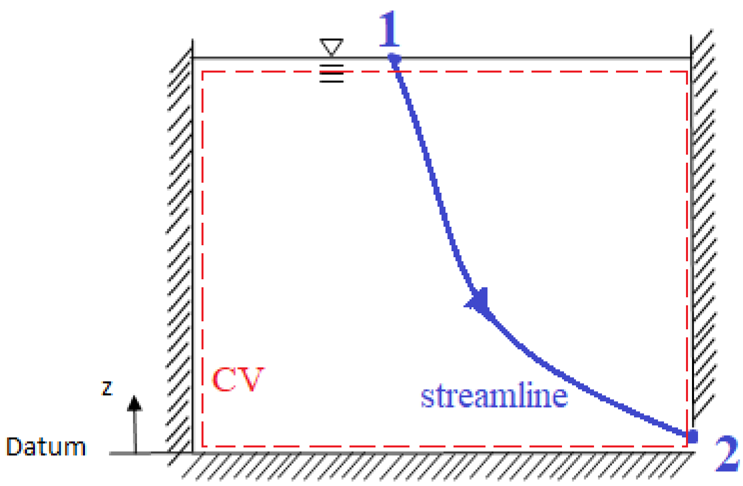
If the density is constant, we obtain the Bernoulli equation

$$\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant}$$

It is subject to **restrictions**:
1. Steady flow
2. Incompressible flow
3. Frictionless flow
4. Flow along a streamline

Example: A hole is pierced at the bottom of a large reservoir, which is initially filled with an incompressible fluid of density ρ to a depth of h, as shown in the figure. As a first approximation, fluid may be considered as inviscid, and the reservoir is large enough so that the change in its level may be neglected. Determine the velocity of the fluid leaving the hole, which is pierced at the bottom of the reservoir.





From continuity equation

$$V_1 A_1 = V_2 A_2$$

The area of the reservoir, A_1, is very large when compared to the area of the hole A_2. For this reason, the velocity of the fluid in the reservoir may approximately be taken as zero, so that,

$$V_1 = 0$$

From Bernoulli equation

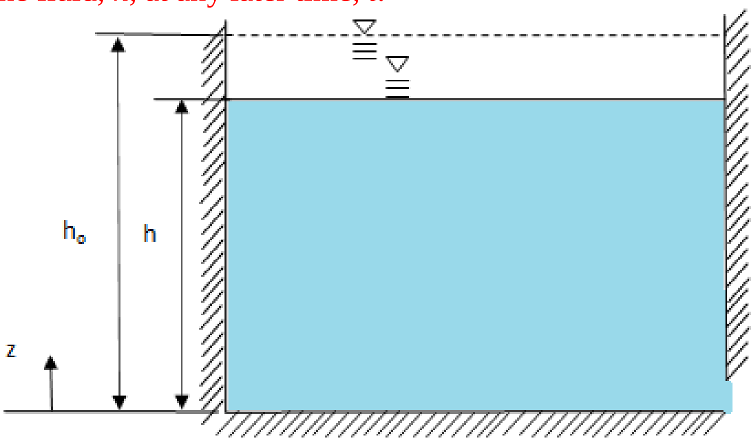
$$\frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2 = \frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1$$

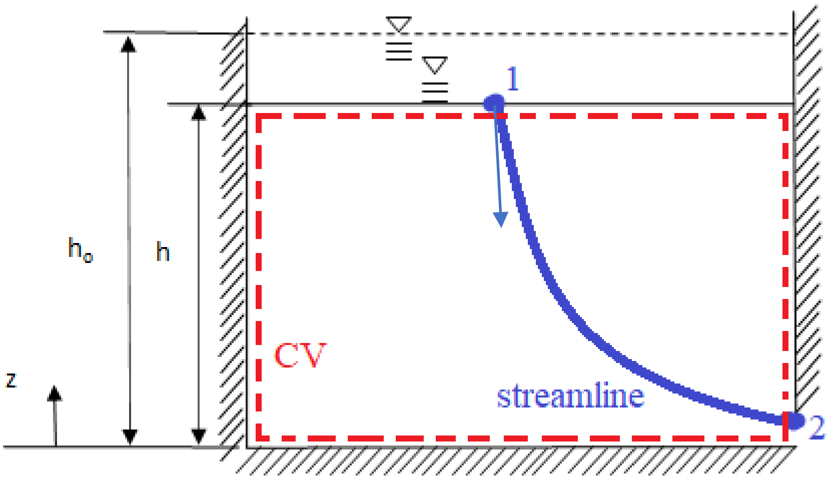
$$p_1 = p_2 = p_{atm} \qquad z_1 = h \qquad z_2 = 0$$

$$\therefore \qquad V_2 \equiv \sqrt{2gh}$$

This equation is first derived by Toricelli, and is often referred as **Torricelli equation**.

Example: A hole pierced at the bottom of a large reservoir, which is initially filled with an incompressible fluid of density ρ to a depth of h_o. The area of the tank and the hole are A_t and A_h, respectively. For the quasi-steady flow of the fluid, develop an expression for the height of the fluid, h, at any later time, t.





The continuity equation may be given

$$V_1 A_1 = V_2 A_2 \qquad \text{but } A_1 = A_t \text{ and } A_2 = A_h$$

so that $$V_2 = V_1 \frac{A_t}{A_h}$$

The Bernoulli equation for the steady flow of an incompressible and inviscid flow is

$$\frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2 = \frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1$$

Since, $p_1 = p_2 = p_{atm},\ z_1 = h, z_2 = 0$

Then, the Bernoulli equation takes the form

$$V_2^2 = V_1^2 + 2gh$$

or $$V_1^2 \left(\frac{A_t}{A_h}\right)^2 = V_1^2 + 2gh$$

$$V_1 = \sqrt{\frac{2gh}{\frac{A_t^2}{A_h^2} - 1}}$$

But, one should note that, $V_1 = -\frac{dh}{dt}$, therefore

$$\frac{dh}{dt} = -\sqrt{\frac{2gh}{\frac{A_t^2}{A_h^2} - 1}}$$

The variables may now be separated as

$$\frac{dh}{(2gh)^{1/2}} = -\frac{dt}{\left(\frac{A_t^2}{A_h^2} - 1\right)^{1/2}}$$

which may be integrated to yield

$$h = \frac{-gt^2}{\left(\frac{A_t^2}{A_h^2} - 1\right)} + C$$

where C is the constant of integration. However, one should observe that $h = h_0$ at $t = 0$, so that $C = h_0$, and

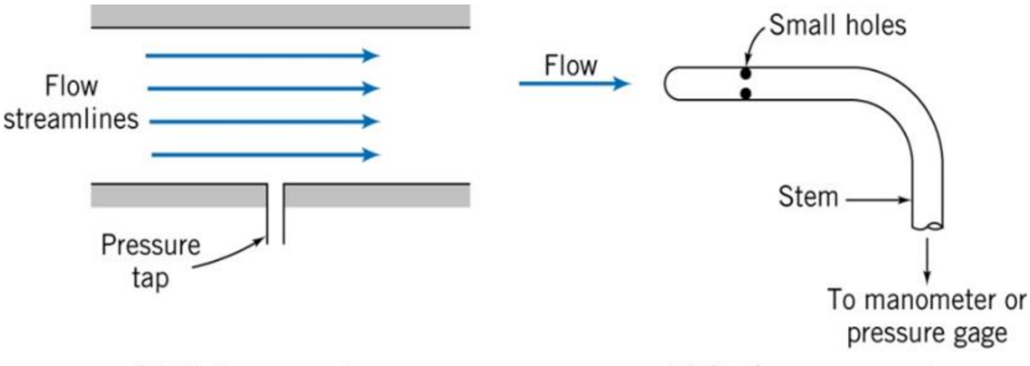
$$h = h_0 - \frac{gt^2}{\left(\frac{A_t^2}{A_h^2} - 1\right)}$$

STATIC, STAGNATION, AND DYNAMIC PRESSURE

Bernoulli equation is

$$\frac{p}{\rho}+\frac{V^2}{2}+gz=\text{constant}$$

In this equation p is called static pressure, because it is the pressure that would be measured by an instrument that is static with respect to the fluid. Of course, if the instrument were static with respect to fluid, it would have move along with the fluid. However, such a measurement rather difficult to make in a practical situation. However, we showed that there was no pressure variation normal to straight streamlines. This fact makes it possible to measure the static pressure in a flowing fluid using a wall pressure "tap" placed in a region where the flow streamlines are straight as shown in the figure. The pressure tap is a small hole, drilled carefully in the wall, with its axis perpendicular to the surface.



(a) Wall pressure tap (b) Static pressure probe

Figure. Measurement of static pressure.

In fluid stream far from a wall, or where streamlines are curved, accurate static pressure measurements can be made by careful use of a static pressure probe, shown in the figure.

When a flowing fluid is decelerated to zero speed by a frictionless process, the pressure is measured at that point is called stagnation pressure.

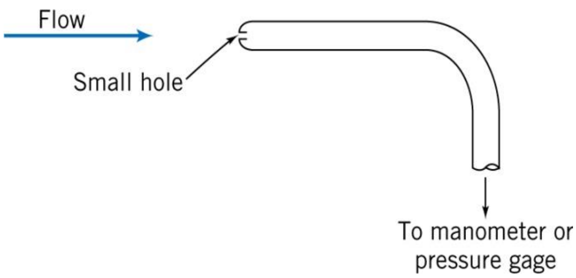


Figure. Measurement of stagnation pressure (Pitot tube).

In incompressible flow, applying Bernoulli equation between points in the free stream and at the nose of tube and taking $z = 0$ at the tube centerline, we get

$$\frac{p_0}{\rho} + \underbrace{\frac{V_0^2}{2}}_{=0} = \frac{p}{\rho} + \frac{V^2}{2}$$

where p_0 is the stagnation pressure, the stagnation speed V_0 is zero.

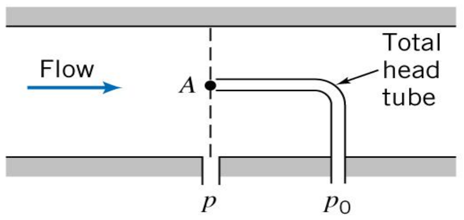
$$\therefore \quad p_0 = p + \frac{1}{2}\rho V^2$$

where p is the static pressure. The term $\frac{1}{2}\rho V^2$ generally is called dynamic pressure. Solving the dynamic pressure gives

$$\frac{1}{2}\rho V^2 = p_0 - p$$

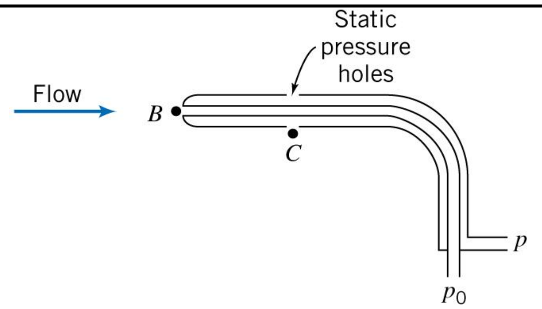
and for the speed

$$V = \sqrt{\frac{2(p_0 - p)}{\rho}}$$



(*a*) Total head tube used with wall static tap

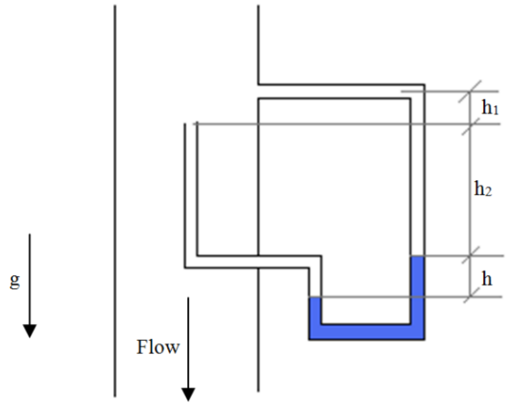
The static pressure corresponds to a point A is read from the wall static pressure tap. The stagnation pressure is measured directly at A by the total head tube.

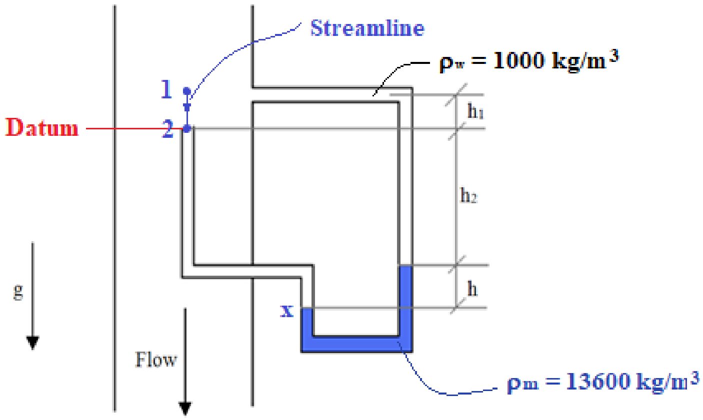


(*b*) Pitot-static tube

Two probes are combined as in pitot-static tube. The inner tube is used to measure the stagnation pressure at point B while the static pressure at C is measured by the small holes in the inner tube.

Example: A simple pitot tube and a piezometer are installed in a vertical pipe as shown in the figure. If the deflection in the mercury manometer is 0.1 m, then determine the velocity of the water at the center of the pipe. The densities of water and mercury are 1000 kg/m^3 and 13600 kg/m^3, respectively.





Applying Bernoulli equation between **points 1** and **2** along the streamline

$$\frac{p_1}{\rho_w}+\frac{V_1^2}{2}+gz_1=\frac{p_2}{\rho_w}+\frac{V_2^2}{2}+gz_2$$

However, from the principles of manometry

$$p_1 = p_x-\rho_w g\left(h_1+h_2\right)-\rho_m gh$$

and

$$p_2 = p_x-\rho_w g\left(h_2+h\right)$$

Also according to the chosen datum $z_1 = h_1$ and $z_2 = 0$.
Finally as long as **point 2** is a stagnation point, then the velocity at this point is zero, that is $V_2 = 0$. Then the Bernoulli equation takes the form

$$\frac{p_x-\rho_w g\left(h_1+h_2\right)-\rho_m gh}{\rho_w}+\frac{V_1^2}{2}+gh_1=\frac{p_x-\rho_w g\left(h_2+h\right)}{\rho_w}$$

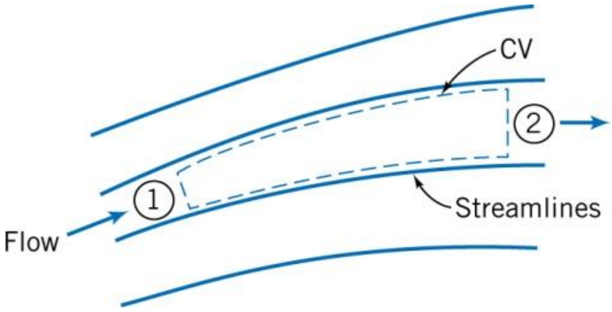
Solving for velocity at **point 1**

$$V_1=\sqrt{2gh\left(\frac{\rho_m}{\rho_w}-1\right)}$$

$$V_1=\sqrt{2(9.81)(0.1)\left(\frac{13600}{1000}-1\right)}=4.97\,\text{m/sec}$$

RELATION BETWEEN THE FIRST LAW OF THERMODYNAMICS AND THE BERNOULLI EQUATION

Consider steady flow in the absence of shear forces. We choose a control volume bounded by streamlines along its periphery. Such a control volume often is called a **streamtube**.



Basic equation (Energy equation)

$$\underset{0}{\dot{Q}} - \underset{0}{\dot{W}_s} - \underset{0}{\dot{W}_{shear}} - \underset{0}{\dot{W}_{other}} = \underbrace{\frac{\partial}{\partial t}\int_{C\forall} e\rho d\forall}_{0} + \int_{CS} \left(e + p\vartheta\right)\rho\vec{V}\cdot d\vec{A}$$

<u>Restrictions</u>: 1) $\dot{W}_s = 0$

2) $\dot{W}_{shear} = 0$

3) $\dot{W}_{other} = 0$

4) Steady flow

5) Uniform flow and properties at each section

Under these restrictions

$$0 = -\left(u_1 + p_1 v_1 + \frac{V_1^2}{2} + gz_1\right)\left\{-\left|\rho_1 V_1 A_1\right|\right\} + \left(u_2 + p_2 v_2 + \frac{V_2^2}{2} + gz_2\right)\left\{-\left|\rho_2 V_2 A_2\right|\right\} - \dot{Q}$$

But from continuity under these restrictions

$$0 = \underbrace{\frac{\partial}{\partial t}\int_{C\forall} \rho d\forall}_{0} + \int_{CS} \rho\vec{V}\cdot d\vec{A}$$

or
$$0 = \left\{-\left|\rho_1 V_1 A_1\right|\right\} + \left\{-\left|\rho_2 V_2 A_2\right|\right\}$$

That is,

$$\dot{m} = \rho_1 V_1 A_1 = \rho_2 V_2 A_2$$

Also,

$$\dot{Q} = \frac{\delta Q}{\delta t} = \frac{\delta Q}{dm}\frac{dm}{dt} = \frac{\delta Q}{dm}\dot{m}$$

Thus, from the energy equation

$$0=\left[\left(p_2v_2+\frac{V_2^2}{2}+gz_2\right)-\left(p_1v_1+\frac{V_1^2}{2}+gz_1\right)\right]\dot{m}+\left(u_2-u_1-\frac{\delta Q}{dm}\right)\dot{m}$$

or

$$p_1v_1+\frac{V_1^2}{2}+gz_1=p_2v_2+\frac{V_2^2}{2}+gz_2+\left(u_2-u_1-\frac{\delta Q}{dm}\right)$$

Under the restriction of incompressible flow $v_1=v_2=\dfrac{1}{\rho}$ and hence

$$\frac{p_1}{\rho}+\frac{V_1^2}{2}+gz_1=\frac{p_2}{\rho}+\frac{V_2^2}{2}+gz_2+\left(u_2-u_1-\frac{\delta Q}{dm}\right)$$

This will reduce to the Bernoulli equation if the term in parentheses were zero. Thus, under the additional restrictions,

6) incompressible flow $v_1=v_2=\dfrac{1}{\rho}=\text{constant}$

7) $\left(u_2-u_1-\dfrac{\delta Q}{dm}\right)=0$

The energy equation reduces to

$$\frac{p_1}{\rho}+\frac{V_1^2}{2}+gz_1=\frac{p_2}{\rho}+\frac{V_2^2}{2}+gz_2=\text{constant}$$

The Bernoulli equation was derived from momentum considerations (Newton's second law), and is valid for steady, incompressible, frictionless flow along a streamline.

In this section, the Bernoulli equation was obtained by applying the first law of thermodynamics to a streamtube control volume, subject to restrictions 1 through 7 above.

Example: Consider the frictionless, incompressible flow with heat transfer.
Show that $u_2-u_1=\dfrac{\delta Q}{dm}$.

In general internal energy *u*, can be expressed as

$$u=u(T,v)$$

For incompressible flow *v = constant*, and $u=u(T)$.
Thus, the internal energy change for any process, u_2-u_1, depends only on the temperatures at the end states.

From the Gibbs (property) equation $Tds=du+pdv$

$$Tds=du+p\underbrace{dv}_{0}$$ since *v = constant* for incompressible flow

Since, frictionless flow is a reversible process

$$\int_1^2 Tds = \frac{\delta Q}{dm}$$

Therefore,

$$\int_1^2 Tds = \int_1^2 u = \frac{\delta Q}{dm} \qquad \Rightarrow \qquad u_2 - u_1 = \frac{\delta Q}{dm}$$

Often it is convenient to represent the mechanical energy level of a flow graphically. The energy equation, that is Bernoulli equation, suggests such a representation. Dividing Bernoulli equation by *g*, we obtain

$$\frac{p}{\rho g} + \frac{V^2}{2g} + z = H = \text{constant}$$

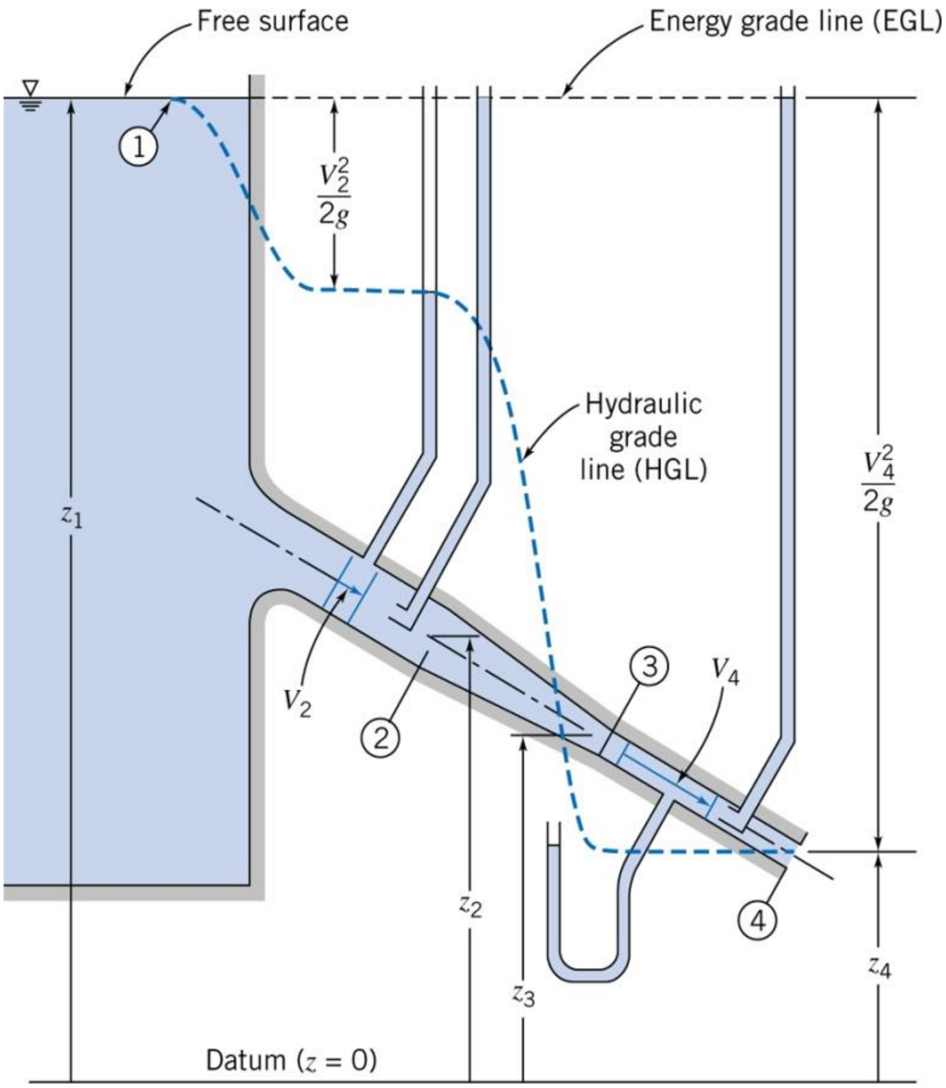
Each term has dimensions of length, or "**head**" of flowing fluid. The individual terms are

$\frac{p}{\rho g}$	is **the head due to local static pressure**
$\frac{V^2}{2g}$	is **the head due to local dynamic pressure**
z	is **elevation head**
H	is **the total head of the flow**

The energy grade line (EGL): The locus of points at a vertical distance, $H = \frac{p}{\rho g} + \frac{V^2}{2g} + z$, measured above a horizontal datum, which is the total head of the fluid.

The hydraulic grade line (HGL): The locus of points at a vertical distance, $\frac{p}{\rho g} + z$, measured above a horizontal datum.

The difference is heights between the EGL and HGL represents, **the dynamic (velocity) head,** $\frac{V^2}{2g}$.



UNSTEADY BERNOULLI EQUATION – INTEGRATION OF EULER'S EQUATION ALONG A STREAMLINE

Consider the streamwise Euler equation in streamline coordinates

$$V\frac{\partial V}{\partial s}+\frac{1}{\rho}\frac{\partial p}{\partial s}+g\frac{\partial z}{\partial s}+\frac{\partial V}{\partial t}=0$$

The above equation may now be integrated along an instantaneous streamline from point 1 to point 2 to yield

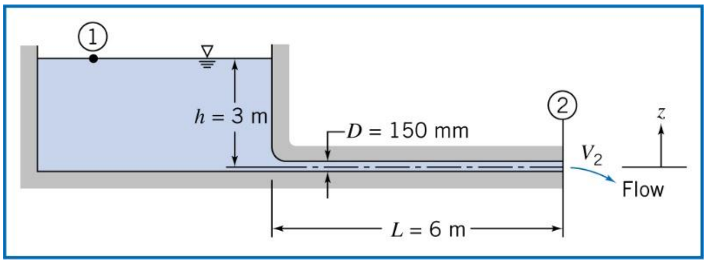
$$\int_1^2 V\frac{\partial V}{\partial s}ds+\int_1^2\frac{1}{\rho}\frac{\partial p}{\partial s}ds+\int_1^2 g\frac{\partial z}{\partial s}ds+\int_1^2\frac{\partial V}{\partial t}ds=0$$

For an incompressible flow, it becomes

$$\frac{p_1}{\rho}+\frac{V_1^2}{2}+gz_1=\frac{p_2}{\rho}+\frac{V_2^2}{2}+gz_2+\int_1^2\frac{\partial V_s}{\partial t}ds$$

Restrictions: 1) Incompressible flow
2) Frictionless flow
3) Flow along a streamline

Example: A long pipe is connected to a large reservoir that initially is filled with water to a depth of 3m. The pipe is 150 mm in diameter and 6 m long. As a first approximation, friction may be neglected. Determine the flow velocity leaving the pipe as a function of time after a cap is removed from its free end. The reservoir is large enough so that the change in its level may be neglected.



Given: $h = 3\ m$
$D = 150\ mm$
$L = 6\ m$

Find: $V_2(t) = ?$
Basic equation:

$$\frac{p_1}{\rho}+\frac{V_1^2}{2}+gz_1=\frac{p_2}{\rho}+\frac{V_2^2}{2}+gz_2+\int_1^2\frac{\partial V_s}{\partial t}ds$$

Assumptions: 1. Incompressible flow
2. Frictionless flow
3. Flow along a streamline from point 1 to point 2
4. $p_1 = p_2 = p_{atm}$
5. $V_1^2 = 0$
6. $z_2 = 0$
7. $z_1 = h = \text{constant}$
8. Neglect velocity in the reservoir, except for small region near the inlet to the tube.

$$\frac{p_{atm}}{\rho}+\underset{0}{\frac{V_1^2}{2}}+gz_1=\frac{p_{atm}}{\rho}+\frac{V_2^2}{2}+g\underset{0}{z_2}+\int_1^2\frac{\partial V_s}{\partial t}ds$$

$$gz_1=gh=\frac{V_2^2}{2}+\int_1^2\frac{\partial V_s}{\partial t}ds$$

In view of assumption 8, the integral becomes

$$\int_1^2 \frac{\partial V_s}{\partial t}\,ds \approx \int_0^L \frac{\partial V_s}{\partial t}\,ds$$

In the tube, $V_s = V_2$ everywhere, so that

$$\int_0^L \frac{\partial V_s}{\partial t}\,ds = \int_0^L \frac{dV_2}{dt}\,ds = L\frac{dV_2}{dt}$$

Substituting gives

$$gh = \frac{V_2^2}{2} + L\frac{dV_2}{dt}$$

Separating varibles, we obtain

$$\frac{dV_2}{2gh - V_2^2} = \frac{dt}{2L}$$

Integrating between limits $V = 0$ at $t = 0$, and $V = V_2$ at $t = t$,

$$\int_0^{V_2} \frac{dV}{2gh - V^2} = \frac{t}{2L}$$

$$\left[\frac{1}{\sqrt{2gh}}\tanh^{-1}\left(\frac{V}{\sqrt{2gh}}\right)\right]_0^{V_2} = \frac{t}{2L}$$

Since $\tanh^{-1}(0) = 0$, we obtain

$$\frac{1}{\sqrt{2gh}}\tanh^{-1}\left(\frac{V_2}{\sqrt{2gh}}\right) = \frac{t}{2L}$$

or
$$\frac{V_2}{\sqrt{2gh}} = \tanh\left(\frac{t}{2L}\sqrt{2gh}\right)$$

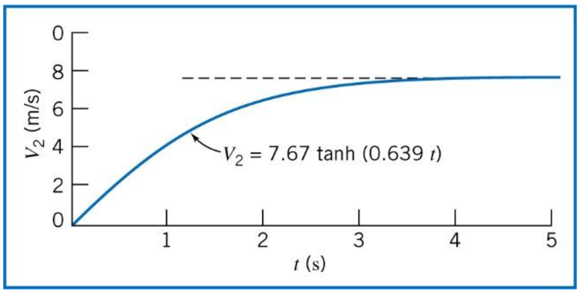
For the given conditions

$$\sqrt{2gh} = \sqrt{2(9.81)(3)} = 7.67\text{ m/s}$$

and

$$\frac{t}{2L}\sqrt{2gh} = \frac{t}{2(6)}(7.67) = 0.639t$$

The result is then $V_2 = 7.67\tanh\left(0.639t\right)$ m/s



IRROTATIONAL FLOW

When the fluid elements moving in a flow field do not undergo any rotation, then the flow is known to be irrotational. For an irrotational flow,

$$\vec{w} = 0 \quad \text{or} \quad \nabla \times \vec{V} = 0$$

that is,

$$\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$$

In cylindrical coordinates,

$$\frac{1}{r}\frac{\partial V_z}{\partial \theta} - \frac{\partial V_\theta}{\partial z} = \frac{\partial V_r}{\partial z} - \frac{\partial V_z}{\partial r} == \frac{1}{r}\frac{\partial r V_\theta}{\partial r} - \frac{\partial V_r}{\partial \theta} = 0$$

BERNOULLI EQUATION APPLIED TO IRROTATIONAL FLOW

Euler equation for steady flow was

$$-\frac{1}{\rho}\nabla p - g\nabla z = \left(\vec{V}\cdot\nabla\right)\vec{V}$$

using vector identity

$$\left(\vec{V}\cdot\nabla\right)\vec{V} = \frac{1}{2}\nabla\left(\vec{V}\cdot\vec{V}\right) - \vec{V}\times\left(\nabla\times\vec{V}\right)$$

We see that for irrotational flow $\nabla\times\vec{V} = 0$; therefore,

$$\left(\vec{V}\cdot\nabla\right)\vec{V} = \frac{1}{2}\nabla\left(\vec{V}\cdot\vec{V}\right)$$

and Euler's equation for irrotational flow can be written as

$$-\frac{1}{\rho}\nabla p - g\nabla z = \frac{1}{2}\nabla\left(\vec{V}\cdot\vec{V}\right) = \frac{1}{2}\nabla\left(V^2\right)$$

During the interval *dt*, a fluid particle moves from the vector position $\vec{r}$ to the position $\vec{r} + d\vec{r}$. Taking the dot product of $d\vec{r} = dx\vec{\imath} + dy\vec{\jmath} + dz\vec{k}$ with each of the terms in above equation, we obtain

$$-\frac{1}{\rho}\nabla p\cdot d\vec{r} - g\nabla z\cdot d\vec{r} = \frac{1}{2}\nabla\left(V^2\right)\cdot d\vec{r}$$

and hence

$$-\frac{dp}{\rho} - gdz = \frac{1}{2}d\left(V^2\right)$$

integrating this equation gives,

$$\int \frac{dp}{\rho} + \frac{V^2}{2} + gz = \text{constant}$$

For incompressible flow, ρ = constant, and

$$\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant}$$

Since $d\vec{r}$ was an arbitrary displacement, this equation is valid between any two points in the flow field. The restrictions are

1. Steady flow
2. Incompressible flow
3. Inviscid flow
4. Irrotational flow

VELOCITY POTENTIAL

We can formulate a relation called the potential function, ϕ, for a velocity field that is irrotational. To do so, we must use the fundamental vector identity

$$\text{curl }(\text{grad } \phi) = \nabla \times (\nabla \phi) = 0$$

which is valid if $\phi(x,y,z,t)$ is a scalar function, having continuous first and second derivatives.

Then, for an irrotational flow in which $\nabla \times \vec{V} = 0$, a scalar function, ϕ, must exist such that the gradient of ϕ is equal to the velocity vector, $\vec{V}$.

$$\vec{V} \equiv \pm \nabla \phi$$

thus,

$$u = -\frac{\partial \phi}{\partial x} \qquad v = -\frac{\partial \phi}{\partial y} \qquad w = -\frac{\partial \phi}{\partial z}$$

In cylindrical coordinates

$$V_r = -\frac{\partial \phi}{\partial r} \qquad V_\theta = -\frac{1}{r}\frac{\partial \phi}{\partial \theta} \qquad V_z = -\frac{\partial \phi}{\partial z}$$

The potential velocity, ϕ, exists only for irrotational flow. Irrotationality may be a valid assumption for those regions of a flow in which viscous forces are negligible. For example, such a region exists outside the boundary layer in the fluid over a solid surface.

All real fluids possess viscosity, but there are many situations in which the assumption of inviscid flow considerably simplifies the analysis and gives meaningful results.

STREAM FUNCTION AND VELOCITY POTENTIAL FOR TWO DIMENSIONAL, IRROTATIONAL INCOMPRESSIBLE FLOW; LAPLACE'S EQUATION

For two dimensional, incompressible, inviscid flow, velocity components *u* and *v* can be expressed in terms of stream function, ψ, and the velocity potential, ϕ,

$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x}$$

$$u = -\frac{\partial \phi}{\partial x} \qquad v = -\frac{\partial \phi}{\partial y}$$

Substituting for *u* and *v* into the irrotational condition

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \qquad \text{we obtain}$$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \tag{A}$$

Substituting for *u* and *v* into the continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

we obtain

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \tag{B}$$

Equations (A) and (B) are forms of Laplace's equation. Any function ψ or ϕ that satisfies Laplace's equation represents a possible two dimensional, incompressible, irrotational flow field.

Along a streamline, stream function ψ is constant, therefore

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = 0$$

The slope of a streamline becomes

$$\left.\frac{dy}{dx}\right)_{\psi} = -\frac{\partial \psi / \partial x}{\partial \psi / \partial y} = -\frac{-v}{u} = \frac{v}{u}$$

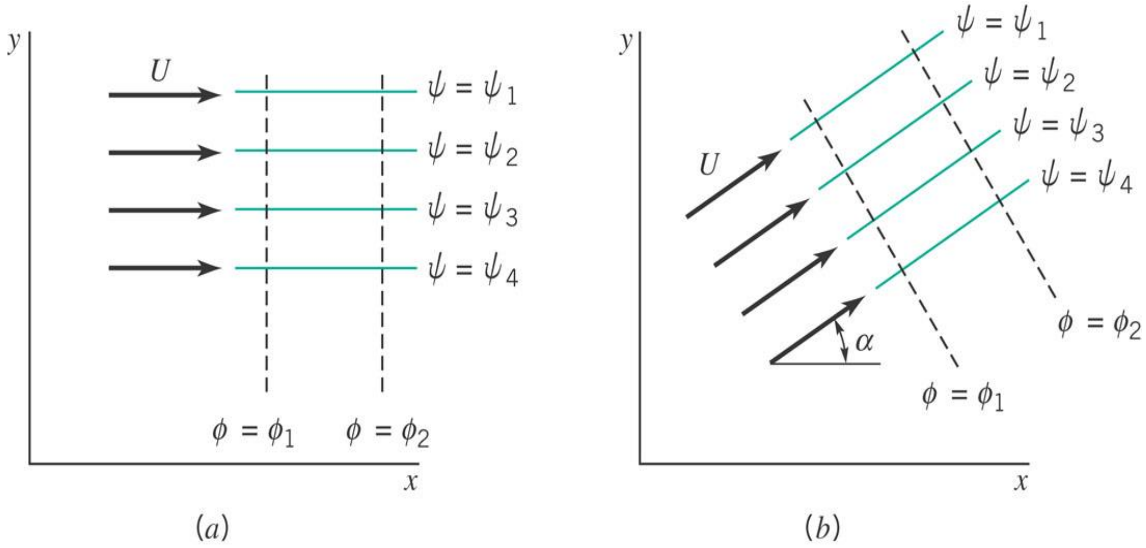
Along a line of constant ϕ, $d\phi = 0$ and

$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy = 0$$

Consequently, the slope of a potential line becomes

$$\frac{dy}{dx} = -\frac{\partial\phi/\partial x}{\partial\phi/\partial y} = -\frac{u}{v}$$

As potential lines and streamlines have slopes that are negative reciprocals; they are perpendicular.



Example: Consider the flow field given by $\phi = 4x^2 - 4y^2$. Show that the flow is irrotational. Determine the stream function for this flow.

NOTE: According to our textbook notation $u = -\dfrac{\partial\phi}{\partial x}$ $\quad v = -\dfrac{\partial\phi}{\partial y}$

$$u = -\frac{\partial\phi}{\partial x} = -8x$$

$$v = -\frac{\partial\phi}{\partial y} = 8y$$

If the flow is irrotational, then $\omega_z = 0$. Since,

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{1}{2}(0-0) = 0$$

$\therefore$ Therefore, the flow is irrotational.

$$u = \frac{\partial\psi}{\partial y} \quad \Rightarrow \quad \frac{\partial\psi}{\partial y} = -8x \quad \Rightarrow \quad \psi = -8xy + f(x)$$

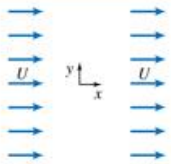
$$v = -\frac{\partial\psi}{\partial x} \quad \Rightarrow \quad 8y = 8y + f'(x)$$

$$f'(x) = 0 \quad \Rightarrow \quad f(x) = C$$

Thus, $\quad \psi = -8xy + C$

ELEMENTARY PLANE FLOWS

A variety of potential flows can be constructed by superposing elementary flow patterns. The ψ and ϕ functions for five elementary two-dimensional flows – a uniform flow, a source, a sink, a vortex and a doublet are summarized in the Table below.

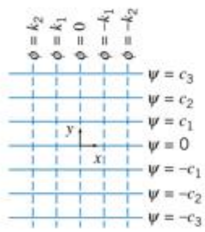


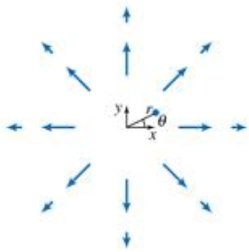
Uniform Flow (positive x direction)

$u = U \qquad \psi = Uy$

$v = 0 \qquad \phi = -Ux$

$\Gamma = 0$ around any closed curve



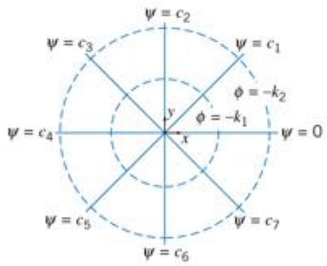


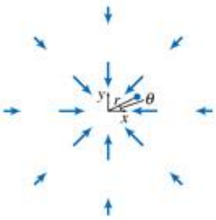
Source Flow (from origin)

$V_r = \dfrac{q}{2\pi r} \qquad \psi = \dfrac{q}{2\pi}\theta$

$V_\theta = 0 \qquad \phi = -\dfrac{q}{2\pi}\ln r$

Origin is singular point
q is volume flow rate per unit depth
$\Gamma = 0$ around any closed curve



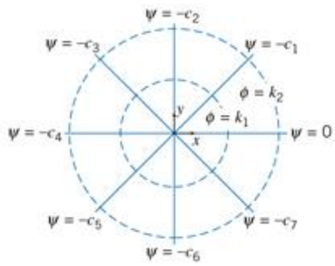


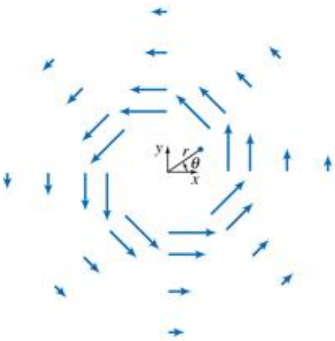
Sink Flow (toward origin)

$V_r = -\dfrac{q}{2\pi r} \qquad \psi = -\dfrac{q}{2\pi}\theta$

$V_\theta = 0 \qquad \phi = \dfrac{q}{2\pi}\ln r$

Origin is singular point
q is volume flow rate per unit depth
$\Gamma = 0$ around any closed curve



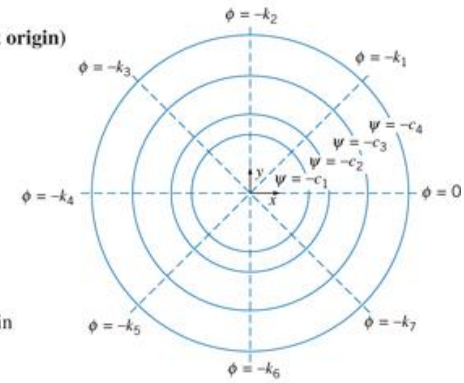


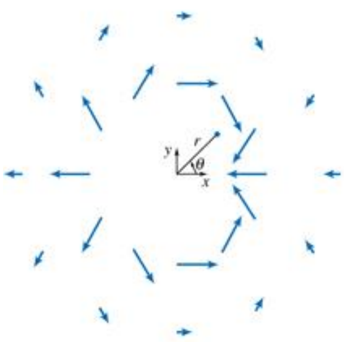
Irrotational Vortex (counterclockwise, center at origin)

$V_r = 0 \qquad \psi = -\dfrac{K}{2\pi}\ln r$

$V_\theta = \dfrac{K}{2\pi r} \qquad \phi = -\dfrac{K}{2\pi}\theta$

Origin is singular point
K is strength of the vortex
$\Gamma = K$ around any closed curve enclosing origin
$\Gamma = 0$ around any closed curve not enclosing origin



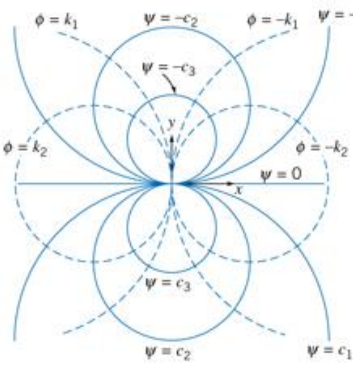


Doublet (center at origin)

$V_r = -\dfrac{\Lambda}{r^2}\cos\theta \qquad \psi = -\dfrac{\Lambda\sin\theta}{r}$

$V_\theta = -\dfrac{\Lambda}{r^2}\sin\theta \qquad \phi = -\dfrac{\Lambda\cos\theta}{r}$

Origin is singular point
Λ is strength of the doublet
$\Gamma =$ around any closed curve

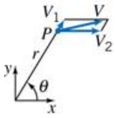


SUPERPOSITION OF ELEMENTARY PLANE FLOWS

We showed that both ϕ and ψ satisfy Laplace's equation for flow that is both incompressible and irrotational. Since Laplace's equation is a linear homogeneous partial differential equation, solutions may be superposed (added together) to develop more complex and interesting patterns of flows.

Table. Superposition of Elementary Plane Flows

Source and Uniform Flow (flow past a half-body)

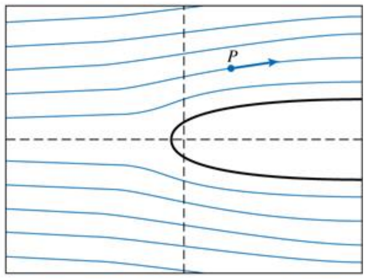


$$\psi = \psi_{so} + \psi_{uf} = \psi_1 + \psi_2 = \frac{q}{2\pi}\theta + Uy$$

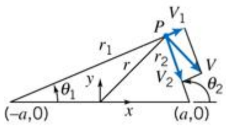
$$\psi = \frac{q}{2\pi}\theta + Ur\sin\theta$$

$$\phi = \phi_{so} + \phi_{uf} = \phi_1 + \phi_2 = -\frac{q}{2\pi}\ln r - Ux$$

$$\phi = -\frac{q}{2\pi}\ln r - Ur\cos\theta$$



Source and Sink (equal strength, separation distance on x axis = 2a)

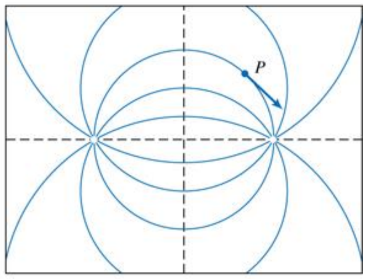


$$\psi = \psi_{so} + \psi_{si} = \psi_1 + \psi_2 = \frac{q}{2\pi}\theta_1 - \frac{q}{2\pi}\theta_2$$

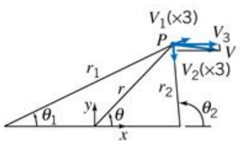
$$\psi = \frac{q}{2\pi}(\theta_1 - \theta_2)$$

$$\phi = \phi_{so} + \phi_{si} = \phi_1 + \phi_2 = -\frac{q}{2\pi}\ln r_1 + \frac{q}{2\pi}\ln r_2$$

$$\phi = \frac{q}{2\pi}\ln\frac{r_2}{r_1}$$



Source, Sink, and Uniform Flow (flow past a Rankine body)

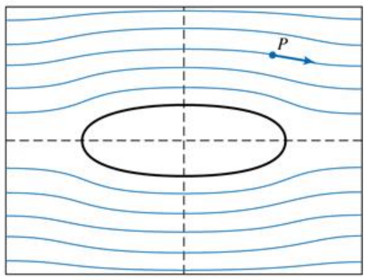


$$\psi = \psi_{so} + \psi_{si} + \psi_{uf} = \psi_1 + \psi_2 + \psi_3$$
$$= \frac{q}{2\pi}\theta_1 - \frac{q}{2\pi}\theta_2 + Uy$$

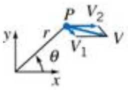
$$\psi = \frac{q}{2\pi}(\theta_1 - \theta_2) + Ur\sin\theta$$

$$\phi = \phi_{so} + \phi_{si} + \phi_{uf} = \phi_1 + \phi_2 + \phi_3$$
$$= -\frac{q}{2\pi}\ln r_1 + \frac{q}{2\pi}\ln r_2 - Ux$$

$$\phi = \frac{q}{2\pi}\ln\frac{r_2}{r_1} - Ur\cos\theta$$



Vortex (clockwise) and Uniform Flow

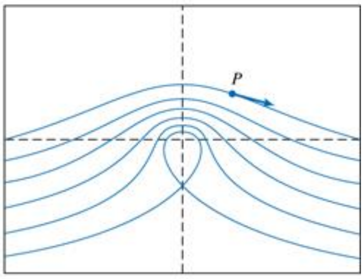


$$\psi = \psi_v + \psi_{uf} = \psi_1 + \psi_2 = \frac{K}{2\pi}\ln r + Uy$$

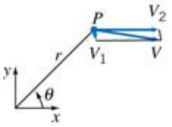
$$\psi = \frac{K}{2\pi}\ln r + Ur\sin\theta$$

$$\phi = \phi_v + \phi_{uf} = \phi_1 + \phi_2 = \frac{K}{2\pi}\theta - Ux$$

$$\phi = \frac{K}{2\pi}\theta - Ur\cos\theta$$



Doublet and Uniform Flow (flow past a cylinder)



$$\psi = \psi_d + \psi_{uf} = \psi_1 + \psi_2 = -\frac{\Lambda\sin\theta}{r} + Uy$$

$$= -\frac{\Lambda\sin\theta}{r} + Ur\sin\theta$$

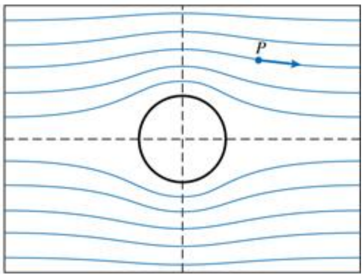
$$\psi = U\left(r - \frac{\Lambda}{Ur}\right)\sin\theta$$

$$\psi = Ur\left(1 - \frac{a^2}{r^2}\right)\sin\theta \quad a = \sqrt{\frac{\Lambda}{U}}$$

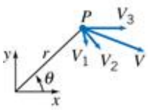
$$\phi = \phi_d + \phi_{uf} = \phi_1 + \phi_2 = -\frac{\Lambda\cos\theta}{r} - Ux$$

$$= -\frac{\Lambda\cos\theta}{r} - Ur\cos\theta$$

$$\phi = -U\left(r + \frac{\Lambda}{Ur}\right)\cos\theta = -Ur\left(1 + \frac{a^2}{r^2}\right)\cos\theta$$



Doublet, Vortex (clockwise), and Uniform Flow (flow past a cylinder with circulation)



$$\psi = \psi_d + \psi_v + \psi_{uf} = \psi_1 + \psi_2 + \psi_3$$

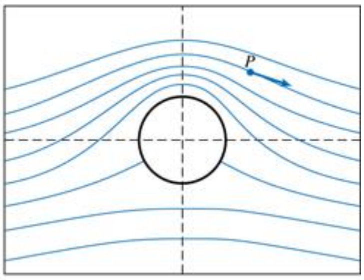
$$= -\frac{\Lambda\sin\theta}{r} + \frac{K}{2\pi}\ln r + Uy$$

$$\psi = -\frac{\Lambda\sin\theta}{r} + \frac{K}{2\pi}\ln r + Ur\sin\theta$$

$$\psi = Ur\left(1 - \frac{a^2}{r^2}\right)\sin\theta + \frac{K}{2\pi}\ln r$$

$$\phi = \phi_d + \phi_v + \phi_{uf} = \phi_1 + \phi_2 + \phi_3$$

$$= -\frac{\Lambda\cos\theta}{r} + \frac{K}{2\pi}\theta - Ux$$

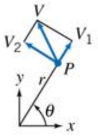


$a = \sqrt{\frac{\Lambda}{U}};\ K < 4\pi a U$

$$\phi = -\frac{\Lambda\cos\theta}{r} + \frac{K}{2\pi}\theta - Ur\cos\theta$$

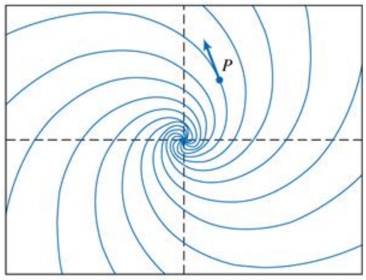
$$\phi = -Ur\left(1 + \frac{a^2}{r^2}\right)\cos\theta + \frac{K}{2\pi}\theta$$

Source and Vortex (spiral vortex)

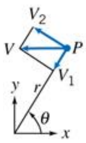


$$\psi = \psi_{so} + \psi_v = \psi_1 + \psi_2 = \frac{q}{2\pi}\theta - \frac{K}{2\pi}\ln r$$

$$\phi = \phi_{so} + \phi_v = \phi_1 + \phi_2 = -\frac{q}{2\pi}\ln r - \frac{K}{2\pi}\theta$$

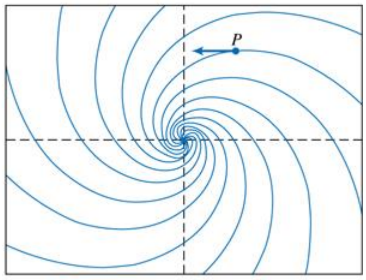


Sink and Vortex

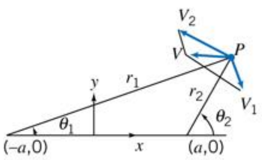


$$\psi = \psi_{si} + \psi_v = \psi_1 + \psi_2 = -\frac{q}{2\pi}\theta - \frac{K}{2\pi}\ln r$$

$$\phi = \phi_{si} + \phi_v = \phi_1 + \phi_2 = \frac{q}{2\pi}\ln r - \frac{K}{2\pi}\theta$$



Vortex Pair (equal strength, opposite rotation, separation distance on x axis = 2a)

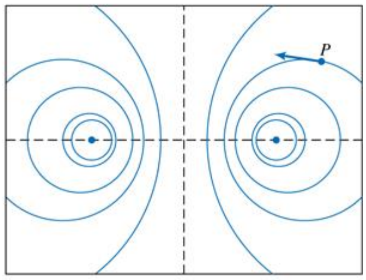


$$\psi = \psi_{v1} + \psi_{v2} = \psi_1 + \psi_2 = -\frac{K}{2\pi}\ln r_1 + \frac{K}{2\pi}\ln r_2$$

$$\psi = \frac{K}{2\pi}\ln\frac{r_2}{r_1}$$

$$\phi = \phi_{v1} + \phi_{v2} = \phi_1 + \phi_2 = -\frac{K}{2\pi}\theta_1 + \frac{K}{2\pi}\theta_2$$

$$\phi = \frac{K}{2\pi}(\theta_2 - \theta_1)$$



Example: A source with strength of **0.2 m^3/sm** and a counterclockwise vortex with strength of **1 m^3/s** are placed on the origin. Obtain streamfunction and velocity potential, and velocity field for the combined flow. Find the velocity at **point (1,0.5)**.

For source

$$\phi_s = -\frac{q}{2\pi}\ln r = -\frac{0.2}{2\pi}\ln r \qquad m^2/s$$

and

$$\psi_s = \frac{q}{2\pi}\theta = \frac{0.2}{2\pi}\theta \qquad m^2/s$$

For vortex

$$\phi_v = -\frac{K}{2\pi}\theta = -\frac{1}{2\pi}\theta \qquad m^2/s$$

and

$$\psi_v = -\frac{K}{2\pi}\ln r = -\frac{1}{2\pi}\ln r \qquad m^2/s$$

$$\therefore \qquad \phi = \phi_s + \phi_v = -\frac{1}{\pi}\left[0.1\ln r + \frac{\theta}{2}\right] \qquad m^2/s$$

$$\psi = \psi_s + \psi_v = -\frac{1}{\pi}\left[0.1\theta - \frac{\ln r}{2}\right] \qquad m^2/s$$

$$V_r = -\frac{\partial \phi}{\partial r} = \frac{1}{10\pi r}$$

$$V_\theta = -\frac{1}{r}\frac{\partial \phi}{\partial \theta} = \frac{1}{2\pi r}$$

$$r = \sqrt{x^2 + y^2} = \sqrt{1^2 + 0.5^2} = 1.117\ m$$

$\therefore$

$$V_r = \frac{1}{10\pi r} = \frac{1}{10\pi(1.117)} = 0.0295\ m/s$$

$$V_\theta = \frac{1}{2\pi r} = \frac{1}{2\pi(1.117)} = 0.143\ m/s$$

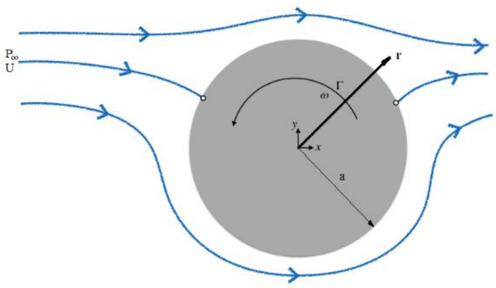
Example: The following stream function represents the flow past a cylinder of radius **"a"** with circulation

$$\psi = Ur\sin\theta - \frac{Ua^2}{r}\sin\theta - aU\ln\left(\frac{r}{a}\right)$$

Determine the pressure distribution over the cylinder.

$$V_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{1}{r}\left[Ur\cos\theta - \frac{Ua^2}{r}\cos\theta\right] = U\left(1 - \frac{a^2}{r^2}\right)\cos\theta$$

$$V_\theta = -\frac{\partial \psi}{\partial r} = -\left[U\sin\theta + \frac{Ua^2}{r^2}\sin\theta - \frac{aU}{r}\right] = U\left[\frac{a}{r} - \left(1 + \frac{a^2}{r^2}\right)\sin\theta\right]$$



We now apply Bernoulli equation to streamlines immediately adjacent to the surface of the cylinder. With $V_r = 0$ at the cylinder surface, Bernoulli equation gives

$$\frac{p_\infty}{\rho}+\frac{1}{2}U^2=\frac{p_s}{\rho}+\frac{1}{2}V_s^2 \qquad \text{but} \qquad V_s=\sqrt{V_{\theta_s}^2+V_{r_s}^2} \text{ and } V_{r_s}=0$$

$$\text{then } V_s=V_{\theta_s}$$

where p_s and V_s are the pressure and velocity on the cylinder surface, respectively.

$$\frac{p_\infty}{\rho}+\frac{1}{2}U^2=\frac{p_s}{\rho}+\frac{U^2}{2}\left[\frac{a}{a}-\left(1+\frac{a^2}{a^2}\right)\sin\theta\right]^2$$

or

$$\frac{p_\infty}{\rho}+\frac{1}{2}U^2=\frac{p_s}{\rho}+\frac{U^2}{2}\left[1-2\sin\theta\right]^2$$

or

$$\frac{p_s-p_\infty}{\rho}=\frac{U^2}{2}\left[1-\left(1-4\sin\theta+4\sin^2\theta\right)\right]$$

DIMENSIONAL ANALYSIS AND SIMILITUDE

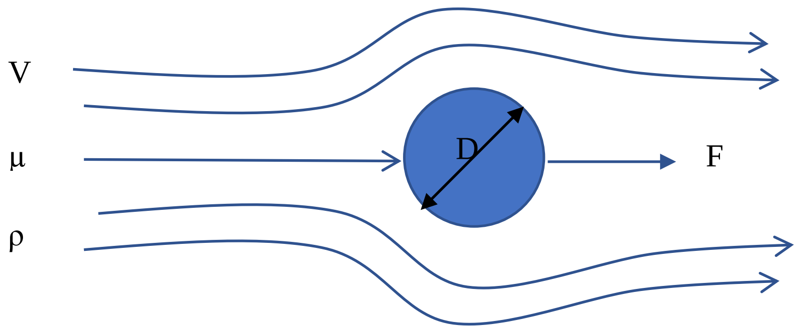
Many real fluid flow problems can be solved, at best, only approximately by using analytical or numerical methods. Therefore, experiments play a crucial role in verifying these approximate solutions.

Solutions of real problems usually involve a combination of analysis and experimental work. First, the real physical flow situation is approximated with a mathematical model that is simple enough to yield a solution. Then experimental measurements are made the check the analytical results. Based on the measurements, refinements are made in the analysis. The experimental results are an essential link in this iterative process.

The experimental work in the laboratory is both time consuming and expensive. The obvious goal is to obtain the most information from the fewest experiments.

The dimensional analysis is an important tool that often helps in achieving this goal. Dimensional analysis is packaging or compacting technique used to reduce the complexity of experimental programs and at the same time increase the generality of experimental information.

Consider the drag force on a stationary smooth sphere immersed in a uniform stream. What experiments must be conducted to determine the drag force on the sphere?



We would expect the drag force, F, depends on diameter of the sphere, D, the fluid velocity, V, fluid viscosity, μ and the fluid density ρ. That is,

$$F = f(D, V, \rho, \mu)$$

Let us imagine a series of experiments to determine the dependence of F on the variables D, V, ρ and μ. To obtain a curve of F versus V for fixed values of ρ, μ and D, we might need tests at 10 values of V. To explore the diameter effect, each test would be repeat for spheres of ten different diameters. If the procedure were repeated for 10 values of ρ and μ in turn, simple arithmetic shows that 10^4 separate tests would be needed. Also we would have to find 100 different fluids. Because we need 10 different ρ's and 10 different μ's. Assuming each test takes ½ hour and we work 8 hours per day, the testing will require 2.5 years to complete.
Dimensional analysis comes to rescue. If we apply dimensional analysis, it reduces to the equivalent form.

$$\frac{F}{\rho V^2 D^2} = f_1\left(\frac{\rho V D}{\mu}\right)$$

The form of function still must be determined experimentally. However, rather than needing to conduct 10^4 experiments, we would establish the nature of function as accurately with only 10 tests.

BUCKINGHAM PI THEOREM

The dimensional analysis is based on the Buckingham Pi theorem. Suppose that in a physical problem, the dependent variable q_1 is a function of n-1 independent variables q_2, q_3, ….., q_n. Then the relationship among these variables may be expressed in the functional form as

$$q_1 = f(q_2, q_3, \dots, q_n)$$

Mathematically, we can express the functional relationship in the equivalent form.

$$g(q_1, q_2, q_3, \dots, q_n) = 0$$

where *g* is an unspecified function, and it is different from the function *f*. For the drag on sphere we wrote the symbolic equation

$$F = f(D, V, \rho, \mu)$$

We could just as well have written

$$g(F, D, V, \rho, \mu) = 0$$

The Buckingham Pi theorem states that, the *n* parameters may be grouped into *n-m* independent dimensionless ratios, or π parameters, expressible in functional form by

$$G(\pi_1, \pi_2, \dots, \pi_{n-m}) = 0$$

or

$$\pi_1 = G_1(\pi_2, \pi_3, \dots, \pi_{n-m})$$

The number m is usually, but not always, equal to the minimum number of independent dimensions required to specify the dimensions of all the parameters, q_1, q_2, ….., q_n.

DETERMING THE Π GROUPS

To determine the π parameters, the steps listed below should be followed.

Step 1

Select all the parameters that affect a given flow phenomenon and write the functional relationship in the form

$$q_1 = f(q_2, q_3, \dots, q_n)$$

or

$$g(q_1, q_2, \dots, q_n) = 0$$

If all the pertinent parameters are not included, a relation may be obtained, but it will not give the complete story. If parameters that actually have no effect on the physical phenomenon are included, either the process of dimensional analysis will show that these do not enter the relation sought, or experiments will indicate that one or more nondimensional groups are irrelevant.

Step 2

List the dimensions of all parameters in terms of the primary dimensions which are the mass, M, the length, L, and the time, t (MLt), or the force, F, the length, L, and the time, t (FLt). Let "r" be the number of primary dimensions.

Step 3

Select a number of repeating parameters, equal to the number of primary dimensions, r, and including all the primary dimensions. As long as, the repeating parameter may appear in all of the nondimensional groups that are obtained, then do not include the dependent parameter among those selected in this step.

Step 4

Set up dimensional equation, combining the parameters selected in step 3 with each of the remaining parameters in turn, to form dimensionless groups. (There will be n-m equations). Solve the dimensional equation to obtain the (n-m) dimensionless groups.

Step 5

Check to see that each group obtained is dimensionless.

Example: The drag force, ***F***, on a smooth sphere, which is moving comparatively slowly through a viscous fluid, depends on the relative velocity, ***V***, the sphere diameter, ***D***, the fluid density, ***ρ***, and the fluid viscosity, ***μ***. Obtain a set of dimensionless groups that can be used to correlate experimental data.

Solution:

Step 1	*F*	*V*	*D*	ρ	μ	n = 5 parameters
Step 2	$\dfrac{ML}{t^2}$	$\dfrac{L}{t}$	L	$\dfrac{M}{L^3}$	$\dfrac{M}{Lt}$	r = 3 primary dimensions

Step 3 Select repeating parameters *ρ, V, D*

Step 4 Then, *n-m = 2 dimensionless groups* will result. Setting up dimensional equations, we obtain,

$$\Pi_1 = \rho^a V^b D^c F = \left[\left(\frac{M}{L^3}\right)^a \left(\frac{L}{t}\right)^b (L)^c \left(\frac{ML}{t^2}\right)\right] = \left[M^0 L^0 t^0\right]$$

Equating the exponents of *M*, *L*, and *t* results in

$$\left.\begin{array}{lll} M: & a+1=0 & a=-1 \\ L: & -3a+b+c+1=0 & c=-2 \\ t: & -b-2=0 & b=-2 \end{array}\right\} \text{Therefore, } \Pi_1 = \frac{F}{\rho V^2 D^2}$$

Similarly,

$$\Pi_2 = \rho^d V^e D^f \mu = \left[\left(\frac{M}{L^3}\right)^d \left(\frac{L}{t}\right)^e (L)^f \left(\frac{M}{Lt}\right)\right] = \left[M^0 L^0 t^0\right]$$

$$\left.\begin{array}{lll} M: & d+1=0 & d=-1 \\ L: & -3d+e+f-1=0 & f=-1 \\ t: & -e-1=0 & e=-1 \end{array}\right\} \text{Therefore, } \Pi_2 = \frac{\mu}{\rho VD}$$

Step 5: Check using *F*, *L*, *t* dimensions

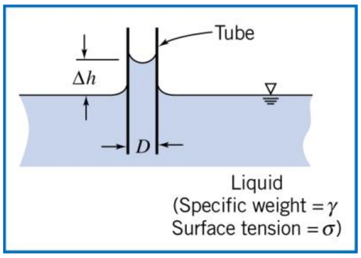
$$\Pi_1 = \frac{F}{\rho V^2 D^2} = \left[F\left(\frac{L^4}{Ft^2}\right)\left(\frac{t}{L}\right)^2\left(\frac{1}{L^2}\right)\right] = [1]$$

$$\Pi_2 = \frac{\mu}{\rho VD} = \left[\left(\frac{Ft}{L^2}\right)\left(\frac{L^4}{Ft^2}\right)\left(\frac{t}{L}\right)\left(\frac{1}{L}\right)\right] = [1]$$

The functional relationship is

$$\Pi_1 = f(\Pi_2) \quad \text{or} \quad \frac{F}{\rho V^2 D^2} = f\left(\frac{\mu}{\rho V D}\right)$$

Example: When a small tube is dipped into a pool liquid, surface tension causes a meniscus to form at the free surface, which is elevated or depressed depending on the contact angle at the liquid-solid-gas interface. Experiments indicate that the magnitude of the capillary effect, $\boldsymbol{\Delta h}$, is a function of the tube diameter, $\boldsymbol{D}$, liquid specific weight, $\boldsymbol{\gamma}$, and surface tension, $\boldsymbol{\sigma}$. Determine the number of independent Π parameters that can be formed and obtain a set.



Solution:

Given: $\Delta h = f(D, \gamma, \sigma)$

Find: Determine the number of independent Π parameters and obtain a set of $\boldsymbol{\pi}$ parameters.

Step 1 Δh D γ σ $n = 4$ parameters

Step 2 Choose primary dimensions, use both *M, L, t* and *F, L, t* dimensions to illustrate the problem in determining *m.*

a) M, L, t

Δh	D	γ	σ
L	L	$\frac{M}{L^2t^2}$	$\frac{M}{t^2}$

r = 3 primary dimensions

b) F, L, t

Δh	D	γ	σ
L	L	$\frac{F}{L^3}$	$\frac{F}{L}$

r = 2 primary dimensions

Thus for each primary set of dimensions we ask, “Is m equal to r?” Let us check each dimensional matrix to find out. The dimensional matrices are,

	Δh	D	γ	σ
M	0	0	1	1
L	1	1	-2	0
t	0	0	-2	-2

	Δh	D	γ	σ
F	0	0	1	1
L	1	1	-3	-1

The rank of a matrix is equal to the order of its largest nonzero determinant.

$$\begin{vmatrix} 0 & 1 & 1 \\ 1 & -2 & 0 \\ 0 & -2 & -2 \end{vmatrix} = 0-(1)(-2)+(1)(-2)$$

$=0$

$$\begin{vmatrix} -2 & 0 \\ -2 & -2 \end{vmatrix} = 4 \neq 0$$

$\therefore$ m = 2

m $\neq$ r

$$\begin{vmatrix} 1 & 1 \\ -3 & -1 \end{vmatrix} = -1+3 = 2 \neq 0$$

$\therefore$ m = 2

m = r

*Alternatively, you may use reduced row echelon form of the matrix to determine the rank of the matrix. The number of nonzero rows of the reduced row echelon matrix give the rank of that matrix.

Step 3

m = 2. Choose **D, γ** as repeating parameters.	m = 2. Choose **D, γ** as repeating parameters.

Step 4

n-m = 2 dimensionless groups will result

$$\Pi_1 = D^a \gamma^b \Delta h$$

$$= \left[(L)^a \left(\frac{M}{L^2 t^2} \right)^b (L) \right] = \left[M^0 L^0 t^0 \right]$$

$$\left.\begin{aligned} M: &\quad b+0=0 \\ L: &\quad a-2b+1=0 \\ t: &\quad -2b-0=0 \end{aligned}\right\} \quad \begin{aligned} b&=0 \\ a&=-1 \end{aligned}$$

Therefore, $\Pi_1 = \dfrac{\Delta h}{D}$

$$\Pi_2 = D^c \gamma^d \sigma$$

$$= \left[(L)^c \left(\frac{M}{L^2 t^2} \right)^d \left(\frac{M}{t^2} \right) \right] = \left[M^0 L^0 t^0 \right]$$

$$\left.\begin{aligned} M: &\quad d+1=0 \\ L: &\quad c-2d=0 \\ t: &\quad -2d-2=0 \end{aligned}\right\} \quad \begin{aligned} d&=-1 \\ c&=-2 \end{aligned}$$

Therefore, $\Pi_2 = \dfrac{\sigma}{D^2 \gamma}$

n-m = 2 dimensionless groups will result

$$\Pi_1 = D^e \gamma^f \Delta h$$

$$= \left[(L)^e \left(\frac{F}{L^3} \right)^f (L) \right] = \left[F^0 L^0 t^0 \right]$$

$$\left.\begin{aligned} F: &\quad f=0 \\ L: &\quad e-3f+1=0 \end{aligned}\right\} \quad \begin{aligned} f&=0 \\ e&=-1 \end{aligned}$$

Therefore, $\Pi_1 = \dfrac{\Delta h}{D}$

$$\Pi_2 = D^g \gamma^h \sigma$$

$$= \left[(L)^g \left(\frac{F}{L^3} \right)^h \left(\frac{F}{L} \right) \right] = \left[F^0 L^0 t^0 \right]$$

$$\left.\begin{aligned} F: &\quad h+1=0 \\ L: &\quad g-3h-1=0 \end{aligned}\right\} \quad \begin{aligned} h&=-1 \\ g&=-2 \end{aligned}$$

Therefore, $\Pi_2 = \dfrac{\sigma}{D^2 \gamma}$

Step 5

Check using *F, L, t* dimensions

$$\Pi_1 = \frac{\Delta h}{D} = \left[\frac{L}{L} \right] = [1]$$

$$\Pi_2 = \frac{\sigma}{D^2 \gamma} = \left[\frac{F}{L} \frac{1}{L^2} \frac{L^3}{F} \right] = [1]$$

Check using *M, L, t* dimensions

$$\Pi_1 = \frac{\Delta h}{D} = \left[\frac{L}{L} \right] = [1]$$

$$\Pi_2 = \frac{\sigma}{D^2 \gamma} = \left[\frac{M}{t^2} \frac{1}{L^2} \frac{L^2 t^2}{M} \right] = [1]$$

Therefore, both systems of dimensions yield the same dimensionless Π parameters. The functional relationship is

$$\Pi_1 = f(\Pi_2) \quad \text{or} \quad \frac{\Delta h}{D} = f\left(\frac{\sigma}{D^2 \gamma} \right)$$

DIMENSIONLESS GROUPS OF SIGNIFICANCE IN FLUID MECHANICS

There are several hundred dimensionless groups in engineering. Following tradition, each such group has been given the name of a prominent scientist or engineer, usually the one who pioneered its use.

Forces encountered in the flowing fluids include those due to inertia, viscosity, pressure, gravity, surface tension, and compressibility. The ratio of any two forces will be dimensionless. We can estimate typical magnitudes of some of these forces in a flow:

$$\text{Inertia force} = m\vec{a} = m\frac{D\vec{V}}{Dt} = m\left[u\frac{\partial u}{\partial x}\ldots\right] \propto \rho L^3 V\frac{V}{L} = \rho L^2 V^2$$

$$\text{Viscous force} = \tau A = \mu\frac{du}{dy}A \propto \mu\frac{V}{L}L^2 = \mu VL$$

$$\text{Pressure force} = (\Delta p)A \propto \Delta p L^2$$

$$\text{Gravity force} = mg \propto g\rho L^3$$

$$\text{Surface tension force} = \sigma L$$

$$\text{Compressibility force} = E_v A \propto E_v L^2$$

$$E_v \equiv \frac{dp}{d\rho/\rho}$$

Inertia forces are important in most fluid mechanics problems. The ratio of the inertia force to each of other forces listed above leads to five fundamental groups encountered in fluid mechanics.

The Reynolds number is the ratio of inertia forces to the viscous forces, and it is named after Osbourne Reynolds (1842 - 1912).

$$\text{Reynolds number} = \text{Re} = \frac{\text{Inertia force}}{\text{Viscous force}} = \frac{\rho V^2 L^2}{\mu VL} = \frac{\rho VL}{\mu} = \frac{VL}{\nu}$$

$$\text{Euler number} = \text{Eu} = f\left(\frac{\text{Pressure force}}{\text{Inertia force}}\right) = \frac{\Delta p L^2}{\frac{1}{2}\rho V^2 L^2} = \frac{\Delta p}{\frac{1}{2}\rho V^2}$$

where Δp is the pressure difference between local pressure and the freestream pressure.

$$\Delta p = p - p_\infty$$

$$\text{Froude number} = \text{Fr} = f\left(\frac{\text{Inertia force}}{\text{Gravity force}}\right) = \left(\frac{\rho V^2 L^2}{\rho g L^3}\right)^{1/2} = \frac{VL}{\sqrt{gL}}$$

$$\text{Weber number} = \text{We} = f\left(\frac{\text{Inertia force}}{\text{Surface tension force}}\right) = \frac{\rho V^2 L^2}{\sigma L} = \frac{\rho V^2 L^2}{\sigma L}$$

$$\text{Mach number} = \text{M} = f\left(\frac{\text{Inertia force}}{\text{Compressibility force}}\right) = \left(\frac{\rho V^2 L^2}{E_\upsilon L^2}\right)^{1/2} = \frac{V}{\sqrt{\frac{E_\upsilon}{\rho}}} = \frac{V}{c}$$

where c is the local sonic speed.

FLOW SIMILARITY AND MODEL STUDIES

When an object, which is in original sizes, is tested in laboratory it is called **prototype**. A **model** is a scaled version of the prototype. A model which is typically smaller than its prototype is economical, since it costs little compared to its prototype. The use of the models is also practical, since environmental and flow conditions can be rigorously controlled. However, models are not always smaller than the prototype. As an example, the flow in a carburetor might be studied in a very large model.

There are three basic laws of similarity of model and prototype flows. All of them must be satisfied for obtaining complete similarity between fluid flow phenomena in a prototype and in a model. These are

a) The geometric similarity,
b) the kinematic similarity, and
c) the dynamic similarity.

Geometric Similarity: The geometric similarity requires that the model and prototype be identical in shape but differ in size. Therefore, ratios of the corresponding linear dimensions in the prototype and in the model are the same.

Kinematic Similarity: The kinematic similarity implies that the flow fields in the prototype and in the model must have geometrically similar sets of streamlines. The velocities at corresponding points are in the same direction and are related in magnitude by a constant scale factor.

Dynamic Similarity: When two flows have force distributions such that identical types of forces are parallel and are related in magnitude by a constant scale factor at all corresponding points, the flows are dynamically similar.

By using Buckingham Π theorem, we can find which dimensionless groups are important for a given flow phenomenon. To achieve dynamic similarity between geometrically similar flows, we must duplicate all of these dimensionless groups.

For example, in considering the drag force on sphere we found that

$$\frac{F}{\rho V^2 D^2} = f_1\left(\frac{\rho V D}{\mu}\right) = f_1(\text{Re})$$

Thus in considering a model flow and prototype flow about a sphere, the flows will be dynamically similar if

$$\left(\frac{\rho VD}{\mu}\right)_{\text{model}} = \left(\frac{\rho VD}{\mu}\right)_{\text{prototype}}$$

that is

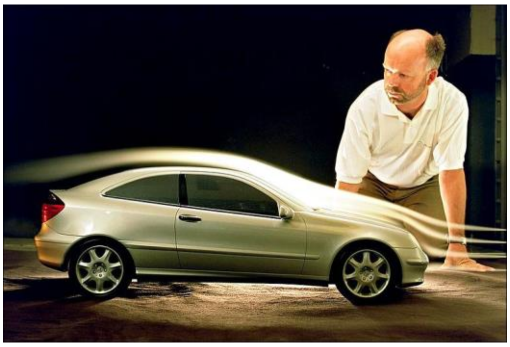
$$\text{Re}_{\text{model}} = \text{Re}_{\text{prototype}}$$

then

$$\left(\frac{F}{\rho V^2 D^2}\right)_{\text{model}} = \left(\frac{F}{\rho V^2 D^2}\right)_{\text{prototype}}$$

The results determined from the model study can be used to predict the drag on the full scale prototype.

Example: A **one-tenth-scale model** of a derby car, shown in the figure, is tested in a wind tunnel. The air speed in the wind tunnel is **70 m/s**, the air drag on the model car is **240 N**, and the air temperature and pressure are identical those expected when the prototype car is racing. Find the corresponding racing speed in still air and the drag on the car.



Solution:

The functional relation for the drag force can be found by applying Buckingham-Π theorem such that

$$\frac{F_D}{\rho V^2 L^2} = f\left(\frac{\rho VL}{\mu}\right) \qquad \text{Re} = \frac{\rho VL}{\mu}$$

and the test should be run at

$$\text{Re}_{\text{model}} = \text{Re}_{\text{prototype}}$$

$$\frac{\rho_m V_m L_m}{\mu_m} = \frac{\rho_p V_p L_p}{\mu_p}$$

to ensure dynamic similarity The problem statements show that $\rho_m = \rho_p$ and $\mu_m = \mu_p$. Then,

$$V_p = V_m \frac{L_m}{L_p}\frac{\rho_m}{\rho_p}\frac{\mu_p}{\mu_m} \qquad \Rightarrow V_p = V_m \frac{L_m}{L_p}$$

$$V_p = V_m \frac{L_m}{L_p} = 70\frac{1}{10} = 7\,\text{m/s}$$

This speed is low enough to neglect compressibility effects. At these test conditions, the model and the prototype flows are dynamically similar. Hence,

$$\left.\frac{F_D}{\rho V^2 L^2}\right)_m = \left.\frac{F_D}{\rho V^2 L^2}\right)_p$$

and

$$F_{Dp} = F_{Dm}\left(\frac{\rho_p}{\rho_m}\right)\left(\frac{V_p^2 L_p^2}{V_m^2 L_m^2}\right) = F_{Dm}\left(\frac{V_p L_p}{V_m L_m}\right)^2$$

$$F_{Dp} = F_{Dm}\left(\frac{V_p L_p}{V_m L_m}\right)^2 = 240\left(\frac{7}{70}\frac{10}{1}\right)^2 = 240\text{N}$$

Example: A jet plane travelling at a velocity of **900 m/s at 6 km altitude**, where the temperature and the pressure are **-24 °C** and **47.22 kPa**, respectively. A **one-tenth scale model** of the jet is tested in a wind tunnel in which **carbon dioxide** is flowing. The gas constant for air and carbon dioxide are **287 J/kg K** and **18.8 J/kgK**, respectively. The specific heat ratios for air and carbon dioxide are **1.4** and **1.28**, respectively. Also the absolute viscosities of the air at -24 °C and carbon dioxide at 20 °C are **1.6×10^{-5} Pa.s** and **1.47×10^{-5} Pa.s**, respectively.

Determine
a)The required velocity in the model, and
b)The pressure required in the wind tunnel.

Solution:

a) As long as the model jet plane is moving in a compressible fluid, then a free surface does not exist. Therefore, it is not necessary to concern either with the wave or surface tension effects. The Froude and the Weber numbers play no role for the dynamic similarity. In order to achieve dynamic similarity, the Reynolds numbers and Mach numbers must be equal on the model and on the prototype.

$$M_p = \frac{V_p}{c_p} = \frac{V_m}{c_m} = M_m$$

Then the velocity of the model jet plane is

$$V_m = V_p\frac{c_p}{c_m} = V_p\left(\frac{k_m R_m T_m}{k_p R_p T_p}\right)^{1/2} = 900\left(\frac{1.28\times187.8\times293}{1.4\times287\times249}\right)^{1/2} = 755.14\,\text{m/s}$$

b) The other requirement for the dynamic similarity is the equality of the Reynolds numbers

$$\mathrm{Re_p} = \frac{\rho_p V_p L_p}{\mu_p} = \frac{\rho_m V_m L_m}{\mu_m} = \mathrm{Re_m}$$

The density of air may be evaluated by using equation of state for a perfect gas

$$\rho_p = \frac{p_p}{R_p T_p} = \frac{47220}{287 \times 249} = 0.661 \ \frac{\mathrm{kg}}{\mathrm{m^3}}$$

Now, required density of the carbon dioxide may be evaluated as

$$\rho_m = \rho_p \frac{L_p V_p \mu_p}{L_m V_m \mu_m} = 0.661 \times 10 \times \frac{900 \times 1.47 \times 10^{-5}}{755.14 \times 1.60 \times 10^{-5}} = 7.24 \ \frac{\mathrm{kg}}{\mathrm{m^3}}$$

Finally, the required pressure of the carbon dioxide is

$$p_m = \rho_m R_m T_{pm} = 7.24 \times 187.7 \times 293 = 398.38 \ \mathrm{kPa}$$

INCOMPLETE SIMILARITY

To achieve complete dynamic similarity between geometrically similar flows all of the dimensionless numbers in prototype and in the model (that is *Re, Eu, Fr, We, M,..*) should be equal.

Fortunately, in most engineering problems, the equality of all of dimensionless groups is not necessary. Since some of forces

i. may not act
ii. may be negligible magnitude or
iii. may oppose other forces in such a way that the effect of both is reduced.

In some cases, complete dynamic similarity may not be attainable. Determining the drag force of surface ship is on example of such a situation. The viscous shear stress and surface wave resistance cause the drag. So that for complete dynamic similarity, both Reynolds and Froude numbers must be equal between model and prototype. This requires that

$$Fr_m = \frac{V_m}{(gL_m)^{\frac{1}{2}}} = Fr_p = \frac{V_p}{(gL_p)^{\frac{1}{2}}}$$

$$\frac{V_m}{V_p} = \left(\frac{L_m}{L_p}\right)^{\frac{1}{2}}$$

To ensure dynamically similar surface wave patterns.

From the Reynolds number requirement

$$Re_m = \frac{V_m L_m}{\nu_m} = Re_p = \frac{V_p L_p}{\nu_p}$$

$$\frac{\nu_m}{\nu_p} = \frac{V_m}{V_p}\frac{L_m}{L_p}$$

If we use the velocity ratio obtained from matching Froude numbers, equality of Reynolds number leads to a kinematic viscosity ratio of

$$\frac{\nu_m}{\nu_p} = \left(\frac{L_m}{L_p}\right)^{\frac{1}{2}}\left(\frac{L_m}{L_p}\right) = \left(\frac{L_m}{L_p}\right)^{\frac{3}{2}}$$

If L_m/L_p equals 1/100 (a typical length scale for ship model tests), then $\nu_m/\ \nu_p$ must be 1/1000. Mercury, which is the only liquid, its kinematic viscosity is less than water. Thus, we cannot simultaneously match Reynolds number and Froude number in the scale-model test. Then one is forced to choose ether the Froude number similarity, or the Reynolds number similarity. For this reason, the experiments with the model are performed so that $Fr_p = Fr_m$ which results $Re_p >> Re_m$. The test results are then corrected by using the experimental data which is dependent on the Reynolds number.

Example: The drag force on a submarine, which is moving on the surface, is to be determined by a test on a model which is scaled down to **one-twentieth of the prototype**. The test is to be carried in a towing tank, where the model submarine is moved along channel of liquid. The density and the kinematic viscosity of the seawater are **1010 kg/m³** and **1.3×10⁻⁶ m²/s** respectively. The speed of the prototype is **2.6 m/s**.

a) Determine the speed at which the model should be moved in the towing tank.
b) Determine the kinematic viscosity of the liquid that should be used in the towing tank.
c) If such a liquid is not available, then the test may be carried out with seawater by neglecting the viscous effects. In this case, determine the ratio of the drag force due to the surface waves in the prototype to the drag force in the model.

a) Because of low speed of the submarine, the compressibility has no effect on the dynamic similarity, and the Mach number plays no role.
The Froude numbers for the prototype and the model may be equated to yield.

$$Fr_p = \frac{V_p}{(gL_p)^{\frac{1}{2}}} = \frac{V_m}{(gL_m)^{\frac{1}{2}}} = Fr_m$$

$$V_m = V_p \left(\frac{L_m}{L_p}\right)^{\frac{1}{2}} = 2.6\left(\frac{1}{20}\right)^{\frac{1}{2}} = 0.58\, m/s$$

b) To determine the kinetic viscosity of the liquid that should be used in the towing tank, one may equate the Reynolds number in the model and prototype.

$$Re_p = \frac{V_p L_p}{\nu_p} = \frac{V_m L_m}{\nu_m} = Re_m$$

Rearranging one may obtain

$$\nu_m = \nu_p \left(\frac{V_m}{V_p}\right)\left(\frac{L_m}{L_p}\right) = 1.3\text{x}10^{-6}\left(\frac{0.58}{2.6}\right)\left(\frac{1}{20}\right) = 1.45\text{x}10^{-8}\, m^2/s$$

c) However, one should note that a liquid with a given kinematic viscosity cannot be practically formed. Then the test in towing tank may be carried out with seawater by neglecting the viscous effects. In this case, only the equality of the Froude number is sufficient for the dynamic similarity and the drag force is only due to the surface waves.
By using Buckingham π theorem one may obtain.

$$\frac{F}{\rho V^2 L^2} = f(Re, Fr)$$

But in this case only the equality of the Froude number is sufficient, then

$$\frac{F}{\rho V^2 L^2} = f(Fr)$$

$Fr_{model} = Fr_{prototype}$

$$\left(\frac{F}{\rho V^2 L^2}\right)_{model} = \left(\frac{F}{\rho V^2 L^2}\right)_{prototype}$$

$$\frac{F_p}{F_m} = \frac{\rho_p}{\rho_m} \frac{{V_p}^2}{{V_m}^2} \frac{{L_p}^2}{{L_m}^2}$$

$$V_m = V_p \left(\frac{L_m}{L_p}\right)^{\frac{1}{2}} \quad \Rightarrow \quad \frac{V_m}{V_p} = \left(\frac{L_m}{L_p}\right)^{\frac{1}{2}}$$

$$\frac{F_p}{F_m} = \frac{L_p}{L_m}\left(\frac{L_p}{L_m}\right)^2 = \left(\frac{L_p}{L_m}\right)^3 = 20^3 = 8000$$

This result must be corrected for viscous effects.