ENM 202
OPERATIONS RESEARCH (I)
OR (I)
6

LECTURE NOTES

DUALITY and SENSITIVITY ANALYSIS

DUALITY and SENSITIVITY ANALYSIS

- Sensitivity Analysis is very important. Because in the real world, decision environments rarely remain static, so we need to determine how the optimal solution changes when the parameters of the model are changed. This is what Sensitivity Analysis does.
- We dealt with Sensitivity Analysis at an elementary level (Graphically and in optimal simplex Tableau)
- In algebraic treatment of the sensitivity analysis we use the Duality Theory.
- Duality is the key concept in the development of the important practical topic of the Sensitivity Analysis.
- In addition, duality theory constitute some new and efficient computational techniques for LP problems.

In linear programming, we use **duality** in a wide variety of both theoretical and practical ways:

1. In some cases, it may be **easier** to solve the dual problem than the primal
2. The dual variables provide **important economic interpretations** of the results obtained when solving a linear programming problem
3. Duality is used as an **aid** when we make the sensitivity analysis of a given linear programming problem
4. Duality can be utilized to solve the problems which don’t have a readily starting basic feasible solution (the technique itself is known as **the dual simplex**)
5. Duality is used to develop a number of **important theoretical results** in linear programming

Definition of the Dual Problem

- From now on we will call the original LP problem as “PRIMAL”

- The dual problem is constructed by using the (original) primal problem directly.

Constructing Dual LP Model

- **<u>1) By using the primal LP in standard form</u>**
  - By using the primal LP in standard form with artificials
- 2) By using the general rules

1) **<u>Constructing Dual LP Model</u>** **<u>by using the primal LP in standard form</u>**

- Remember
- In the standard form, all constraints are equations with nonnegative RHSs and all the variables are nonnegative

The general standard form of the primal problem is as follows:

$$\text{Max (or Min) } z = \sum_{j=1}^{n} c_j x_j$$

$s.t.$

$$\sum_{j=1}^{n} a_{ij} x_j = b_i \qquad i = 1,2,\ldots.,m$$

$$x_j \geq 0 \qquad j = 1,2,\ldots.,n$$

or

$$\text{P:} \quad Max \text{ (or Min) } c\ x$$

$$s.t. \quad Ax = b$$

$$x \leq 0$$

Standard primal objective	Dual objective	Dual Constraint type	Dual Variable sign
Max	Min	≥	urs
Min	Max	≤	urs

The dual is obtained symmetrically from the primal according to the following rules:

1) For every primal constraint there is a dual variable.

2) For every primal variable there is a dual constraint.

3) The constraint coefficients of a primal variable form the LHS coefficients of the dual constraint and the objective coefficient of the same variable becomes the RHS of the dual constraint.

4)The objective coefficients of the dual equal the right-hand-side of the primal constraint equations.

- So, the dual problem will have ***m*** variables and ***n*** equations.
- The type of the constraints and the sign of the dual variables is summarized on the left.

The general form of the primal problem is as follows:

$$\text{Max } Z = \sum_{j=1}^{n} c_j x_j$$

$s.t.$

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i \qquad i = 1,2,....,m$$

$$x_j \ge 0 \qquad j = 1,2,....,n$$

or

P: $Max\ cx$

$$s.t. \quad Ax \le b$$

$$x \ge 0$$

The general form of the dual problem is as follows:

$$\text{Min W} = \sum_{i=1}^{m} b_i y_i$$

$s.t.$

$$\sum_{i=1}^{m} a_{ij} y_i \ge c_j$$

$$y_i \ge 0$$

or

D: $Min\ yb$

$$s.t. \quad y\ A \ge c$$

$$y \ge 0$$

Example 1

Primal LP	Primal LP in standard Form	Dual LP
Max Z=5x1+12x2+4x3 st x1+2x2 +x3 ≤ 10 2x1 -x2+3x3 = 8 x1,x2,x3≥0	Max Z=5x1+12x2+4x3 st x1+2x2 +x3 +x4=10 2x1 -x2+3x3 = 8 x1,x2,x3, x4 ≥0	Min W=10y1+8y2 st y1+2y2 ≥5 2y1-y2 ≥12 y1+3y2 ≥ 4 y1 ≥0 y1,y2 urs

Example 2

Primal LP	Primal LP in standard Form	Dual LP
Min z=15x1+12x2 St x1+2x2 ≥ 3 2x1- 4x2 ≤ 5 x1,x2 ≥0	Min z=15x1+12x2 St x1+2x2 –x3 =3 2x1- 4x2 +x4= 5 x1,x2,x3,x4 ≥0	Max W=3y1+5y2 St y1+2y2 ≤15 2y1-4y2 ≤12 -y1 ≤ 0 y2 ≤ 0 y1,y2 urs

NOTE THAT: The dual of the dual problem is itself the primal problem

Example 2

Primal LP	Primal LP in standard Form	Dual LP
Max $z=5x1+6x2$ st $x1+2x2 = 5$ $-x1+5x2 \geq 3$ $4x1+7x2 \leq 8$ $x1$ urs, $x2 \geq 0$	Max $z= 5x_1^+ - 5x_1^- +6x2$ st $x_1^+ - x_1^- +2x2 \quad =5$ $-x_1^+ + x_1^- +5x2-x3 \quad = 3$ $4x_1^+ - 4x_1^- +7x2 \quad +x4= 8$ $x_1^+ , x_1^-, x2, x3, x4 \geq 0$ where $x1= x_1^+ - x_1^-$	Min $W=5y1+3y2+8y3$ st $y1-y2+4y3 \geq 5$ $-y1+y2-4y3 \geq -5$ $2y1+5y2+7y3 \geq 6$ $-y2 \geq 0$ $y3 \geq 0$ y1,y2,y3 urs

NOTE THAT: The general rule is that an unrestricted primal variable always corresponds to an equality dual constraints. Conversely, a primal equation produces an urs dual variable

Constructing the Dual LP by using LP in standard Form with artificials Example 1

Primal LP	Primal LP in standard Form with artificials	Dual LP
Max z=5x1+12x2+4x3 st x1+2x2 +x3 ≤ 10 2x1 -x2+3x3 = 8 x1,x2,x3≥0	Max z=5x1+12x2+4x3-Mx5+0x4 st x1+2x2 +x3 +x4 =10 2x1 -x2+3x3 +x5= 8 x1,x2,x3, x4,x5 ≥0	Min Y=10y1+8y2 st y1+2y2 ≥5 2y1-y2 ≥12 y1+3y2 ≥ 4 y1 ≥0 y2 ≥-M y1,y2 urs

General Rules for constructing Dual Problem

Maximization Problem	Minimization Problem
Constraints	Variables
$\geq$ $\leftrightarrow$	≤ 0
$\leq$ $\leftrightarrow$	≥ 0
$=$ $\leftrightarrow$	urs
Variables	Constraints
≥ 0 $\leftrightarrow$	$\geq$
≤ 0 $\leftrightarrow$	$\leq$
urs $\leftrightarrow$	$=$

Apply the rules for Example 1

Primal LP	Primal LP in standard Form	Dual LP
Max $z=5x1+12x2+4x3$ st $x1+2x2+x3 \leq 10$ $2x1-x2+3x3 = 8$ $x1,x2,x3 \geq 0$	Max $z=5x1+12x2+4x3$ st $x1+2x2+x3 + x4=10$ $2x1-x2+3x3 = 8$ $x1,x2,x3 \geq 0$	Min $Y=10y1+8y2$ st $y1+2y2 \geq 5$ $2y1-y2 \geq 12$ $y1+3y2 \geq 4$ $y1 \geq 0$ $y2$ urs

Apply the rules for Example 1

Primal LP	Dual LP
Max $z=5x_1+12x_2+4x_3$ st $x_1+2x_2+x_3 \le 10$ $2x_1-x_2+3x_3 = 8$ $x_1,x_2,x_3 \ge 0$	Min $Y=10y_1+8y_2$ st $y_1+2y_2 \ge 5$ $2y_1-y_2 \ge 12$ $y_1+3y_2 \ge 4$ $y_1 \ge 0$ y_2 urs

Apply the rules for Example 2

Primal LP		Dual LP
Max z=5x1+6x2 st x1+2x2 = 5 -x1+5x2 ≥ 3 4x1+7x2 ≤ 8 x1 urs, x2 ≥0		Min W=5y1+3y2+8y3 st y1-y2+4y3 = 5 2y1+5y2+7y3 ≥ 6 y1 urs y2 ≤ 0 y3 ≥ 0

Apply the rules, Example 3

Primal LP	Dual LP
Max z=2x1+x2 st x1+2x2 = 2 2x1 -x2 ≥ 3 x1 -x2 ≤ 1 x1 ≥0, x2 urs	Min W=2y1+3y2+y3 st y1+2y2 +y3 ≥ 2 2y1-y2-y3 =1 y1 urs y2 ≤ 0 y3 ≥ 0

Apply the rules, Example 4

Primal LP	Dual LP
Min W=2y1+4y2+6y3 st y1+2y2+y3 ≥ 2 y1- y3 ≥ 1 y2+ y3 = 1 2y1+y2 ≤3 y1 urs y2, y3 ≥ 0	Max z=2x1+x2+x3+3x4 st x1+x2 +2x4 = 2 2x1+ x3 +x4 ≤ 4 x1+x2+x3 ≤ 6 x1,x2 ≥0, x3 urs , x4≤0

Constructing the Dual LP by using LP in standard Form with artificials
Example 1

Primal LP	Primal LP in standard Form with artificials	Dual LP
Max z=5x1+12x2+4x3 st x1+2x2 +x3 ≤ 10 2x1 -x2+3x3 = 8 x1,x2,x3≥0	Max z=5x1+12x2+4x3-Mx5+0x4 st x1+2x2 +x3 +x4 =10 2x1 -x2+3x3 +x5= 8 x1,x2,x3, x4 ≥0	Min Y=10y1+8y2 st y1+2y2 ≥5 2y1-y2 ≥12 y1+3y2 ≥ 4 y1 ≥0 y2 ≥-M y1,y2 urs

PRIMAL DUAL RELATIONSHIPS

- The primal and dual solutions are so closely related.
- **<u>Properties for any pair of primal and dual problems:</u>**
- 1)For any pair of FEASIBLE primal and dual solutions:
- Objective value in max problem ≤ objective value in min problem
- 2) At the optimum solution for both problems
- Objective value in max problem = objective value in min problem

The general form of the primal problem is as follows:

$$\text{Max } z = \sum_{j=1}^{n} c_j x_j$$

$s.t.$

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i \qquad i = 1,2,....,m$$

$$x_j \ge 0 \qquad j = 1,2,....,n$$

or

P: $Max\, c\ x$

$s.t. \quad Ax \le b$

$x \ge 0$

The general form of the dual problem is as follows:

$$\text{Min } Y = \sum_{i=1}^{m} b_i y_i$$

$s.t.$

$$\sum_{i=1}^{m} a_{ij} y_i \ge c_j$$

$$y_i \ge 0$$

or

D: $Min\ y\ b$

$s.t. \quad y\ A \ge c$

$y \ge 0$

Lets see

P: $Max\, c\, x$ D: $Min\, yb$

$s.t. \quad Ax \le b$ $s.t. \quad yA \ge c$

$x \ge 0$ $y \ge 0$

1) If x is a feasible solution to (P) then

$Ax \le b$

$x \ge 0$

2)If y is a feasible solution to (D) then

$yA \ge c$

$y \ge 0$

since $y \ge 0$ multiplying the vector inequality $Ax \le b$ by y yields the scalar inequality yAx $\le$ yb

since $x \ge 0$ multiplying the vector inequality $yA \ge c$ by x yields the scalar inequality $yA\,x \ge cx$

combining these two yields

$$yb \ \ge \ yA\,x \ge cx$$

$$cx \le yAx \le yb$$

WEAK DUALITY THEOREM

WEAK DUALITY THEOREM

- $cx \le yAx \le yb$
- The objective value of any feasible solution to (D) is an upper bound to the objective value of any feasible solution to (P)

The general form of the primal problem is as follows:

$$
\begin{aligned}
\mathrm{P}:\ & \mathit{Max}\ cx \\
s.t.\ & Ax \le b \\
& x \ge 0
\end{aligned}
$$

The general form of the dual problem is as follows:

$$
\begin{aligned}
\mathrm{D}:\ & \mathit{Min}\ yb \\
s.t.\ & yA \ge c \\
& y \ge 0
\end{aligned}
$$

WEAK DUALITY THEOREM

$cx \le yAx \le yb$

- Weak Duality Theorem is very important result that forms the foundation for several other duality relationships.
- First of all notice that :
- Each feasible solution to the max problem provides a LB for the objective of the minimization problem.
- Likewise, each feasible solution to the min problem provides a UB for the objective of the maximization problem.

$$\begin{aligned} \text{P}: \quad & Max\ cx \\ s.t. \quad & Ax \le b \\ & x \ge 0 \end{aligned} \qquad\qquad \begin{aligned} \text{D}: \quad & Min\ yb \\ s.t. \quad & yA \ge c \\ & y \ge 0 \end{aligned}$$

STRONG DUALITY THEOREM

$$cx^* = y^*Ax^* = y^*b$$

- At the optimal point of (P) and (D) problem, the optimal objective value of the (P) is equal to the optimal objective value of (D)

Any pair of primal-dual problems will assume one of the following four states:

- (1) If the (P) objective is unbounded THEN the (D) problem is infeasible
- (2) If the (D) objective is unbounded THEN the (P) problem is infeasible
- (3) If the (P) problem is infeasible THEN the (D) is either infeasible or has an unbounded objective
- (4) If the (D) problem is infeasible THEN the (P) is either infeasible or has an unbounded objective

(1) If the (P) objective is unbounded THEN the (D) problem is infeasible

- (P) :

max $z = x$

st

$-x \leq 1$

$x \geq 0$

This problem is unbounded because x can be made arbitrarily large and still remain feasible

- (D):

Min $w = y$

st

$-y \geq 1$

$y \geq 0$

The corresponding dual problem is obviously infeasible

(2) If the (D) objective is unbounded THEN the (P) problem is infeasible

- At first glance, you can conclude that the converse of each corollary is true. However, this is not the case, because if one problem is infeasible, it is also possible for the other to be infeasible.

(3) (P) Infeasible (D) Infeasible

- Example
- (P) Max $z=x1+2x2$
- St
- $-x1+2x2 \leq -2$
- $x1-2x2 \leq -2$
- $x1,x2 \geq 0$

- (D) Min $w= -2y1-2y2$
- St
- $-y1+y2 \geq 1$
- $2y1-2y2 \geq 2$
- $y1,y2 \geq 0$

Any pair of primal-dual problems will assume one of the following four states:

	Primal	**Dual**
1	Feasible	Feasible
2	Unbounded	Infeasible
3	Infeasible	Unbounded
4	Infeasible	Infeasible

Finding Optimal Dual Solution While we know the Primal Optimal Tableau

- The optimal dual solution can be obtained directly from the optimal simplex tableau.

- Let us understand the Simplex Tableau Layout and identify the "**<u>Inverse Matrix"</u>**

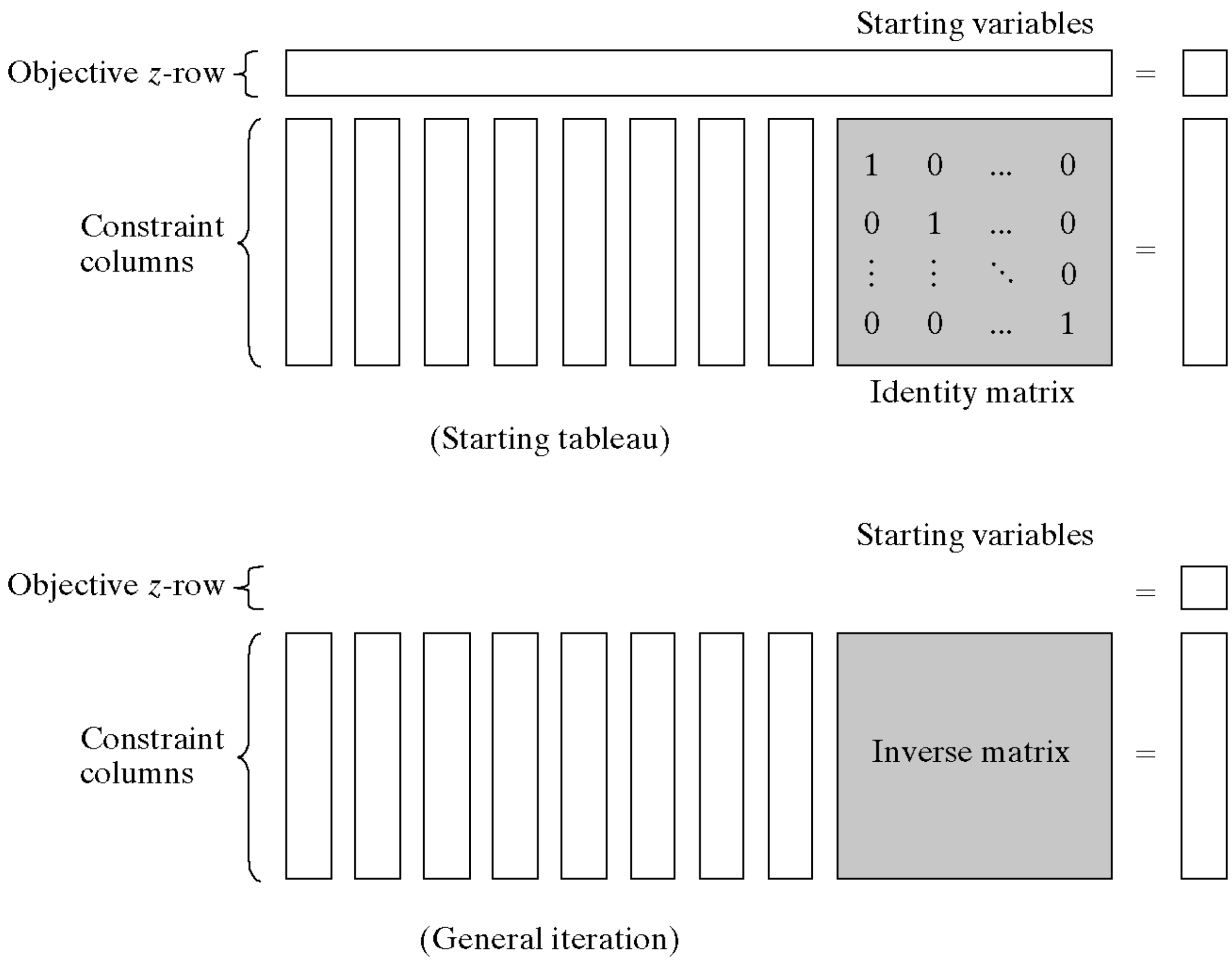

Figure 4.1

Schematic representation of the starting and general simplex tableaus.

LP:
Min z= -x1-3x2
s.t.
2x1 + 3x2 ≤ 6
-x1 + x2 ≤ 1
x1, x2 ≥ 0

Simplex Algorithm, Minimization Example

Iteration	Basic	x1	x2	S1	S2	RHS	MRT
(0) starting	z	1	3	0	0	0	
	S1	2	3	1	0	6	6/3=2
x2 enters S2 leaves	S2	-1	1	0	1	1	1/1=1*
(1)	z	4	0	0	-3	-3	
x1 enters S1 leaves	S1	5	0	1	-3	3	3/5*
	x2	-1	1	0	1	1	ignore
(2) optimum	z	0	0	-4/5	-3/5	-27/5	
	x1	1	0	1/5	-3/5	3/5	
	x2	0	1	1/5	2/5	8/5	

Big M Method, Example

- LP in Standard Form:

Min $z=4x1+x2+0x3+0x4$

St

$3x1+x2=3$

$4x1+3x2-x3=6$

$x1+2x2+x4=4$

$x1,x2 \geq 0$

- LP in Standard Form with Artificials:

Min$z=4x1+x2+0x3+0x4+MR1+MR2$

St

$3x1+x2+R1=3$

$4x1+3x2-x3+R2=6$

$x1+2x2+x4=4$

$x1,x2, x3, x4, R1, R2 \geq 0$

Iteration	Basic	x1	x2	x3	R1	R2	x4	RHS
(0)	z	-4+7M	-1+4M	-M	0	0	0	9M
X1 enters, R1 leaves	R1	3	1	0	1	0	0	3
	R2	4	3	-1	0	1	0	6
	x4	1	2	0	0	0	1	4
(1)	z	0	(1+5M)/3	-M	(4-7M)/3	0	0	4+2M
	x1	1	1/3	0	1/3	0	0	1
x2 enters, R2 leaves	R2	0	5/3	-1	-4/3	1	0	2
	x4	0	5/3	0	-1/3	0	1	3
(2)	z	0	0	1/5	(8/5)-M	-(1/5)-M	0	18/5
	x1	1	0	1/5	3/5	-1/5	0	3/5
	x2	0	1	-3/5	-4/5	3/5	0	6/5
x3 enters, x4 leaves	x4	0	0	1	1	-1	1	1
(3)	z	0	0	0	(7/5)-M	-M	-1/5	17/5
optimum	x1	1	0	0	2/5	0	-1/5	2/5
	x2	0	1	0	-1/5	0	3/5	9/5
Big M	x3	0	0	1	1	-1	1	1

Reddy Mikks Example

LP:

Max z= 5x1+4x2

s.t.

$6x_1 + 4x_2 \leq 24$

$x_1 + 2x_2 \leq 6$

$-x_1 + x_2 \leq 1$

$x_2 \leq 2$

$x_1, x_2 \geq 0$

LP in Standard Form:

Max z= 5x1+4x2+0S1+0S2+0S3+0S4

s.t.

$6x_1 + 4x_2 + S1 = 24$

$x_1 + 2x_2 + S2 = 6$

$-x_1 + x_2 + S3 = 1$

$x_2 + S4 = 2$

$x_1, x_2, S1, S2, S3, S4 \geq 0$

z function is in equation form

z-5x1-4x2-0S1-0S2-0S3-0S4=0

Reddy Mikks Example
Optimal Simplex Tableau

BV	z	x1	x2	S1	S2	S3	S4	RHS
z	1	0	0	3/4	1/2	0	0	21
x1	0	1	0	1/4	-1/2	0	0	3
x2	0	0	1	-1/8	3/4	0	0	3/2
S3	0	0	0	3/8	-5/4	1	0	5/2
S4	0	0	0	1/8	-3/4	0	1	1/2

INVERSE MATRIX, B^{-1}

- In the starting Tableau, the constraint coefficients under the starting variables form an Identity Matrix.
- According to this arrangement, subsequent iterations of the Simplex Tableau generated by the Gauss Jordan elementary row operations will modify the elements of the identity matrix to produce what is known as the inverse matrix
- Actually, the inverse matrix is the key to computing all the elements of the associated Simplex Tableau.

Method I

Finding optimal dual solution while we know the primal optimal tableau

$$\begin{bmatrix} \text{Optimal values of} \\ \text{Dual Variables} \end{bmatrix} = \begin{bmatrix} \text{Row vector of} \\ \text{original objective} \\ \text{coefficients of optimal} \\ \text{primal basic variables} \end{bmatrix} * \begin{bmatrix} \text{Optimal} \\ \text{primal} \\ \text{inverse} \end{bmatrix}$$

$$y = c_B * B^{-1}$$

Note that: This formula can be applied at any iteration .
All we need is the associated inverse matrix

- **Question :** For LP problem on the right, the optimal simplex Tableau is given below. Find the corresponding optimal dual solution by using Method I

- OR
- **Question:** Solve the following LP problem on the right by using Simplex Algorithm. Then according to the optimal simplex Tableau, find the corresponding optimal dual solution by using Method I

EXAMPLE 1

Primal LP

Min z=5x1+12x2+4x3

St

x1+2x2 + x3 ≤ 10

2x1 -x2 +3x3 = 8

x1,x2,x3 ≥ 0

- Optimal Tableau:

Iteration	Basic	x1	x2	x3	x4	R	RHS
(3)	z	0	0	3/5	29/5	-(2/5)+M	274/5
optimum	x2	0	1	-1/5	2/5	-1/5	12/5
	x1	1	0	7/5	1/5	2/5	26/5

(P) LP:	(P) LP in standard form with artificials	(D) LP:
Max $z=5x1+12x2+4x3$ St $x1+2x2+x3 \leq 10$ $2x1 - x2 + 3x3 = 8$ $x1,x2,x3 \geq 0$	Max $z=5x1+12x2+4x3+0x4-MR$ St $x1+2x2 + x3 + x4 = 10$ $2x1 - x2 + 3x3 + R = 8$ $x1, x2, x3, x4, R \geq 0$	Min $W= 10y1+8y2$ St $y1+2y2 \geq 5$ $2y1-y2 \geq 12$ $y1+3y2 \geq 4$ $y1 \geq 0$ y2 urs

Optimal Tableau

Iteration	Basic	x1	x2	x3	x4	R	RHS
(3)	z	0	0	3/5	29/5	-(2/5)+M	274/5
optimum	x2	0	1	-1/5	2/5	-1/5	12/5
	x1	1	0	7/5	1/5	2/5	26/5

- (D)
- Min W= 10y1+8y2
- St
- y1+2y2 ≥5
- 2y1-y2 ≥12
- y1+3y2 ≥4
- y1 ≥0
- y2 urs

$$y = c_B B^{-1}$$

$$y = \begin{bmatrix} y_1 & y_2 \end{bmatrix}$$

row vector of original obj. coefficients of optimal primal basic variables:

$$c_B = \begin{bmatrix} c_{x2} & c_{x1} \end{bmatrix} = \begin{bmatrix} 12 & 5 \end{bmatrix}$$

$$B^{-1} = \begin{bmatrix} 2/5 & -1/5 \\ 1/5 & 2/5 \end{bmatrix}$$

$$\begin{bmatrix} y_1 & y_2 \end{bmatrix} = \begin{bmatrix} 12 & 5 \end{bmatrix} \begin{bmatrix} 2/5 & -1/5 \\ 1/5 & 2/5 \end{bmatrix}$$

$$= \begin{bmatrix} 29/5 & -2/5 \end{bmatrix}$$

$$y_1 = 29/5$$

$$y_2 = -2/5$$

Check that the optimal objective value of the dual problem is equal to 274/5 because of the Strong Duality Theorem

Easy way to determine the values of the dual variables from the optimal primal Simplex Tableau

- 1) If the LP has all slacks in the standard form, then the dual variables are z-row coefficients under the slacks because our starting variables are our slack variables.
- 2) If the LP has not all slacks in standard form then look at z-row coefficients under the starting variables omit the M components

EXAMPLE 2

(P) LP:	(P) LP in standard form with artificials	(D) LP:
Max z= 3x1+2x2+5x3 St x1+3x2+2x3 ≤ 15 2x2 -x3 ≥ 5 2x1+x2- 5x3 =10 x1,x2,x3≥ 0	Max z= 3x1+2x2+5x3-Mx6 -Mx7 St x1+3x2+2x3 +x4 =15 2x2 -x3 –x5+x6 = 5 2x1+x2-5x3 +x7 =10 x1,x2,x3,x4,x5,x6,x7≥ 0	Min w=15y1+5y2+10y3 St y1 + 2y3 ≥ 3 3y1+2y2 + y3 ≥ 2 2y1 - y2 - 5y3 ≥ 5 y1 ≥0 y2≤ 0 y3 urs

Optimal Tableau

	x1	x2	x3	x4	x5	x6	x7	RHS
z	0	0	0	51/23	58/23	M-(58/23)	M+(9/23)	565/23
x3	0	0	1	4/23	5/23	-5/23	-2/23	15/23
x2	0	1	0	2/23	-9/23	9/23	-1/23	65/23
x1	1	0	0	9/23	17/23	-17/23	7/23	120/23

- Min w=15y1+5y2+10y3
- St
- y1+2y3 ≥3
- 3y1+2y2+y3≥ 2
- 2y1-y2-5y3 ≥5
- y1 ≥0
- y2≤ 0
- y3 urs

$$y = c_B B^{-1}$$

$$y = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}$$

row vector of original obj. coefficients of optimal primal basic variables:

$$c_B = \begin{bmatrix} c_{x3} & c_{x2} & c_{x1} \end{bmatrix} = \begin{bmatrix} 5 & 2 & 3 \end{bmatrix}$$

$$B^{-1} = \begin{bmatrix} 4/23 & -5/23 & -2/23 \\ 2/23 & 9/23 & -1/23 \\ 9/23 & -17/23 & 7/23 \end{bmatrix}$$

$$\begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} = \begin{bmatrix} 5 & 2 & 3 \end{bmatrix} \begin{bmatrix} 4/23 & -5/23 & -2/23 \\ 2/23 & 9/23 & -1/23 \\ 9/23 & -17/23 & 7/23 \end{bmatrix}$$

$$= \begin{bmatrix} 51/23 & -58/23 & 9/23 \end{bmatrix}$$

$$y_1 = 51/23$$

$$y_2 = -58/23$$

$$y_3 = 9/23$$

Check that the optimal objective value of the dual problem is equal to 565/23 because of the Strong Duality Theorem

EXAMPLE 3

(P) LP:	(P) LP in standard form with artificials	(D) LP:
Min z= 3x1+2x2+x3 St x1+x2+x3≥4 x2-x3 ≤2 x1+x2+2x3 =6 x1,x2,x3≥ 0	Min z= 3x1+2x2+x3 St x1+x2+x3-e1+a1 =4 x2-x3 +S2=2 x1+x2+2x3 +a3=6 x1,x2,x3,e1,a1,S2, a3≥ 0	Max w=4y1+2y2+6y3 St y1+y3 ≤3 y1+y2+y3 ≤ 2 y1-y2+2y3 ≤ 1 y1 ≥ 0 y2 ≤ 0 y3 urs

Optimal Tableau

	x1	x2	x3	e1	S2	a1	a3	RHS
z	-1	0	0	-3	0	3-M	-1-M	6
x2	1	1	0	-2	0	2	-1	2
S2	-1	0	0	3	1	-3	2	2
x3	0	0	1	1	0	-1	1	2

- Max w=4y1+2y2+6y3
- St
- y1+y3 ≤3
- y1+y2+y3 ≤ 2
- y1-y2+2y3 ≤ 1
- y1 ≥ 0
- y2 ≤ 0
- y3 urs

$$y = c_B B^{-1}$$

$$y = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}$$

row vector of original obj. coefficients of optimal primal basic variables:

$$c_B = \begin{bmatrix} c_{x2} & c_{S2} & c_{x3} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \end{bmatrix}$$

$$B^{-1} = \begin{bmatrix} 2 & 0 & -1 \\ -3 & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & -1 \\ -3 & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 & \text{-}1 \end{bmatrix}$$

$$y_1 = 3$$

$$y_2 = 0$$

$$y_3 = -1$$

Check that the optimal objective value of the dual problem is equal to 6 because of the Strong Duality Theorem

- ECONOMIC INTERPRETATION
- OF
- DUALITY

- 1)Economic Interpretation of Dual Variables (Shadow Prices-Simplex Multipliers)
- 2)Economic Interpretation of Dual Constraints

1)Economic Interpretation of Dual Variables (Shadow Prices-Simplex Multipliers)

(P):

$$\text{Max } z = \sum_{j=1}^{n} c_j x_j$$

$s.t.$

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i \qquad i = 1,2,\ldots.,m$$

$$x_j \ge 0 \qquad j = 1,2,\ldots.,n$$

(D):

$$\text{Min } w = \sum_{i=1}^{m} b_i y_i$$

$s.t.$

$$\sum_{i=1}^{m} a_{ij} y_i \ge c_j \qquad \text{j} = 1,2,..,\text{n}$$

$$y_i \ge 0 \qquad \text{i} = 1,2,...,\text{m}$$

or

P: $Max\ cx$

$$s.t. \quad Ax \le b$$
$$x \ge 0$$

or

D: $Min\ yb$

$$s.t. \quad yA \ge c$$
$$y \ge 0$$

Strong Duality Theorem

$$cx = yAx = yb$$

So, we can write

$$z = w = \sum_{i=1}^{m} b_i y_i = b_1 y_1 + b_2 y_2 + \ldots.. + b_m y_m \quad \text{at the optimum}$$

The rate of change of z with respect to b_i can be written as derivative of z with respect to the b_i

$$\frac{\partial z}{\partial b_i} = y_i$$

In other words, y_i represents the unit worth of resource i. They also refered as shadow price i.

the shadow price of the ith constraint is the amount by which the optimal z value changes if we increase b_i by 1(from b_i to b_i +1)

2)Economic Interpretation of Dual Constraints

- In any simplex iteration, the objective coefficient of primal variable xj is the difference between the LHS and RHS of the j^{th} dual constraint
- Z-row computations
- METHOD II

$$\begin{bmatrix} \text{primal z equation} \\ \text{coefficient of any} \\ \text{variable } \mathrm{x}_j \end{bmatrix} = \begin{bmatrix} \text{LHS of the jth} \\ \text{dual constraint} \end{bmatrix} - \begin{bmatrix} \text{RHS of the jth} \\ \text{dual constraint} \end{bmatrix}$$

$$z_j - c_j \qquad = \qquad \sum_{i=1}^{m} a_{ij} y_i \qquad - \qquad c_j$$

Imputed Cost
Unit usage of avaliable resources for activity j

(P) LP:	(P) LP in standard form with artificials	(D) LP:
Max z=5x1+12x2+4x3 St x1+2x2 + x3 ≤ 10 2x1 -x2 +3x3 = 8 x1,x2,x3 ≥ 0	Max z= 5x1 + 12x2 + 4x3 + 0x4 - MR St x1+2x2 +x3 +x4 =10 2x1 -x2 +3x3 +R = 8 x1,x2, x3, x4, R ≥ 0	Min W = 10y1+8y2 St y1+2y2 ≥5 2y1-y2 ≥12 y1+3y2 ≥4 y1 ≥ 0 y2 ≥ -M y1, y2 urs

Optimal Tableau

Iteration	Basic	x1	x2	x3	x4	R	RHS
(3)	z	0	0	3/5	29/5	-(2/5)+M	274/5
optimum	x2	0	1	-1/5	2/5	-1/5	12/5
	x1	1	0	7/5	1/5	2/5	26/5

REDUCED COST

- y1=29/5, y2=-2/5

- z coefficient of x1= z_1-c_1 =y1+2y2-5=(29/5)+2(-2/5)-5=0
- z coefficient of x2= z_2-c_2 =2y1-y2-12=2(29/5)-(-2/5)-12=0
- z coefficient of x3= z_3-c_3 =y1+3y2-4=(29/5)+3(-2/5)-4=3/5

- **REDUCED COST:** For any NBV, the reduced cost for the variable is the amount by which the NBV's objective function coefficient must be improved before that variable will be a BV in some optimal solution to the LP

- z coefficient of x4=y1-0=(29/5)-0=29/5
- z coefficient of R=y2-(-M)=(-2/5)-(-M)=-2/5+M

(P) LP:	(D) LP:
Min z= $5x1 + 12x2 + 4x3$ St $x1+2x2 + x3 \le 10$ $2x1 - x2 + 3x3 = 8$ $x1,x2,x3 \ge 0$	Min W= $10y1+8y2$ St $y1+2y2 \ge 5$ $2y1-y2 \ge 12$ $y1+3y2 \ge 4$ $y1 \ge 0$ $y2 \ge -M$ y1, y2 urs

z coefficient of x1=

$y1+2y2-5=(29/5)+2(-2/5)-5=0$

z coefficient of x2=

$2y1-y2-12=2(29/5)-(-2/5)-12=0$

z coefficient of x3=

$y1+3y2-4=(29/5)+3(-2/5)-4=3/5$

If we increase 4 by 3/5 $(4+(3/5))=23/5$

Then the z becomes $5x1+12x2+(23/5)x3$

In the optimal solution z=274/5

X*(0,3.14,3.71)

Optimal Tableau

Iteration	Basic	x1	x2	x3	x4	R	RHS
(3)	z	0	0	3/5	29/5	-(2/5)+M	274/5
optimum	x2	0	1	-1/5	2/5	-1/5	12/5
	x1	1	0	7/5	1/5	2/5	26/5

REDUCED COST ANALYSIS

- x3 (activity) 3 is unprofitable since z_3-c_3 is positive (3/5).
- If the company management wants to make activity 3 profitable, How they can do?
- x3 becomes attractive economically only if z_3-c_3<0 (in max problem, optimality condition) or equivalently $z_3 < c_3$

<u>(Either) increase the profit per unit (c_3)</u> **This is not possible to remain competitive in the market**	**<u>(Or) decrease the imputed cost of the used resources</u>** (z_3) This is possible by making improvement in production operations to reduce their unit usage of available resources

- Let r1, r2 represent the proportions by which the unit usage of available resources is reduced, the problem requires determining r1,r2 such that the new imputed cost, z3 of the two resources falls below the unit profit c3- that is
- $1(1-r1)y_1+3(1-r2)y_2<4$

- $1(1-r1)(29/5)+3(1-r2)(-2/5)<4$
- $(29/5)-(29/5)r1-(6/5)+(6/5)r2<4$
- $(23/5)-(29/5)r1+(6/5)r2<4$
- $-29r1+6r2<20-23$

- $29r1-6r2>3$

- Thus any values of r1 and r2 between 0 and 1 that satisfy the condition above should make x3 profitable

Toyco Example, Taha OR Book PP 135 for reduced cost analysis

- Toyco assembles three types of toys: trains, trucks and cars using three operations. The daily limits on the available times for the three operations are 430,460, and 420 minutes, respectively; and the profits per toy train, truck and car are $3, $2, and $5, respectively. The assembly times per train at the three operations are 1, 3 and 1 minutes, respectively. The corresponding times per truck and per car are (2, 0, 4) and (1, 2, 0) minutes (a zero time indicates that the operation is not used.)

Mathematical Model

- Decision variables
- $x1$: The daily number of units assembled of trains
- $x2$: The daily number of units assembled of trucks
- $x3$: The daily number of units assembled of cars
- LP:
- Max $z= 3x1+2x2+5x3$
- St
- $x1+2x2 +x3 \leq 430$
- $3x1 \quad +2x3 \leq 460$
- $x1+4x2 \quad \leq 420$
- $x1,x2,x3 \geq 0$

- (D):
- Min $w= 430y1+460y2+420y3$
- St
- $y1+3y2+y3 \geq 3$
- $2y1 \quad +4y3 \geq 2$
- $y1+2y2 \quad \geq 5$
- $y1,y2,y3 \geq 0$

Solution by Simplex Algorithm

Iteration	Basic	x1	x2	x3	x4	x5	x6	RHS	MRT
(0)	z	-3	-2	-5	0	0	0	0	
	x4	1	2	1	1	0	0	430	430/1
	x5	3	0	2	0	1	0	460	460/2*
	x6	1	4	0	0	0	1	420	
(1)	z	9/2	-2	0	0	5/2	0	1150	
	x4	-1/2	2	0	1	-1/2	0	200	200/2*
	x3	3/2	0	1	0	1/2	0	230	
	x6	1	4	0	0	0	1	420	420/4
(2)	z	4	0	0	1	2	0	1350	
Optimum	x2	-1/4	1	0	1/2	-1/4	0	100	
	x3	3/2	0	1	0	1/2	0	230	
	x6	2	0	0	-2	0	1	20	

Reduced Cost Analysis

- The optimal primal solution says that activity 2 and 3 profitable but activity 1 is not profitable.
- Q. How can the company management make activity 1 profitable (producing product 1)?
- A. x1 become attractive economically only if $z_1 - c_1 < 0$ or equivalently $z_1 < c_1$

1) <u>(Either) increase the profit per unit (c_1)</u>

- This is not possible to remain competitive in the market

2) <u>(Or) decrease the imputed cost of the used resources (z_1)</u>

- This is possible by making improvement in production operations to reduce unit usage of available resources for x1.

Q. How much can the company management reduce unit usage of available resources for x1?

A. Let r1, r2, r3 represent the proportions by which the unit usage of available resources is reduced, the problem requires determining r1,r2, r3 such that the new imputed cost, z_1 of the three resources falls below the unit profit c_1. that is

- $1(1-r1)y_1+3(1-r2)y_2+1(1-r3)y_3<3$ For $y_1=1$, $y_2=2$ and $y_3=0$
- $1(1-r1)1+3(1-r2)2+1(1-r3)0<3$ reorganize the inequality:
- $r1+6r2>4$
- Thus any values of r1 and r2 between 0 and 1 that satisfy the condition above should make x_1 profitable