ENM 202

OPERATIONS RESEARCH (I)

ARTIFICIAL STARTING SOLUTION

ARTIFICIAL STARTING SOLUTION

- The simplex algorithm requires a starting bfs.
- In the problems we have solved so far, we found a starting bfs by using the **slack variables** as our basic variables.
- If an LP has any ≥ or = constraints, however, a starting bfs may not be readily apparent.
- When a bfs is not readily apparent,

  The Big M Method **or**

  The Two-phase Simplex method

  may be used to solve the problem.

We illustrate this point by the following example:

$$\textit{Minimize} \quad Z = 4x_1 + x_2$$

$$\textit{subject to}$$

$$3x_1 + x_2 = 3 \qquad (1)$$

$$4x_1 + 3x_2 \geq 6 \qquad (2)$$

$$x_1 + 2x_2 \leq 4 \qquad (3)$$

$$x_1,\ x_2 \geq 0$$

$$\text{Standard Form:}$$

$$\textit{Minimize} \quad Z = 4x_1 + x_2$$

$$\textit{subject to}$$

$$3x_1 + x_2 = 3 \qquad (1)$$

$$4x_1 + 3x_2 - x_3 = 6 \qquad (2)$$

$$x_1 + 2x_2 + x_4 = 4 \qquad (3)$$

$$x_1,\ x_2 \geq 0$$

- NOTE THAT:
- Thus, constraints of the type ≥ cannot physically represent a resource restriction, rather, they imply that the solution must meet certain requirements, such as satisfying minimum demand or minimum specifications.

THE BIG M METHOD
(The Method of Penalty)

- It is a version of Simplex Algorithm

- It first finds a bfs by adding “artificial” variables to the problem.

THE BIG M METHOD (The Method of Penalty)

- The idea of using artificial variables is quite simple. It calls for adding a nonnegative variable to the LHS of each equation that has no obvious starting basic variables. The added variable will play the same role as that of a slack variable, in providing a starting basic variable.

- However, since such artificial variables have no physical meaning, the procedure will be valid only if we force these variables to be zero when the optimum is reached.

THE BIG M METHOD (The Method of Penalty)

- Given M, a sufficiently large positive value (Mathematically $M \to \infty$) the objective coefficient of an artificial variable represent an appropriate penalty if artificial objective coefficient is equal to $-M$ in max Problems, and $+M$ in min Problems.

- In other words, we penalize the artificial varibles in objective function.

- Using this penalty, the optimization process will automatically force the artificials to zero.

Big M Method, Example

- LP:
- Min $z=4x1+x2$
- St
- $3x1+x2=3$
- $4x1+3x2 \geq 6$
- $x1+2x2 \leq 4$
- $x1,x2 \geq 0$

- LP in Standard Form:
- Min $z=4x1+x2+0x3+0x4$
- St
- $3x1+x2=3$
- $4x1+3x2 - x3=6$
- $x1+ 2x2 +x4= 4$
- $x1,x2, x3, x4 \geq 0$

Big M Method, Example

- LP in Standard Form:

$\text{Min } z=4x1+x2+0x3+0x4$

St

$3x1+x2=3$

$4x1+3x2 - x3=6$

$x1+ 2x2 +x4= 4$

$x1,x2 \geq 0$

- LP in Standard Form with Artificials:

$\text{Min} z=4x1+x2+0x3+0x4+MR1+MR2$

St

$3x1+x2+R1=3$

$4x1+3x2 - x3+R2=6$

$x1+ 2x2 +x4= 4$

$x1,x2, x3, x4, R1, R2 \geq 0$

Big M Method, Example

Iteration	Basic	x1	x2	x3	R1	R2	x4	RHS
İnconsistent	z	-4	-1	0	-M	-M	0	0
	R1	3	1	0	1	0	0	3
	R2	4	3	-1	0	1	0	6
	x4	1	2	0	0	0	1	4
(0)	z	-4+7M	-1+4M	-M	0	0	0	9M
	R1	3	1	0	1	0	0	3
	R2	4	3	-1	0	1	0	6
	x4	1	2	0	0	0	1	4

Iteration	Basic	x1	x2	x3	R1	R2	x4	RHS
(0)	z	-4+7M	-1+4M	-M	0	0	0	9M
X1 enters, R1 leaves	R1	3	1	0	1	0	0	3
	R2	4	3	-1	0	1	0	6
	x4	1	2	0	0	0	1	4
(1)	z	0	(1+5M)/3	-M	(4-7M)/3	0	0	4+2M
	x1	1	1/3	0	1/3	0	0	1
x2 enters, R2 leaves	R2	0	5/3	-1	-4/3	1	0	2
	x4	0	5/3	0	-1/3	0	1	3
(2)	z	0	0	1/5	(8/5)-M	-(1/5)-M	0	18/5
	x1	1	0	1/5	3/5	-1/5	0	3/5
	x2	0	1	-3/5	-4/5	3/5	0	6/5
x3 enters, x4 leaves	x4	0	0	1	1	-1	1	1
(3)	z	0	0	0	(7/5)-M	-M	-1/5	17/5
optimum	x1	1	0	0	2/5	0	-1/5	2/5
	x2	0	1	0	-1/5	0	3/5	9/5
Big M	x3	0	0	1	1	-1	1	1

Drawback of The Big M Method

The possible computational error that could result from assigning a very large value to the constant M

Optimal Solution and Other Cases

- 1) If all the artificial variables are equal to zero in the optimal solution, then we have found the optimal solution
- 2) If any artificial variables are positive in the optimal solution, then the original problem is infeasible.
- 3) If the final tableau indicate that the LP is unbounded and all artificial variables in this tableau are equal to zero, then the original LP is unbounded.

TWO PHASE METHOD

- Because of the computational drawbacks of the Big M Method, the Two Phase Method was suggested.

- The Two Phase Simplex Method is an alternative to the Big M Method

TWO PHASE METHOD

- **Phase 1**
- 1) Put the LP model in standard form
- 2) Add the necessary artificial variables
- 3) Find a bfs of the resulting equations that minimizes the sum of the artificial variables.
- 4) If the minimum value of the sum is positive the LP problem has no feasible solution, which ends the process,
- Otherwise, i.e, (the minimum value of the sum is zero and no artificials are in the optimal phase 1 basis) go to phase 2
- **Phase 2**
- 1)Use the bfs from phase 1 as a starting bfs for the original problem and solve it.

Example

- LP:

- Min $z=4x1+x2$
- St
- $3x1+x2=3$
- $4x1+3x2 \geq 6$
- $x1+2x2 \leq 4$
- $x1,x2 \geq 0$

- LP in standard form with artificials:

- Min $z=4x1+x2$
- St
- $3x1+x2+R1=3$
- $4x1+3x2 -x3+R2= 6$
- $x1+2x2 +x4= 4$
- $x1,x2,x3,x4,R1,R2 \geq 0$

- LP in standard form with artificials:

- Min $z=4x_1+x_2$
- St
- $3x_1+x_2+R_1=3$
- $4x_1+3x_2-x_3+R_2=6$
- $x_1+2x_2+x_4=4$
- $x_1,x_2,x_3,x_4,R_1,R_2 \geq 0$

- **LP for phase 1**

- **Min $r=R_1+R_2$**
- St
- $3x_1+x_2+R_1=3$
- $4x_1+3x_2-x_3+R_2=6$
- $x_1+2x_2+x_4=4$
- $x_1,x_2,x_3,x_4,R_1,R_2 \geq 0$

- r in equation form
- $r-R_1-R_2=0$

Iteration	Basic	x1	x2	x3	R1	R2	x4	RHS
	r	0	0	0	-1	-1	0	0
inconsistent	R1	3	1	0	1	0	0	3
	R2	4	3	-1	0	1	0	6
	x4	1	2	0	0	0	1	4
(0)	r	7	4	-1	0	0	0	9
X1 enters R1 leaves	R1	3	1	0	1	0	0	3
	R2	4	3	-1	0	1	0	6
	x4	1	2	0	0	0	1	4
(1)	r	0	5/3	-1	-7/3	0	0	2
	x1	1	1/3	0	1/3	0	0	1
X2 enters R2 leaves	R2	0	5/3	-1	-4/3	1	0	2
	x4	0	5/3	0	-1/3	0	1	3
(2)	r	0	0	0	-1	-1	0	0
optimum	x1	1	0	1/5	3/5	-1/5	0	3/5
	x2	0	1	-3/5	4/5	3/5	0	6/5
Phase 1	x4	0	0	1	1	-1	1	1

- We found a bfs.
- We can start to solve our original problem

- The original problem is written as

- Min $z = 4x_1+x_2$
- St
- $x_1+(1/5)x_3=(3/5)$
- $x_2-(3/5)x_3=(6/5)$
- $x_3+x_4=1$
- $x_1,x_2,x_3,x_4\geq 0$

- z in equation form
- $z-4x_1-x_2=0$

Iteration	Basic	x1	x2	x3	x4	RHS
inconsistent	z	-4	-1	0	0	0
	x1	1	0	1/5	0	3/5
	x2	0	1	-3/5	0	6/5
	x4	0	0	1	1	1
(0)	z	0	0	1/5	0	18/5
	x1	1	0	1/5	0	3/5
	x2	0	1	-3/5	0	6/5
x3 enters, x4 leaves	x4	0	0	1	1	1
(1)	z	0	0	0	-1/5	17/5
Optimum	x1	1	0	0	-1/5	2/5
	x2	0	1	0	3/5	9/5
Phase 2	x3	0	0	1	1	1