ENM 202
OPERATIONS RESEARCH (I)
OR (I)
7

LECTURE NOTES

COMPLEMENTARY SLACKNESS THEOREM

Remember

The general form of the primal problem is as follows:

$$\text{Max } z = \sum_{j=1}^{n} c_j x_j$$

$s.t.$

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i \qquad i = 1,2,....,m$$

$$x_j \ge 0 \qquad j = 1,2,....,n$$

or

P: $Max\, cx$

$$s.t. \quad Ax \le b$$
$$x \ge 0$$

The general form of the dual problem is as follows:

$$\text{Min } Y = \sum_{i=1}^{m} b_i y_i$$

$s.t.$

$$\sum_{i=1}^{m} a_{ji} y_i \ge c_j \qquad j = 1,2,....,n$$

$$y_i \ge 0 \qquad i = 1,2,....,m$$

or

D: $Min\ yb$

$$s.t. \quad yA \ge c$$
$$y \ge 0$$

Remember

WEAK DUALITY THEOREM

cx ≤ yAx ≤ yb

- Weak Duality Theorem is very important result that forms the foundation for several other duality relationships.
- First of all notice that :
- Each feasible solution to the max problem provides a LB for the objective of the minimization problem.
- Likewise, each feasible solution to the min problem provides a UB for the objective of the maximization problem.

$$\begin{aligned} \mathrm{P:}\quad & Max\ cx \\ s.t.\quad & Ax \le b \\ & x \ge 0 \end{aligned} \qquad\qquad \begin{aligned} \mathrm{D:}\ & Min\ yb \\ s.t.\quad & yA \ge c \\ & y \ge 0 \end{aligned}$$

Remember

STRONG DUALITY THEOREM

$$cx^* = y^*Ax^* = y^*b$$

- At the optimal point of (P) and (D) problem, the optimal objective value of the (P) is equal to the optimal objective value of (D)

From the Strong Duality Theorem

$cx = yAx = yb$

- Lets consider the left two terms in this relationship first:
- $cx = yAx$
- $0= yAx- cx$
- $0= (yA- c) x$

- Lets consider the right two terms in this relationship first:
- $yAx = yb$
- $0= yb – yAx$
- $y (b – Ax) =0$

•Complementary slackness conditions

- $0= (yA- c) x$

Equal to the surplus (excess) variables İn the dual problem

$e_j x_j=0$ for $j=1,2,\dots,n$

- $y (b – Ax) =0$

Equal to the slack variables in the primal Problem

$y_i S_i=0$ for $i=1,2,\dots,m$

COMPLEMENTARY SLACKNESS THEOREM

- THEOREM:
- Let $x=[x_1\ x_2\ \ldots\ldots x_n]^T$ be a feasible primal solution and $y=[y_1\ y_2\ \ldots\ldots y_m]$ be a feasible dual solution. Then x is primal optimal and y is dual optimal if and only if

$e_j x_j=0$ for $j=1,2,\ldots,n$

$S_i y_i=0$ for $i=1,2,\ldots,m$

Example

- (P):
- Max z= 60x1+30x2+20x3
- st
- 8x1+6x2+x3 ≤48
- 4x1+2x2+1.5x3≤ 20
- 2x1+1.5x2+0.5x3≤ 8
- x1,x2,x3 ≥0

- QUESTION:
- If the Optimal Primal Solution of the LP problem above is as follows, find the corresponding optimal dual solution without using the simplex iterations
- z=280
- x*(2,0,8)

- SOLUTION
- 1) Construct the dual problem
- (D):
- Min w=48y1+20y2+8y3
- st
- 8y1+4y2+2y3 ≥ 60
- 6y1+2y2+1.5y3 ≥ 30
- y1+1.5y2+0.5y3 ≥ 20
- y1, y2, y3 ≥ 0
- 2) Find the slack variable values of the primal optimal problem
- 8x1+6x2+x3 +S1=48
- 8(2)+6(0)+8+S1=48, **S1=24**
- 4x1+2x2+1.5x3+S2= 20
- 4(2)+2(0)+1.5(8)+S2=20, **S2=0**
- 2x1+1.5x2+0.5x3+S3= 8
- 2(2)+1.5(0)+0.5(8)+S3=8, **S3=0**

$e_j x_j=0$ for $j=1,2,\dots,n$

$x_1=2>0$	then	$e_1=0$ the 1st dual constraint is binding
$x_2=0$	then	$e_2>0$ the 2nd dual constraint is nonbinding
$X_3=8>0$	then	$e_3=0$ the 3rd dual constraint is binding

$S_i y_i=0$ for $i=1,2,\dots,m$

$S_1=24$	then	$y_1=0$
$S_2=0$	then	y_2 may be found
$S_3=0$	then y	y_3 may be found

- (D) Min $w=48y_1+20y_2+8y_3$
- st
- $8y_1+4y_2+2y_3 \geq 60$
- $6y_1+2y_2+1.5y_3 \geq 30$
- $y_1+1.5y_2+0.5y_3 \geq 20$
- $y_1,\ y_2,\ y_3 \geq 0$

- (D) Min $w=48(0)+20y_2+8y_3$
- st
- $8(0)+4y_2+2y_3 = 60$
- $(0)+1.5y_2+0.5y_3 = 20$
- $y_2=10,\ y_3=10$

DUAL SIMPLEX METHOD

- The Dual Simplex Method is developed by Lemke in 1954.

- It is useful tool for dealing with sensitivity analysis in linear programming.

- From now on, we will refer to the original simplex algorithm as the **primal simplex algorithm**

- The primal simplex algorithm is an algorithm that always deals with a bfs (basic feasible solution).

- Primal optimality is precisely the same as the dual feasibility.
- The (P) simplex algorithm starts a bfs but nonoptimal
- The (D) simplex algorithm starts better than optimal but a basic infeasible solution.

- In the dual Simplex Algorithm, each iteration is associated with a basic solution again but it is not required as a feasible one

DUAL SIMPLEX METHOD

- Primal optimality corresponds dual feasibility

- Primal feasibility corresponds dual optimality

DUAL SIMPLEX METHOD

- **Feasibility Condition: (Determining the leaving variable)**
- The leaving variable, x_r is the basic variable having the most negative value. If all the basic variables are nonnegative then the algorithm ends.
- **Optimality Condition: (Determining the entering variable)**
- The entering variable is selected from among the NBVs as follows:

$$\min_{NBx_j}\left\{\left|\frac{z_j - c_j}{\alpha_{rj}}\right|, \alpha_{rj} < 0 \right\} \text{ where}$$

$z_j - c_j$ is the objective coefficient of the z-row in the tableau

α_{rj} is the negative constraint coefficient of the tableau associated with the row of the leaving variable x_r, and the column of the NB variable x_j

•Ignore the ratios associated with positive and zero denominators.

•(If all the denominators are zero or positive, the problem has no feasible solution.)

- **To start the Dual Simplex Algorithm the following requirements must be satisfied:**
- **1)** The objective function must satisfy the optimality condition of the regular simplex method.

  a)In a maximaxtion problem all z-row coefficients must be nonnegative

  b)In a minimization problem all z-row coefficients must be nonpositive
- **2)** All the constraints must be type of (≤)

  a)If there exist a (≥) type constraint then multiply both sides by -1

  b)If there exist a (=) type constraint replaced it by two equations just like in the following example:

$$x1+x2=2$$

can be replaced by the following two constriants:

$$x1+x2 \leq 2$$

$$-x1-x2 \leq -2$$

3) After converting all constraints, if and only if at least one of the RHSs of the inequalities is strictly negative START THE DUAL SIMPLEX METHOD.

ELSE (If z row satisfy the optimality condition and none of the RHSs are negative then there is no need to apply the dual simplex algorithm. Because the starting solution is already optimal and feasible

Example 1

LP:	LP for dual simplex algorithm	LP in equation form for dual simplex algorithm
Min z= 2x1+x2 St 3x1+x2≥ 3 4x1+3x2≥ 6 x1+2x2 ≤3 x1,x2≥0	Min z= 2x1+x2 St -3x1-x2 ≤ -3 -4x1-3x2 ≤ -6 x1+2x2 ≤3 x1,x2≥0	Min z= 2x1+x2 St -3x1 -x2 +x3 = -3 -4x1-3x2 +x4 = -6 x1+2x2 +x5 = 3 x1,x2,x3,x4,x5≥0

Example

(P) LP:	(D) LP:
Min $z = 2x_1 + x_2$ St $3x_1 + x_2 \geq 3$ $4x_1 + 3x_2 \geq 6$ $x_1 + 2x_2 \leq 3$ $x_1, x_2 \geq 0$	Max $w = 3y_1 + 6y_2 + 3y_3$ St $3y_1 + 4y_2 + y_3 \leq 2$ $y_1 + 3y_2 + 2y_3 \leq 1$ $y_1 \geq 0$ $y_2 \geq 0$ $y_3 \leq 0$

MRT_1	min	{(|-2/-4|),	(|-1/-3|*)}	=	1			
MRT_2	min	{(|(-2/-3).(-3/5)|*),	entering variable		(|(-1/-3).(-3/1)| }			
Iteration	Basic	x1	x2	x3	x4	x5	RHS	
(0)	z	-2	-1	0	0	0	0	
	x3	-3	-1	1	0	0	-3	
x2 enters x4 leaves	x4	-4	-3	0	1	0	-6	Most negative Leaving variable
	x5	1	2	0	0	1	3	
(1)	z	-2/3	0	0	-1/3	0	2	
x1enters x3 leaves	x3	-5/3	0	1	-1/3	0	-1	Most negative Select arbitrarily
	x2	4/3	1	0	-1/3	0	2	
	x5	-5/3	0	0	2/3	1	-1	Most negative Select arbitrarily
(2)	z	0	0	-2/5	-1/5	0	12/5	
optimum	x1	1	0	-3/5	1/5	0	3/5	
Degenerate optimal solution	x2	0	1	4/5	-3/5	0	6/5	
DualSimplex	x5	0	0	-1	1	1	0	

Example 2

LP:	LP for dual simplex algorithm	LP in equation form for dual simplex algorithm
Min z= 3x1+2x2 St 3x1+x2≥ 3 4x1+3x2≥ 6 x1+x2 ≤3 x1,x2≥0	Min z= 3x1+2x2 St -3x1-x2 ≤ -3 -4x1-3x2 ≤ -6 x1+x2 ≤3 x1,x2≥0	Min z= 3x1+2x2 St -3x1 -x2 +x3 = -3 -4x1-3x2 +x4 = -6 x1+x2 +x5 = 3 x1,x2,x3,x4,x5≥0

MRT_1	min	{(|-3/-4|),	(|-2/-3|*)}	=	1			
MRT_2	min	{(|(-1/-3).(-3/5)|*),	entering variable		(|(-2/-3).(-3/1)| }			
Iteration	Basic	x1	x2	x3	x4	x5	RHS	
(0)	z	-3	-2	0	0	0	0	
	x3	-3	-1	1	0	0	-3	
x2 enters x4 leaves	x4	-4	-3	0	1	0	-6	Most negative Leaving variable
	x5	1	1	0	0	1	3	
(1)	z	-1/3	0	0	-2/3	0	4	
x1enters x3 leaves	x3	-5/3	0	1	-1/3	0	-1	Most negative Leaving variable
	x2	4/3	1	0	-1/3	0	2	
	x5	-1/3	0	0	1/3	1	1	
(2)	z	0	0	-1/5	-3/5	0	21/5	
optimum	x1	1	0	-3/5	1/5	0	3/5	
	x2	0	1	4/5	-3/5	0	6/5	
DualSimplex	x5	0	0	-1/5	2/5	1	6/5	