Solution Cases:

- 1. Unique Optimal Solution
- 2. Alternative Optimal Solutions
- 3. Infeasible solution Case
- 4. Unbounded Solution Case
- 5. Degenerate Optimal Solution Case

- **1. Unique Optimal Solution**
- **Reddy Mikks Example**
- **Diet Problem**

2. Alternative Optimal Solutions (Infinity of Solutions) (Multiple Optimal Solutions)

Example (Taha OR Book pp 106)

LP: Max $Z=2x_1+4x_2$

St

$x_1+\ x_2 \leq 4$ (1)

$x_1+2x_2 \leq 5$ (2)

$x_1 \geq 0,\ x_2 \geq 0$ (3,4)

B(0,5/2)

C(3,1)

WHAT IS THE MEANING IN PRACTICE?

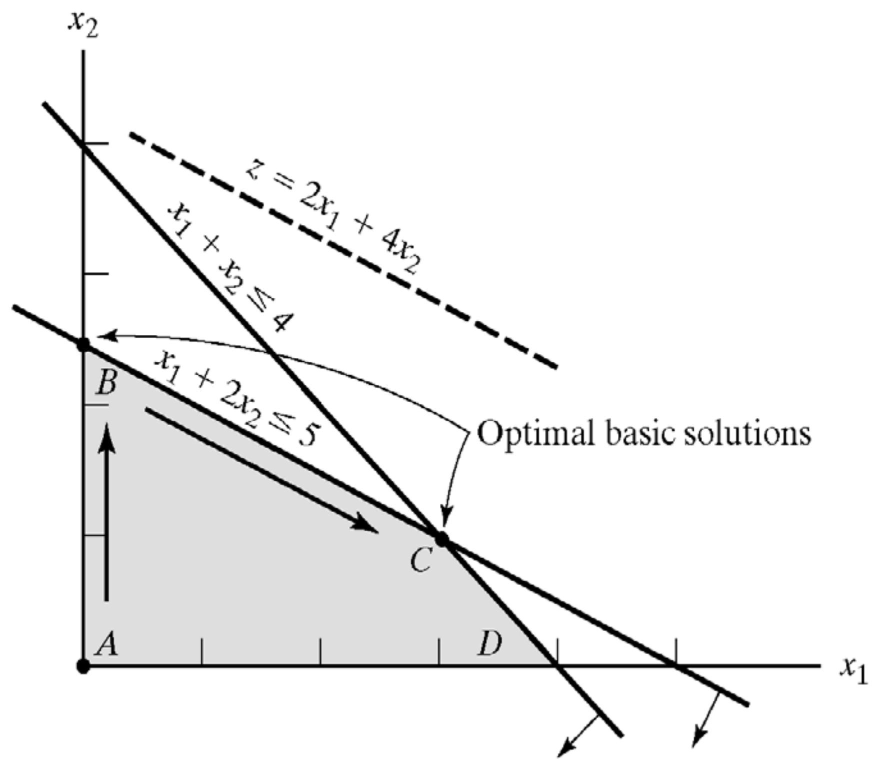

Figure 3.9

LP alternative optima in Example 3.5-2.

- We can determine all the points (x_1, x_2) on the line segment BC as a nonnegative weighted average of the points B and C .
- Thus given $0 \le \alpha \le 1$ and
- B: $x_1=0$, $x_2=5/2$
- C: $x_1=3$, $x_2=1$
- Then all the points on the line segment BC are given by

$$x_1^{'} = \alpha(0) + (1-\alpha)(3) = 3 - 3\alpha$$

$$x_2^{'} = \alpha(5/2) + (1-\alpha)(1) = 1 + (3/2)\alpha$$

3. Infeasible solution case

It is possible for an LP's feasible region to be empty (contain no points), resulting in an infeasible LP . Since the optimal solution to an LP is the best point in the feasible region, an infeasible LP has no optimal solution.

Example (Taha OR Book pp 111)

Max Z=3x1+2x2

St

3x1+4x2 ≥12 (1)

2x1+x2 ≤2 (2)

x1≥0, x2≥0 (3,4)

WHAT WE SHOULD DO
IN SUCH A CASE?

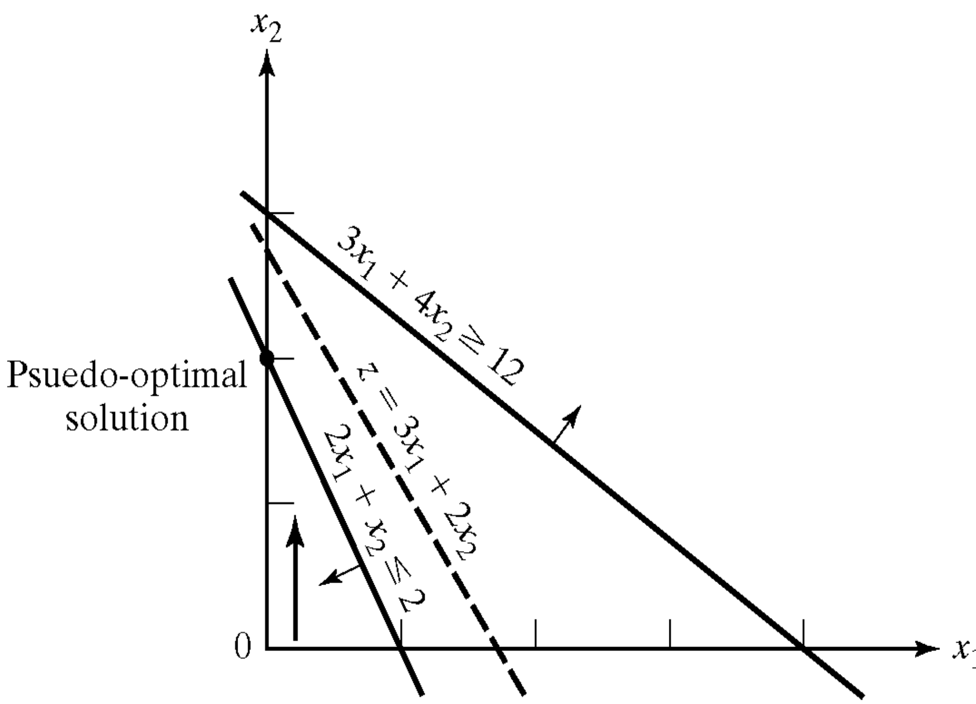

Figure 3.11
Infeasible solution of Example 3.5-4.

4. Unbounded Solution Case

For a maximization problem, an unbounded LP occurs if it is possible to find points in the feasible region with arbitrary large Z-values, corresponding to decision maker earning arbitrarily large revenues or profits.

For a minimization problem, an LP is unbounded if there are points in the feasible region with arbitrarily small Z-values.

•Example (Taha OR Book pp 109)

Max Z=2x1+x2
St
x1- x2 ≤10 (1)
2x1 ≤40 (2)
x1, x2 ≥0 (3,4)

IN A CORRECTLY MODELLED LP, IT SHOULD NOT OCCUR

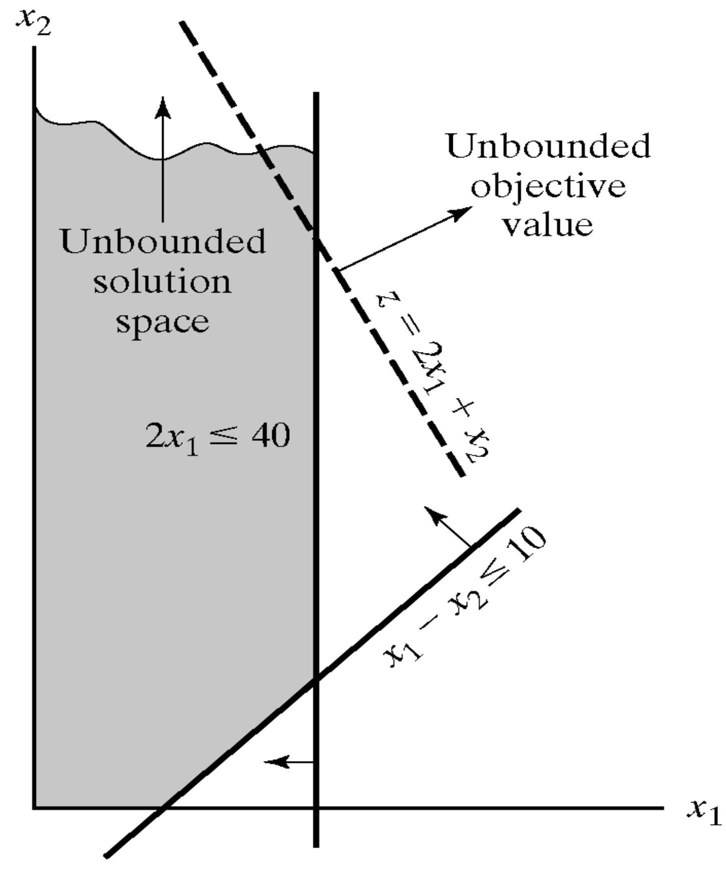

Figure 3.10
LP unbounded solution in Example 3.5-3.

5. Degenerate Optimal Solution

Max $z = 3x_1 + 9x_2$

s.t.

$x_1 + 4x_2 \leq 8$

$x_1 + 2x_2 \leq 4$

$x_1, x_2 \geq 0$

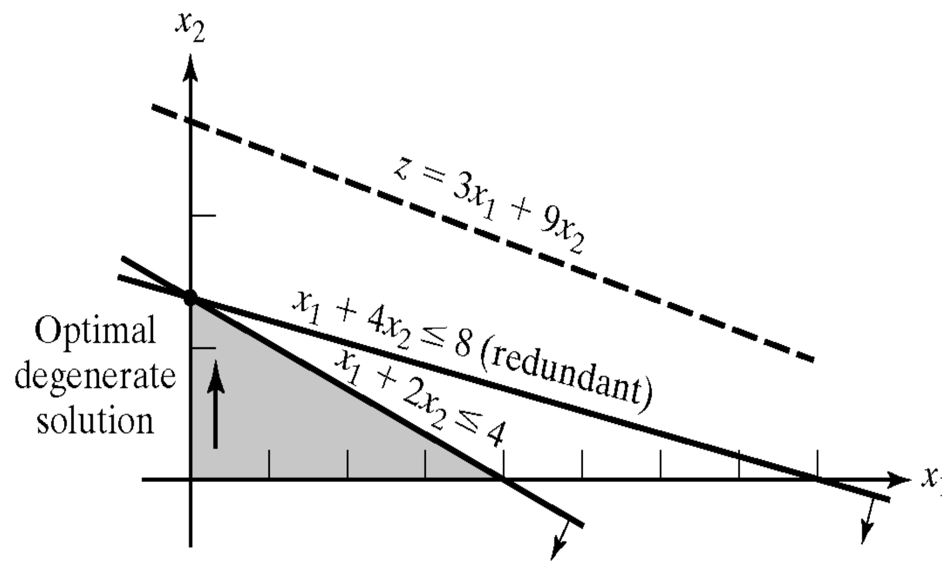

Figure 3.7
LP degeneracy in Example 3.5-1.

TELL YOUR COMMENTS RELATED WITH THIS CASE

Graphical Sensitivity Analysis- Post Optimality Analysis

- This section investigates two cases of sensitivity analysis based on the graphical LP solution
- **(1)** Changes in the objective function coefficients
- **(2)** Changes in the RHS of the constraints

(1) Changes in the objective function coefficients

- The general objective function in a two variable LP problem can be written as

$$\max(\text{or } \min)\ Z = c_1x_1 + c_2x_2$$

- Changes in the coefficients will change the slope of Z, so possibly the optimal corner point. But there is a range of variation for both c_1 and c_2 that will keep the current optimum unchanged.

- Specifically, we are interested in determining the range of optimality for the ratio (c_1/c_2) or (c_2/c_1) that will keep the current optimum unchanged.

Graphical Sensitivity Analysis

- **Example (Reddy Mikks Company, Taha's OR Book, 7nd Ed. pp. 12):**
- Reddy Mikks produces both interior and exterior paints from two raw materials, M1 and M2. The table below provides the basic data of the problem. A market survey indicates that the daily demand for interior paint cannot exceed that of exterior paint by more than 1 ton. Also, the maximum daily demand of interior paint is 2 tons. Reddy Mikks wants to determine the optimum (best) product mix of interior and exterior paints that maximizes the total daily profit.

	Tons of raw material per ton of		
	Exterior Paint	**Interior Paint**	**Maximum daily Availability (tons)**
Raw Material, M1	6	4	24
Raw Material, M2	1	2	6
Profit per ton ($1000)	5	4	

Max Z=5x1+4x2

st

$$
\begin{aligned}
6x1+4x2 &\le 24 \\
x1+2x2 &\le 6 \\
-x1+\ x2 &\le 1 \\
x2 &\le 2 \\
x1,x2 &\ge 0
\end{aligned}
$$

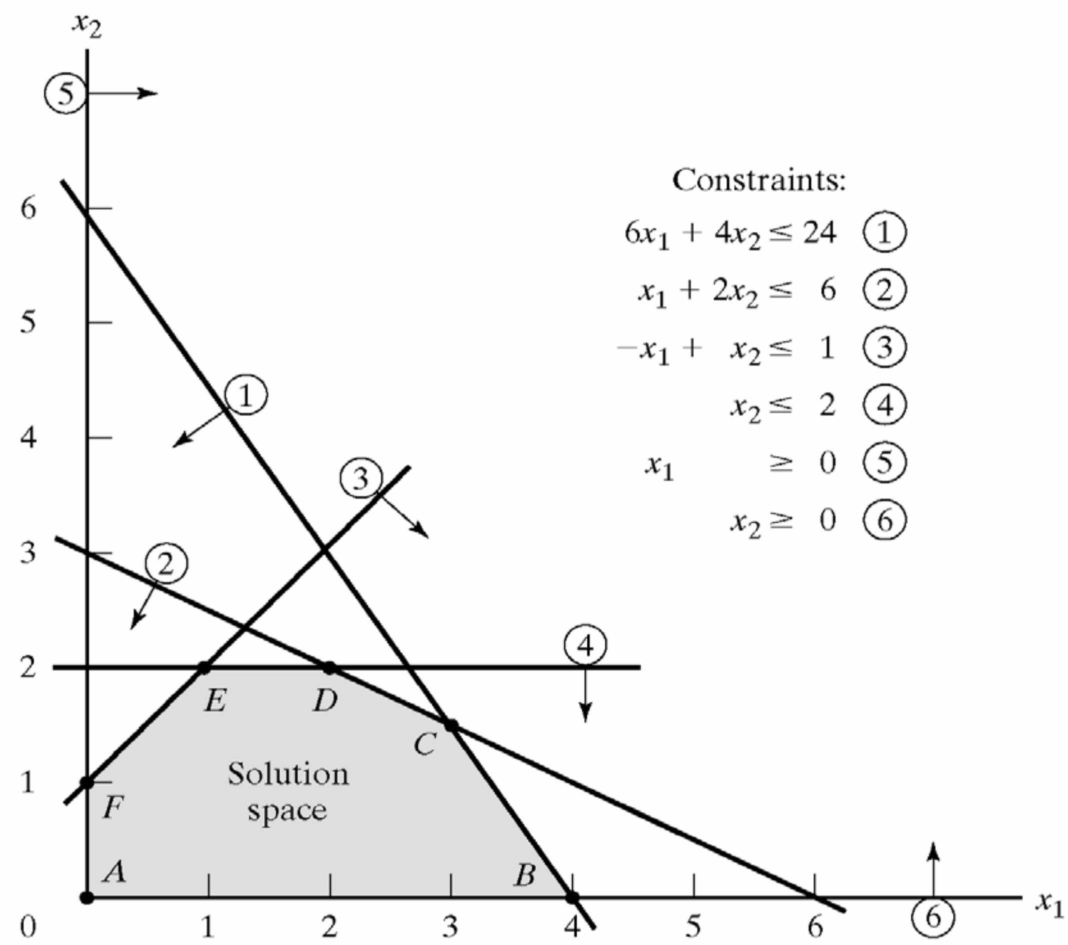

Figure 2.1
Feasible space of the Reddy Mikks model.

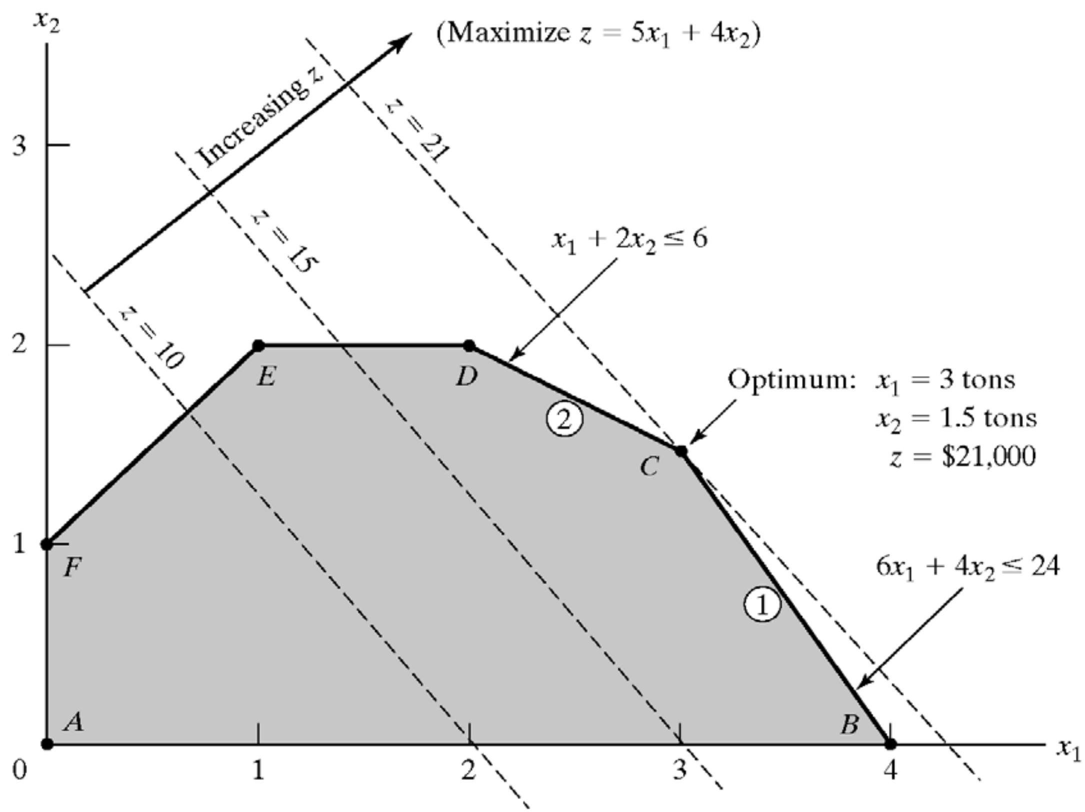

Figure 2.2
Optimum solution of the Reddy Mikks model.

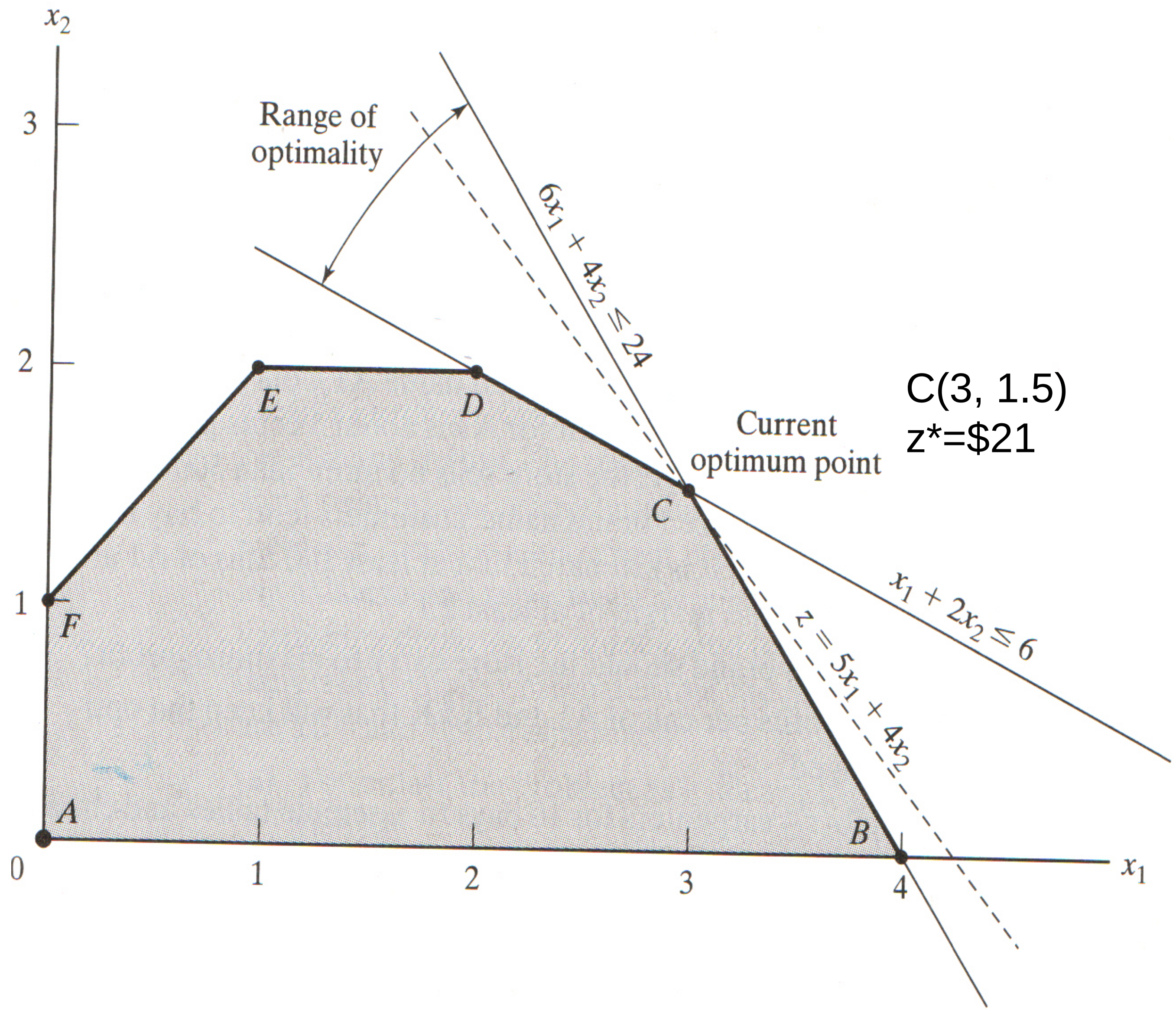

FIGURE 2.5

Range of optimality for the Reddy Mikks model

- <u>If the slope of z lies between the slopes of line (1) and line (2) then the optimal solution will remain same.</u>

- **How can we represent this relationship algebraically?**

- <u>**Condition (1)**</u>
- if $c_1 \neq 0$ then $(4/6) \leq (c_2/c_1) \leq (2/1)$
- <u>**Condition (2)**</u>
- if $c_2 \neq 0$ then $(1/2) \leq (c_1/c_2) \leq (6/4)$

- As long as the (c_2/c_1) ratio is within the (4/6) and (2/1) limits and
- As long as the (c_1/c_2) ratio is within the (1/2) and (6/4) limits then the optimum solution remains unchanged at C(3, 1.5) point

<u>1)What if c_1/c_2 is equal to 6/4?</u>
BC LINE SEGMENT GIVES THE ALTERNATIVE OPTIMAL SOLUTION
<u>2) What if c_1/c_2 is equal to ½?</u>
DC LINE SEGMENT GIVES THE ALTERNATIVE OPTIMAL SOLUTION

IN BOTH CASES C STILL REMAIN OPTIMAL

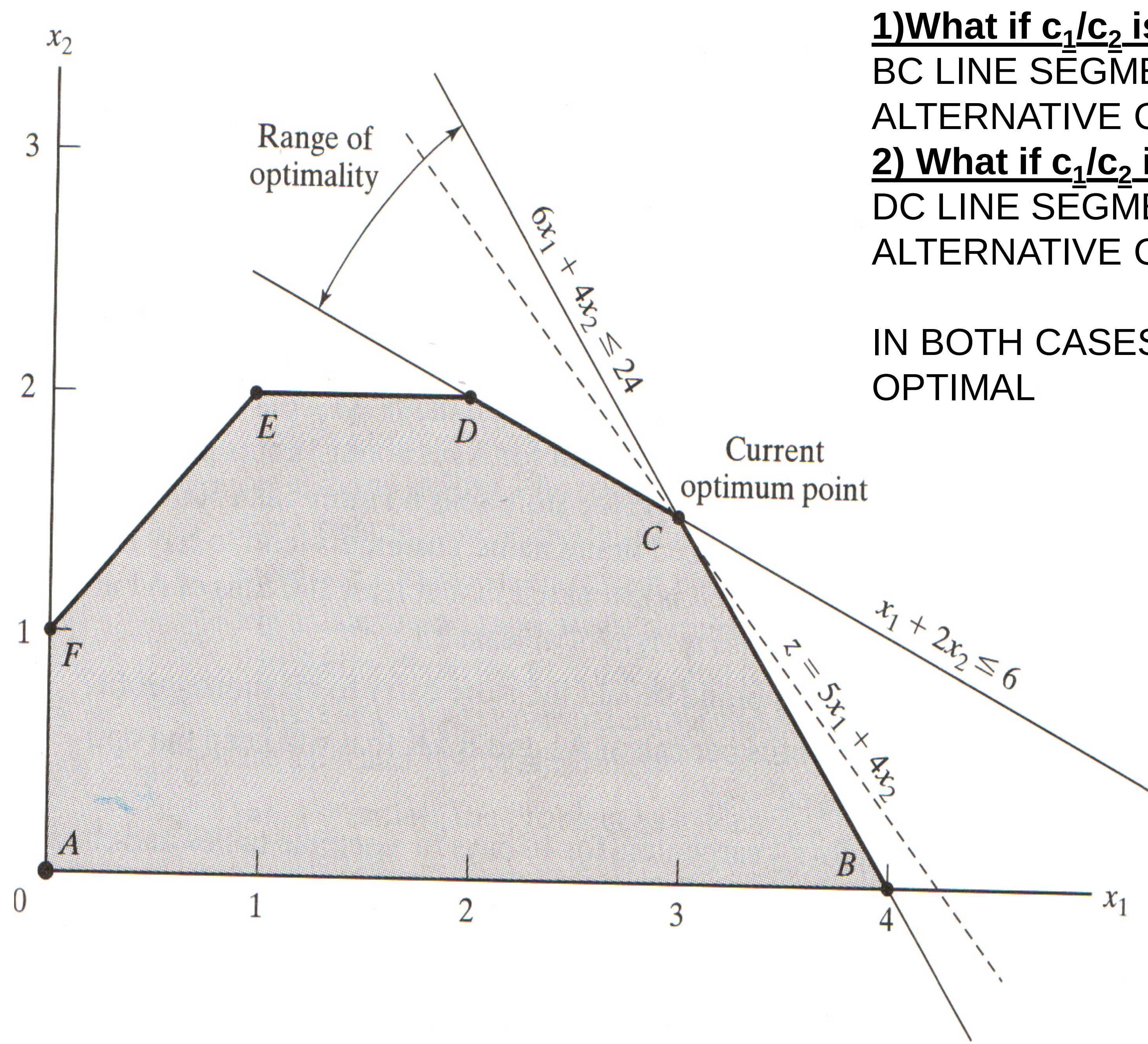

FIGURE 2.5
Range of optimality for the Reddy Mikks model

Calculation of the optimal range for one of the objective coefficient while the other remains same

- **Condition (1)** if $c_1 \neq 0$ then $(4/6) \leq (c_2/c_1) \leq (2/1)$
- **Condition (2)** if $c_2 \neq 0$ then $(1/2) \leq (c_1/c_2) \leq (6/4)$

LP: Max z= 5x1+4x2

s.t.

$6x_1 + 4x_2 \leq 24$

$x_1 + 2x_2 \leq 6$

$-x_1 + x_2 \leq 1$

$x_2 \leq 2$

$x_1, x_2 \geq 0$

- 1) If c_2 remains 4 find the range for c_1
- By substituting $c_2=4$ into the the condition (2)
- $(1/2) \leq (c_1/4) \leq (6/4) \rightarrow 2 \leq c_1 \leq 6$
- 2) If c_1 remains 5 find the range for c_2
- By substituting $c_1=5$ into the the condition (1)
- $(4/6) \leq (c_2/c_1) \leq (2/1) \rightarrow (10/3) \leq c_2 \leq 10$

Graphical Sensitivity Analysis

- (1) Changes in the objective function coefficients
- **(2) Changes in the RHS of the constraints**

(2) Changes in the RHS of the constraints

- Change in availability of scarce resources
- Determining the feasibility range of the scarce resources
  - In LP models, constraints, directly or indirectly represent the usage of limited resources.
  - In this case, the RHS can be thought of as representing limits on the availability of resources.
  - This section investigates the sensitivity of the optimum solution to making changes in the amount of available resources.

- In the sensitivity analysis that investigates the RHS of the constraints, we search for the range that keep the current solution feasible.
- That means, in that range the optimum solution occur at the intersection of the same constraints-but the values of the variables and the value of the objective function may change.

- The current optimum occurs at C which is the intersection of line (1) and line (2) associated with raw materials M1 and M2.
- M1 (current level=24) and M2 (current level=6) are scarce resources
- **<u>QUESTION</u>**
- **<u>Find the feasibility range of M1 given M2=6 tons.</u>**

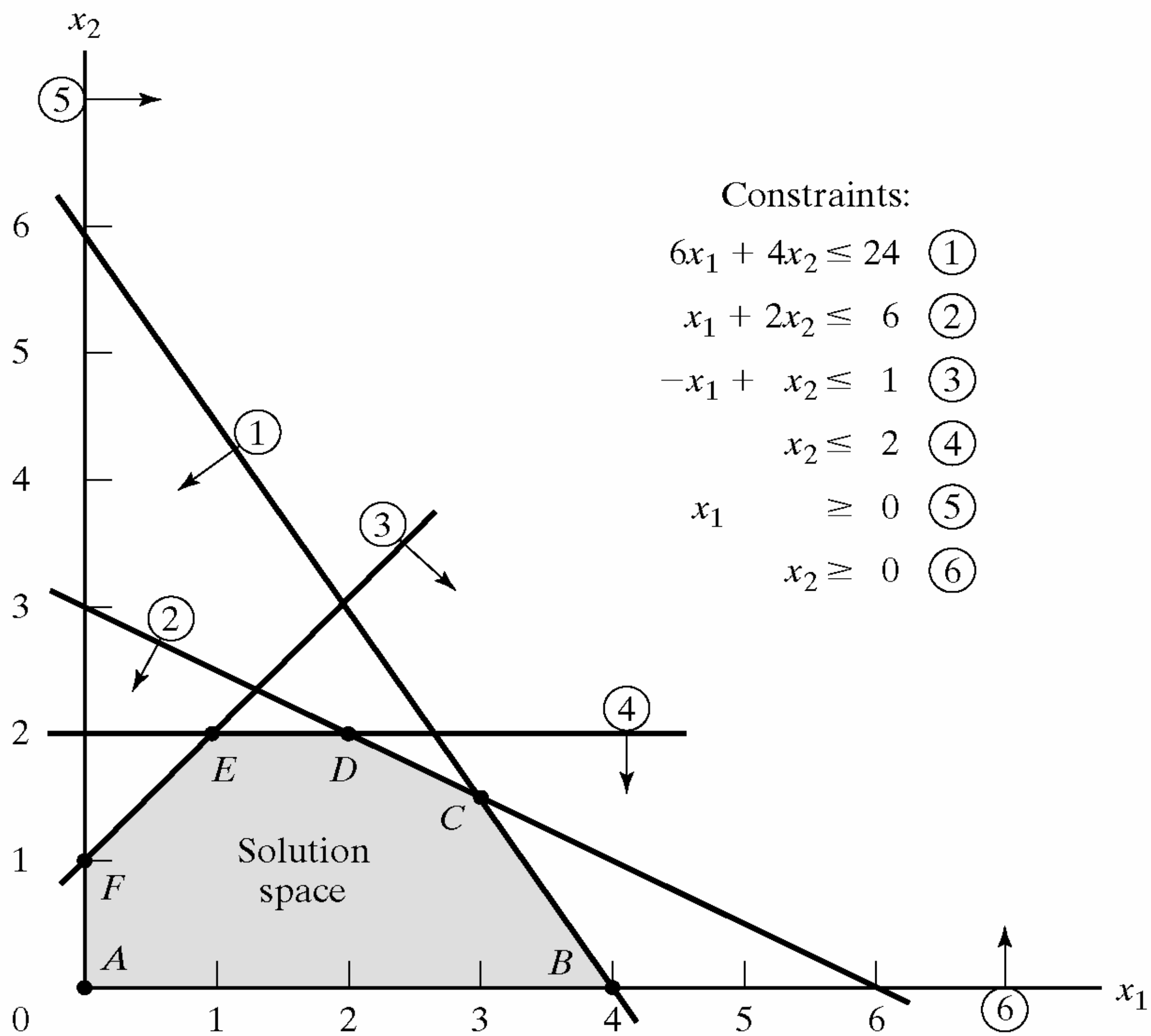

Figure 2.1

Feasible space of the Reddy Mikks model.

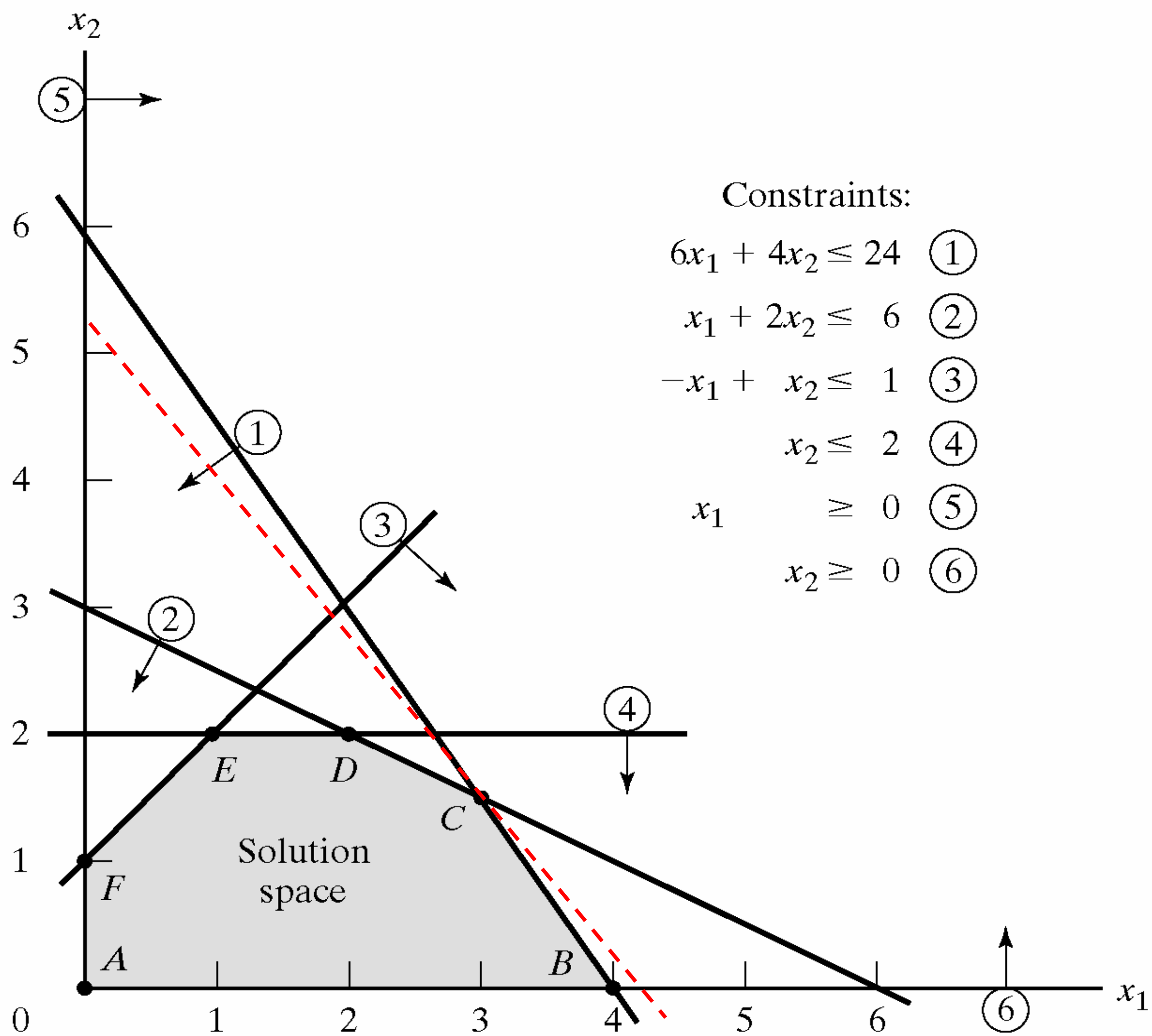

Figure 2.1

Feasible space of the Reddy Mikks model.

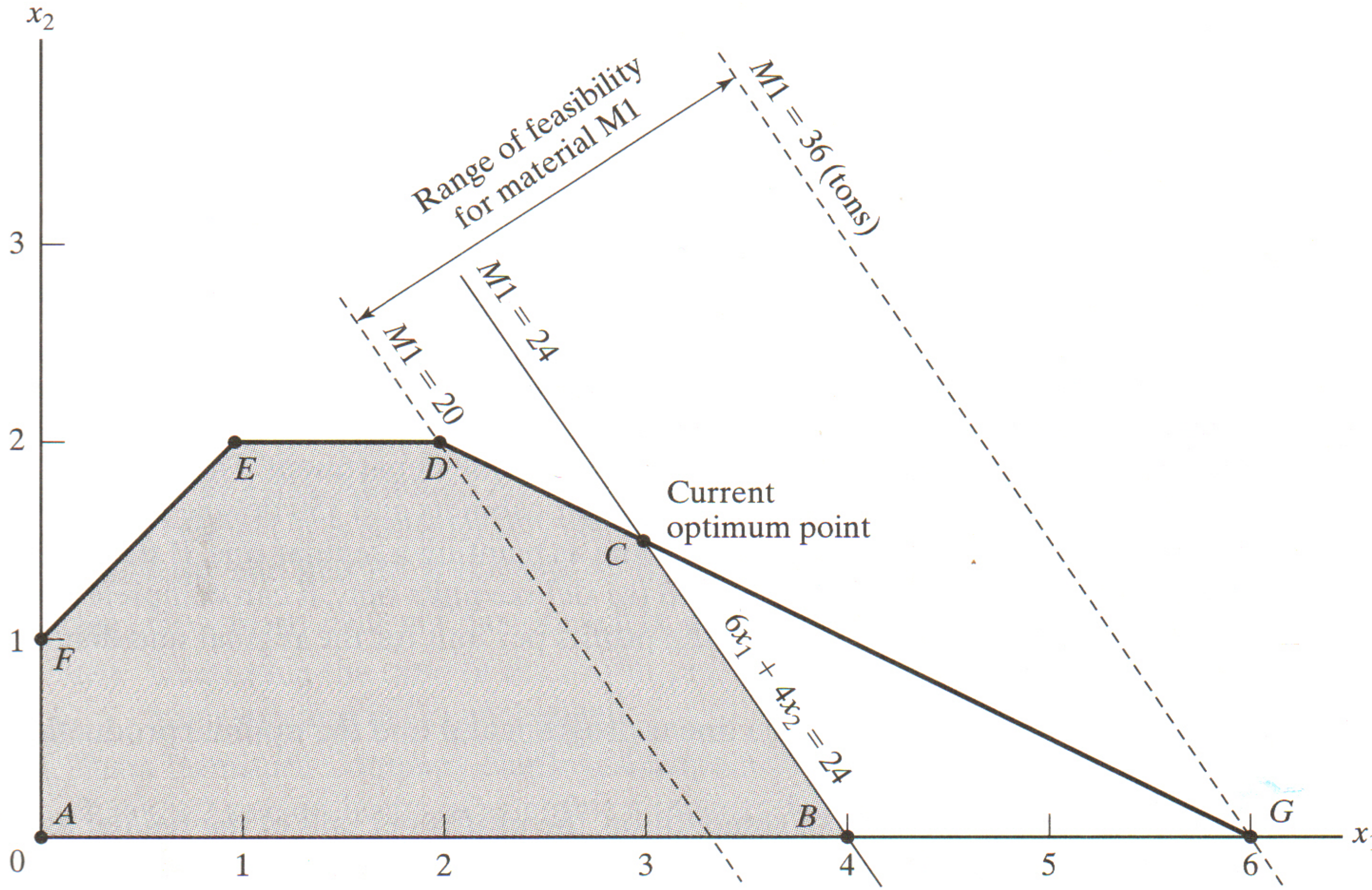

FIGURE 2.6

Range of feasibility for raw material $M1$ in the Reddy Mikks model

$$6x_1 + 4x_2 \leq 24$$

- Amount of M1 at D(2,2) $6x_1+4x_2=6(2)+4(2)=20$
- Amount of M1 at G(6,0) $6x_1+4x_2=6(6)+4(0)=36$
- The meaning of this:
- The amount of raw material 1 (M1) can be decreased by as much as 4 tons or increased as much as 12 tons

- **Then the <u>feasibility range of M1 given M2=6 tons</u>**

$$\mathbf{20 \leq M1 \leq 36}$$

- **Given M2=6 tons, the associated general solution is obtained in terms of M1 as follows:**

$6x_1 + 4x_2 = M_1$

$\underline{x_1 + 2x_2 = 6}$

$6x_1 + 4x_2 = M_1$

$-2/ \quad \underline{x_1 + 2x_2 = 6}$

$6x_1 + 4x_2 = M_1$

$-2\underline{x_1 - 4x_2 = 6}$

$4X_1=-12+M_1$

$\mathbf{X_1=(1/4)M_1-3}$

$\mathbf{X_2=-(1/8)M_1+(9/2)}$

WHERE $20 \le M_1 \le 36$

$$x_1 + 2x_2 \leq 6$$

- **<u>QUESTION</u>**
- **<u>Find the feasibility range of M2 given M1=24 tons.</u>**

LP: Max z= 5x1+4x2

s.t.

$6x_1 + 4x_2 \leq 24$

$x_1 + 2x_2 \leq 6$

$-x_1 + x_2 \leq 1$

$x_2 \leq 2$

$x_1, x_2 \geq 0$

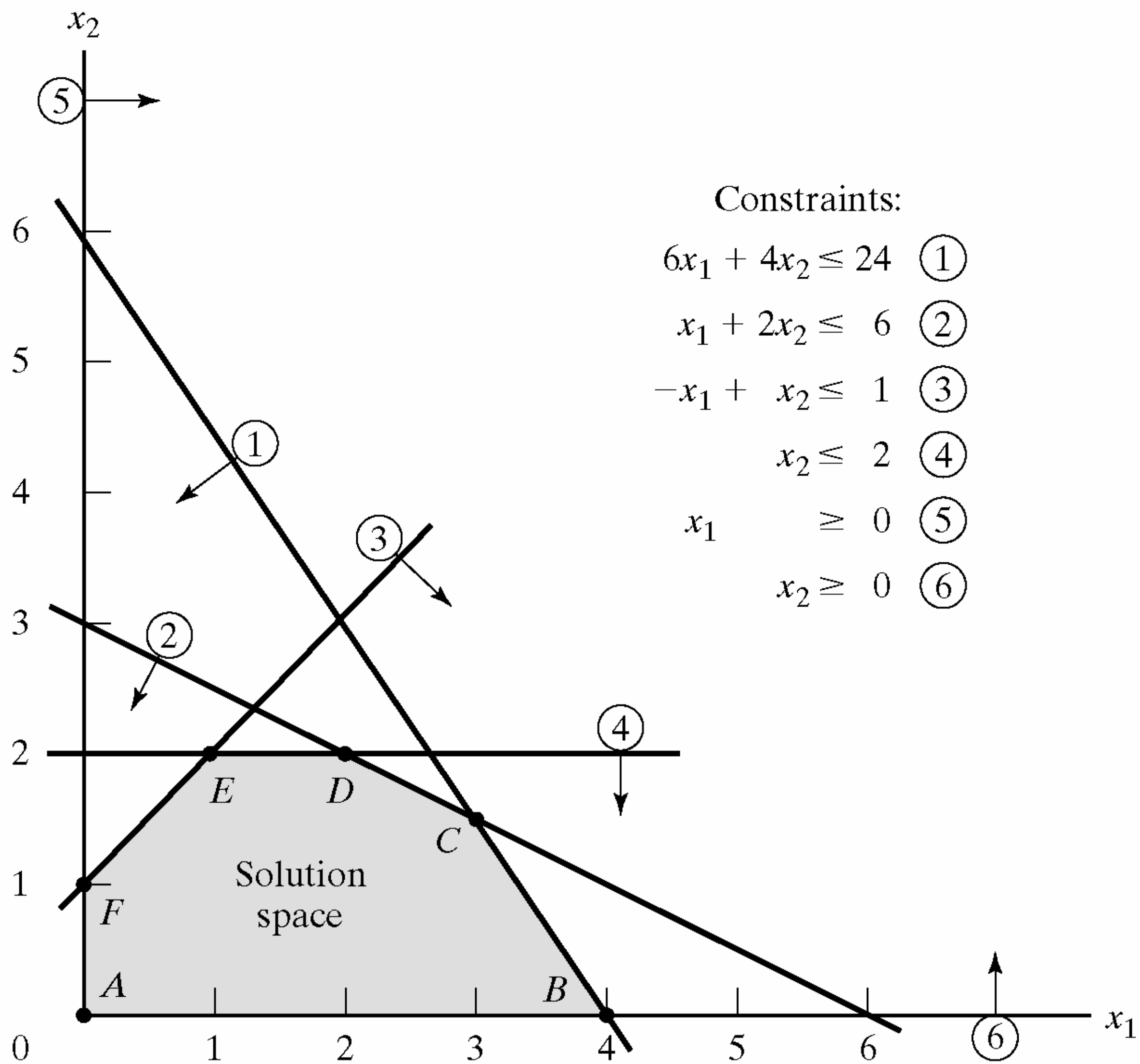

Figure 2.1
Feasible space of the Reddy Mikks model.

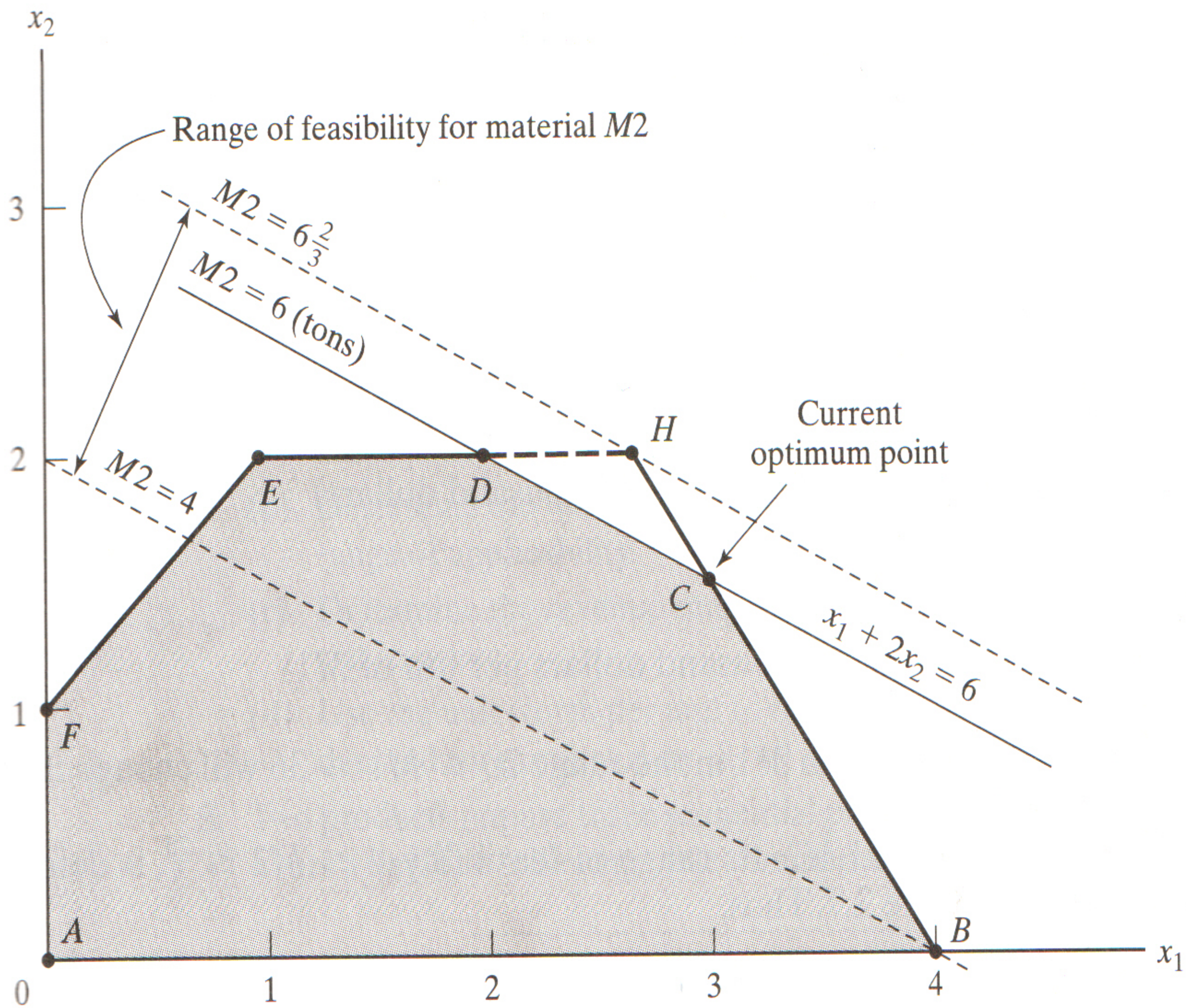

FIGURE 2.7

Range of feasibility for raw material $M2$ in the Reddy Mikks model

$$x_1 + 2x_2 \le 6$$

- Amount of M2 at B(4,0) $x_1+2x_2=(4)+4(0)=4$
- Amount of M2 at H(8/3,2)

  $$x_1+2x_2=(8/3)+4(2)=20/3$$
- The meaning of this:
- The amount of raw material 2 (M2) can be decreased by as much as 2 tons or increased as much as 2/3 tons

- **Then the <u>feasibility range of M2 when given M1=24 tons</u>**

$$\mathbf{4 \le M2 \le (20/3)}$$

- **Given M1=24tons, the associated general solution is obtained in terms of M2 as follows:**

$$6x_1 + 4x_2 = 24$$

$$\underline{x_1 + 2x_2 = M_2}$$

$$6x_1 + 4x_2 = 24$$

$$-2/ \quad \underline{x_1 + 2x_2 = M_2}$$

$$6x_1 + 4x_2 = 24$$

$$-2\underline{x_1 - 4x_2 = -2M_2}$$

$$4X_1=24-2M_2$$

$$\mathbf{X_1=6-(1/2)M_2}$$

$$\mathbf{X_2=-3+(3/4)M_2}$$

WHERE $4 \le M_2 \le (20/3)$

Unit worth of a Resource

DUAL PRICE
SHADOW PRICE
SIMPLEX MULTIPLIERS

- Specifically, we seek to determine the unit worth of a resource, which is defined as the rate of change in the optimum objective value that results from making changes in the available amount of a resource.

- Let y_i represents the worth per unit of resource i, the associated formula for computing this measure

$$y_i = \frac{\text{(change in value of } z \text{ corresponding to the feasible range of resource i)}}{\text{(feasible range of resource i)}}$$

Example

The feasible range for M1, given $M2 = 6$ is $20 \le M1 \le 36$

and this range is delineated by points D(2,2) and G(6,0)

$$y_1 = \frac{\text{Change in } z \text{ from D to G}}{\text{Change in M1 from D to G}}$$

z value at $D = 5x1 + 4x2 = 5(2) + 4(2) = 18$ (thousand dollars)

z value at $G = 5x1 + 4x2 = 5(6) + 4(0) = 30$ (thousand dollars)

$$y1 = \frac{30 - 18}{36 - 20} = \frac{12}{16} = 0.75 \text{ thousand dollars per ton of M1}$$

"a 1-ton change in M1 in the range of $20 \le M1 \le 36$ will change the optimum value of z by \$750"

Consider raw material M2

The feasible range for M2, given $M1 = 24$ is $4 \le M2 \le (20/3)$

and this range is delineated by points B(4,0) and H(8/3,2)

$$y_2 = \frac{\text{Change in z from B to H}}{\text{Change in M2 from B to H}}$$

z value at $B = 5x1 + 4x2 = 5(4) + 4(0) = 20$ (thousand dollars)

z value at $H = 5x1 + 4x2 = 5(8/3) + 4(2) = (64/3)$ (thousand dollars)

$$y_2 = \frac{(64/3) - 20}{(20/3) - 4} = 0.5 \text{ thousand dollars per ton of M2}$$

a 1-ton change in M2 in the range of $4 \le M2 \le (20/3)$ will change the optimum value of z by \$500"