ENM 202 OPERATIONS RESEARCH (I) OR (I) LECTURE NOTES **COMPLEMENTARY SLACKNESS THEOREM**

Remember

The general form of the primal problem is as follows:

Max
$$z = \sum_{j=1}^{n} c_j x_j$$

s.t.

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i \qquad i = 1, 2, ..., m$$
$$x_j \ge 0 \qquad j = 1, 2, ..., n$$

The general form of the dual problem is as follows:

$$\operatorname{Min} \mathbf{Y} = \sum_{i=1}^{m} b_i y_i$$

s.t.

or

$$\sum_{i=1}^{m} a_{ji} y_i \ge c_j \qquad j = 1, 2, ..., n$$
$$y_i \ge 0 \qquad i = 1, 2, ..., m$$

or

P: Max cx $s.t. Ax \le b$ $x \ge 0$ D: Min yb $s.t. yA \ge c$ $y \ge 0$

Remember WEAK DUALITY THEOREM $cx \le yAx \le yb$

- Weak Duality Theorem is very important result that forms the foundation for several other duality relationships.
- First of all notice that :
- Each feasible solution to the max problem provides a LB for the objective of the minimization problem.
- Likewise, each feasible solution to the min problem provides a UB for the objective of the maximization problem.

P :	Max	CX	D:Min	D:Min yb		
	<i>s.t</i> .	$Ax \leq b$	<i>s.t</i> .	$yA \ge c$		
		$x \ge 0$		$y \ge 0$		

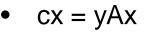
STRONG DUALITY THEOREM cx* = y*Ax* = y*b

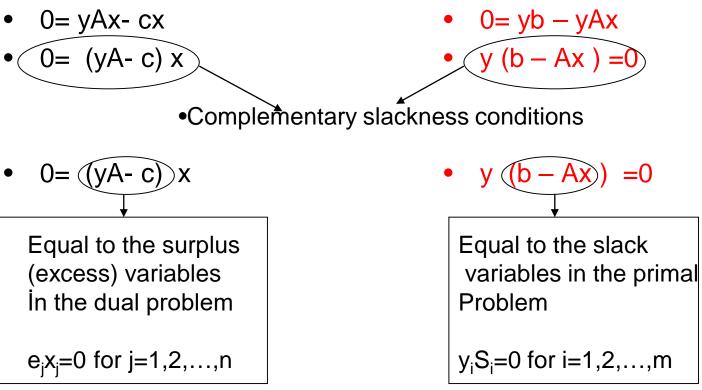
At the optimal point of (P) and (D) problem, the optimal objective value of the (P) is equal to the optimal objective value of (D)

From the Strong Duality Theorem cx = yAx = yb

• yAx = yb

- Lets consider the left two terms in this relationship first:
- Lets consider the right two terms in this relationship first:





COMPLEMENTARY SLACKNESS THEOREM

- THEOREM:
- Let x=[x₁ x₂x_n]^T be a feasible primal solution and y= [y₁ y₂y_m] be a feasible dual solution. Then x is primal optimal and y is dual optimal if and only if e_jx_j=0 for j=1,2,...,n S_iy_i=0 for i=1,2,...,m

- (P):
- Max z= 60x1+30x2+20x3
- st
- 8x1+6x2+x3 ≤48
- 4x1+2x2+1.5x3≤ 20
- 2x1+1.5x2+0.5x3≤ 8
- x1,x2,x3 ≥0
- QUESTION:
- If the Optimal Primal Solution of the LP problem above is as follows, find the corresponding optimal dual solution without using the simplex iterations
- z=280
- x*(2,0,8)

- SOLUTION
- 1) Construct the dual problem
- (D):
- Min w=48y1+20y2+8y3
- st
- 8y1+4y2+2y3 ≥ 60
- 6y1+2y2+1.5y3 ≥ 30
- y1+1.5y2+0.5y3 ≥ 20
- y1, y2, y3 ≥ 0
- 2) Find the slack variable values of the primal optimal problem
- 8x1+6x2+x3 +S1=48
- 8(2)+6(0)+8+S1=48, <u>**S1=24**</u>
- 4x1+2x2+1.5x3+S2=20
- 4(2)+2(0)+1.5(8)+S2=20, **<u>S2=0</u>**
- 2x1+1.5x2+0.5x3+S3= 8
- 2(2)+1.5(0)+0.5(8)+S3=8, <u>S3=0</u>

ejxj=0 for j=1,2,...,n

x1=2>0	then	e1=0 the 1st dual constraint is binding
x2=0	then	e2>0 the 2nd dual constraint is nonbinding
X3=8>0	then	e3=0 the 3rd dual constraint is binding

Siyi=0 for i=1,2,...,m

S1=24	then	y1=0
S2=0	then	y2 may be found
S3=0	then y	y3 may be found

- (D) Min w=48y1+20y2+8y3
- st
- $8y1+4y2+2y3 \ge 60$
- 6y1+2y2+1.5y3 ≥ 30
- y1+1.5y2+0.5y3 ≥ 20
- y1, y2, y3 ≥ 0

- (D) Min w=48(0)+20y2+8y3
- st
- $8(0)+4y^2+2y^3 = 60$
- (0)+1.5y2+0.5y3 = 20
- y2=10, y3=10

DUAL SIMPLEX METHOD

- The Dual Simplex Method is developed by Lemke in 1954.
- It is useful tool for dealing with sensitivity analysis in linear programming.
- From now on, we will refer to the original simplex algorithm as the primal simplex algorithm
- The primal simplex algorithm is an algorithm that always deals with a bfs (basic feasible solution).
- Primal optimality is precisely the same as the dual feasibility.
- The (P) simplex algorithm starts a bfs but nonoptimal
- The (D) simplex algorithm starts better than optimal but a basic infeasible solution.
- In the dual Simplex Algorithm, each iteration is associated with a basic solution again but it is not required as a feasible one

DUAL SIMPLEX METHOD

- Primal optimality corresponds dual feasibility
- Primal feasibility corresponds dual optimality

DUAL SIMPLEX METHOD

• Feasibility Condition: (Determining the leaving variable)

 The leaving variable, x_r is the basic variable having the most negative value. If all the basic variables are nonnegative then the algorithm ends.

• Optimality Condition: (Determining the entering variable)

• The entering variable is selected from among the NBVs as follows:

$$\min_{NBx_{j}} \left\{ \left| \frac{z_{j} - c_{j}}{\alpha_{rj}} \right|, \alpha_{rj} < 0 \right\} \text{ where }$$

 $z_j - c_j$ is the objective coefficient of the z - row in the tableau

 α_{rj} is the negative constraint coefficient of the tableau associated with the row of the leaving variable x_r , and the column of the NB variable x_i

•Ignore the ratios associated with positive and zero denominators.

•(If all the denominators are zero or positive, the problem has no feasible solution.)

- <u>To start the Dual Simplex Algorithm the following</u> requirements must be satisfied:
- <u>1</u>) The objective function must satisfy the optimality condition of the regular simplex method.

a)In a maximaxtion problem all z-row coefficients must be nonnegative b)In a minimization problem all z-row coefficients must be nonpositive

<u>2)</u> All the constraints must be type of (≤)

 a)If there exist a (≥) type constraint then multiply both sides by -1
 b)If there exist a (=) type constraint replaced it by two equations just like in the following example:

x1+x2=2

can be replaced by the following two constriants:

- 3) After converting all constraints, if and only if at least one of the RHSs of the inequalities is strictly negative START THE DUAL SIMPLEX METHOD.
- ELSE (If z row satisfy the optimality condition and none of the RHSs are negative then there is no need to apply the dual simplex algorithm. Because the starting solution is already optimal and feasible

LP:	LP for dual simplex algorithm	LP in equation form for dual simplex algorithm
Min z = 2x1 + x2	Min z = 2x1 + x2	Min = 2x1 + x2
St	St	St
3x1+x2≥ 3	-3x1-x2 ≤ -3	-3x1 - x2 + x3 = -3
4x1+3x2≥ 6	-4x1-3x2 ≤ -6	-4x1-3x2 + x4 = -6
x1+2x2 ≤3	x1+2x2 ≤3	x1+2x2 + x5 = 3
x1,x2≥0	x1,x2≥0	x1,x2,x3,x4,x5≥0

(P) LP:	(D) LP:			
Min z= 2x1+x2	Max w=3y1+6y2+3y3			
St	St			
3x1 +x2 ≥ 3	3y1+4y2+y3 ≤2			
4x1+3x2 ≥ 6	y1+3y2+2y3 ≤1			
x1+2x2 ≤ 3	y1 ≥0			
x1,x2≥0	y2 ≥0			
	y3 ≤0			

MRT ₁	min	{(-2/-4),	(-1/- <mark>3</mark> *)}	=	1			
MRT ₂	min	{ ((-2/-3).(-3/5) *),	entering variable		((-1/-3).(-3/1) }			
Iteration	Basic	x1	x2	x3	x4	x5	RHS	
(0)	Z	-2	-1	0	0	0	0	
	x3	-3	-1	1	0	0	-3	
x2 enters x4 leaves	x4	-4	-3	0	1	0	-6	Most negative Leaving variable
	x5	1	2	0	0	1	3	
(1)	Z	-2/3	0	0	-1/3	0	2	
x1enters x3 leaves	x3	-5/3	0	1	-1/3	0	-1	Most negative Select arbitrarily
	x2	4/3	1	0	-1/3	0	2	
	x5	-5/3	0	0	2/3	1	-1	Most negative Select arbitrarily
(2)	Z	0	0	-2/5	-1/5	0	12/5	
optimum	x1	1	0	-3/5	1/5	0	3/5	
Degenerate optimal solution	x2	0	1	4/5	-3/5	0	6/5	
DualSimplex	x5	0	0	-1	1	1	0	

LP:	LP for dual simplex algorithm	LP in <mark>equation</mark> form for dual simplex algorithm
Min $z = 3x1 + 2x2$	Min z= 3x1+2x2	Min z= 3x1+2x2
St	St	St
3x1+x2≥ 3	-3x1-x2 ≤ -3	-3x1 - x2 + x3 = -3
4x1+3x2≥ 6	-4x1-3x2 ≤ -6	-4x1-3x2 + x4 = -6
x1+x2 ≤3	x1+x2 ≤3	x1+x2+x5 = 3
x1,x2≥0	x1,x2≥0	x1,x2,x3,x4,x5≥0

MRT ₁	min	{(-3/-4),	(-2/-3 *)}	=	1			
MRT ₂	min	{ ((-1/-3).(-3/5) *),	entering variable		((-2/-3).(-3/1) }			
Iteration	Basic	x1	x2	x3	x4	x5	RHS	
(0)	Z	-3	-2	0	0	0	0	
	x3	-3	-1	1	0	0	-3	
x2 enters x4 leaves	x4	-4	-3	0	1	0 (-6	Most negative Leaving variable
	x5	1	1	0	0	1	3	
(1)	Z	-1/3	0	0	-2/3	0	4	
x1enters x3 leaves	x3	-5/3	0	1	-1/3	0	(-1)	Most negative Leaving variable
	x2	4/3	1	0	-1/3	0	2	
	x5	-1/3	0	0	1/3	1	1	
(2)	Z	0	0	-1/5	-3/5	0	21/5	
optimum	x1	1	0	-3/5	1/5	0	3/5	
	x2	0	1	4/5	-3/5	0	6/5	
DualSimplex	x5	0	0	-1/5	2/5	1	6/5	