ENM 202E
OPERATIONS RESEARCH (I)
OR (I)

LECTURE NOTES

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References

1. *ref.sabanciuniv.edu/sites/ref.sabanciuniv.edu/.../gu2_euroma_plenary.ppt*
2. *www.pitt.edu/~jrclass/or/or-intro.html*

Brief History of OR in World
www.pitt.edu/~jrclass/or/or-intro.html

While there is no clear date about the birth of O.R., it is generally accepted that the field originated in England during World War II. The impetus for its origin was the development of radar defense systems for the Royal Air Force, and the first recorded use of the term Operations Research is attributed to a British Air Ministry official named A. P. Rowe who constituted teams to do "operational researches" on the communication system and the control room at a British radar station. The studies had to do with improving the operational efficiency of systems (an objective which is still one of the cornerstones of modern O.R.).

Brief History of OR in World

This new approach of picking an "operational" system and conducting "research" on how to make it run more efficiently soon started to expand into other arenas of the war. Perhaps the most famous of the groups involved in this effort was the one led by a physicist named P. M. S. Blackett which included physiologists, mathematicians, astrophysicists, and even a surveyor. This multifunctional team focus of an operations research project group is one that has carried forward to this day. Blackett's biggest contribution was in convincing the authorities of the need for a scientific approach to manage complex operations, and indeed he is regarded in many circles as the original operations research analyst.

Brief History of OR in World

O.R. made its way to the United States a few years after it originated in England. Its first presence in the U.S. was through the U.S. Navy's Mine Warfare Operations Research Group; this eventually expanded into the Antisubmarine Warfare Operations Research Group that was led by Phillip Morse, which later became known simply as the Operations Research Group. Like Blackett in Britain, Morse is widely regarded as the "father" of O.R. in the United States, and many of the distinguished scientists and mathematicians that he led went on after the end of the war to become the pioneers of O.R. in the United States.

A BRIEF HISTORY OF OR IN TURKEY

ref.sabanciuniv.edu/sites/ref.sabanciuniv.edu/.../gu2_euroma_plenary.ppt

The first OR unit in Turkey was established in the General Staff of Armed Forces under the title Scientific Consultation Directorate on August 19, 1954. Mainly reserve officers with suitable background served in this unit. Later in 1958 the R&D Laboratories in the Air Force were attached to this unit. It continued to serve in the General Staff of Armed Forces until 1970 when it was transferred to the Ministry of Defence.

In 1973, another unit, which was first called Defense Research Directorate and then Armament and Defense Directorate, was established in the General Staff of Armed Forces.

A BRIEF HISTORY OF OR IN TURKEY

In the civilian sector, the first attempt took place on September 1, 1965. The Operations Research Unit was founded within the Scientific and Technical Research Council of Turkey and continued to operate on the campus of Middle East Technical University until 1973. In 1973, it was transferred as a Unit to the Marmara Scientific and Industrial Research Institute in Gebze, Kocaeli. Later in 1992 it was dissolved.

A BRIEF HISTORY OF OPERATIONS RESEARCH IN TURKEY

The first course on OR was offered in the Faculty of Mechanical Engineering at Istanbul Technical University in 1962-1963 academic year. Later two courses were initiated at the Middle East Technical University in 1964-1965 in the Mathematics Department. A graduate degree program was established in the same Department starting in 1965-1966 academic year.

Later OR courses became part of the fundamental course work in the Industrial Engineering Departments and Management Departments. Besides industrial engineering and management fields, OR courses are also included in the mathematics, statistics, econometrics, and regional and city planning curricula among others.

A BRIEF HISTORY OF OR IN TURKEY

Operations Research Association was established in 1975 and organized the first Operations Research National Conference the same year. Starting with the 15th National Congress in July 1993 the title of the Conference was changed to Operations Research and Industrial Engineering. National Conference.

Brief History of Operations Research

- Operations Research (OR) studies firstly started in 1940s.

- The application of mathematics and the scientific method to military operations was called operations research.

- **Definition (Operations Research):**

Today, the term operations research (or, often management science) means, a scientific approach to decision making, which seeks to determine how best to design and operate a system, usually under conditions requiring the allocation of scarce resources.

- **Definition**

Industrial Engineering is an engineering which deals with the solution of decision making problems in an organization which produces product and or service by using the scarce resources efficiently.

- **Definition (Operations Research):** Today, the term operations research (or, often management science) means, a scientific approach to decision making, which seeks to determine how best to design and operate a system, usually under conditions requiring the allocation of scarce resources.
- **Definition (Industrial Engineering**) is an engineering which deals with the solution of decision making problems in an organization which produces product and or service by using the scarce resources efficiently.

- **Definition (Operations Research):** Today, the term operations research (or, often management science) means, a ***scientific approach*** to decision making, which seeks TO DETERMINE HOW BEST to design and operate a system, usually under conditions requiring the allocation of scarce resources.
- **Definition (Industrial Engineering**) is an ***engineering which deals with*** the solution of decision making problems in an organization which produces product and or service by using the scarce resources EFFICIENTLY.

- OR is used to solve **decision making problems**.

- OR is both SCIENCE and an ART.

- **OR Team**
- **-**OR team consists of both the OR analysts and the client who has a problem.
- -OR analysts have experience in technical judgement, modeling and solving the problem.
- -Client has experience

- To solve a decision making problem Industrial Engineers use **the Methodology of OR**.

The Methodology of OR (Winston)

(Steps) Phases of an OR Study (Taha)

1. Definition of the problem
2. Construction of the model
3. Solution of the model
4. Validation of the model
5. Implementation of the solution

1.Definition of the problem

•Entire OR team work together

•The problem is defined verbally

•Analyst observes the system, collects data, estimates the values of parameters which affect the problem.

<u>The principle elements of a decision problem</u> <u>are identified</u>.

-decision alternatives / decision variables

-limitations

-objective criterion

2. Construction of the Model

The verbally defined problem is translated into **mathematical relationships**

•the resulting model may fit into one of the standard mathematical model such as

•linear programming

•*Integer Programming*

•**Dynamic Programming*

•**Network Programming*

•**Nonlinear programming*

•Otherwise, the OR team may use a heuristic approach, or the team may consider the use of simulation if appropriate.

• In some cases, a combination of mathematical, simulation and heuristic models may be needed to solve the decision problem.

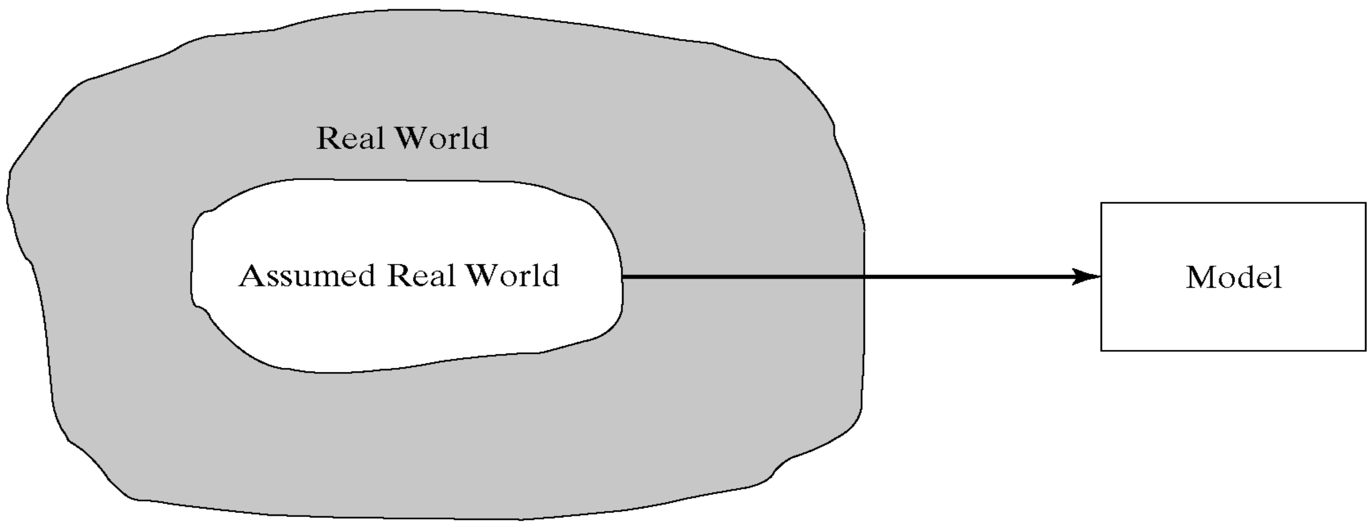


Figure 1.1
Levels of abstraction in model development.

3. Model Solution

- -The optimum solution is found by using **the well-defined optimization algorithms**.

- -Another important aspect in this phase is **Sensitivity Analysis**

*Sensitivity analysis deals with obtaining additional information about the behavior of the optimum solution when the model undergoes <u>some parameter changes.</u>

4.Model Validity

-OR team or the analyst tries to determine if the mathematical model developed in phase 2 is an **<u>accurate representation of reality</u>**

- -How we can check this?
- By using the <u>historical data.</u>
- By using <u>the simulation</u>

5. Implementation

- Analyst present the solution of the model in the understandable form to the decision making individual or group

- This presentation also includes the operating instructions of this solution.
- it may be presented several alternatives and one of them is chosen.
- Sometimes there exist some recommendations. This may result to return the phases 1,2,3,4

INTRODUCTION TO LINEAR PROGRAMMING

- -The most well-known OR technique is Linear Programming (LP)

- -LP is a tool for solving optimization problems (models) in which the objective and constraint functions are strictly **linear.**

Definition (Linear Function):

- A function $f(x_1,x_2,\ldots.,x_n)$ of $x_1,x_2,\ldots..,x_n$ is a linear function if and only if for some set of constants $c_1,c_2,\ldots.,c_n$,

$f(x_1,x_2,\ldots,x_n)=c_1x_1+c_2x_2+\ldots..+c_nx_n$

- <u>Example</u> $f(x_1,x_2)=2x_1+3x_2$ is a linear function

  $f(x_1,x_2)=2(x_1)^2\ 3x_2$ is not a linear function

- **Definition (Linear Inequalities):**
- For any linear function $f(x_1,x_2,\ldots,x_n)$, and any linear number b, the inequalities

  $f(x_1,x_2,\ldots,x_n)\leq b$ and

  $f(x_1,x_2,\ldots,x_n)\geq b$ are linear inequalities

A Linear Programming (LP) Problem:

- A linear programming problem is an optimization problem for which we do the following:

- 1) The **objective function** is linear function of decision variables. We want to maximize or minimize the objective function

- 2) The values of decision variables must satisfy **a set of constraints**. Each constraint must be a linear equation or linear inequality

- 3) A **sign restriction** is associated with each variable. For any variable x_j, the sign restriction specifies that x_i must be either nonnegative or **u**n**r**estricted in **s**ign (u.r.s)

General Representation of an Mathematical Model of an LP Problem

- Maximize (or minimize) $Z= c_1x_1+c_2x_2+\ldots.+c_nx_n$

  Subject to

  $a_{11}x_1+a_{12}x_2+\ldots\ldots\ldots+a_{1n}x_n \{\leq,=,\geq\}b_1$
  $a_{21}x_1+a_{22}x_2+\ldots\ldots\ldots+a_{2n}x_n \{\leq,=,\geq\}b_2$
  .. .. ………. .. .. ..
  $a_{m1}x_1+a_{m2}x_2+\ldots\ldots\ldots+a_{mn}x_n\{\leq,=,\geq\}b_m$
  $x_1\geq0,\ x_2\geq0,\ \ldots\ldots,\ x_n\geq0$

- We have n unknowns(decision variables), m equations(constraints)
- Subject to means:that the values of the decision variables x1,x2,….,xn must satisfy all constraints and all sign restrictions

The closed form of the LP formulation can be written in summation notation:

$$\max(or \min) \;\; z = \sum_{j=1}^{n} c_j x_j$$

$$st.$$

$$\sum_{j=1}^{n} a_{ij} x_j \{\leq, =, \geq\} \; b_i \qquad \mathrm{i = 1,2,...,m}$$

$$\mathrm{x_j \geq 0} \qquad \mathrm{j = 1,2,...,n}$$

- **Decision Variables:** Principal elements of a decision making problem
- **Technological Coefficients:**
- The coefficients of the decision variables in the constraints are called technological coefficients.
- Because they often reflect the technology used to produce different products.
- **The Right Hand Side (RHS) of each Constraint:** The number on the RHS of each constraint is called the constraint's RHS.
- RHS represents (often) the quantity of a **resource** that is available.
- **Sign Restrictions:**
- If a decision variable x_i can only assume nonnegative values, then we add the sign restrictions $x_i \geq 0$
- If a decision variable x_i can assume both positive and negative (or zero) values, then we say that x_i is unrestricted in sign (**u**n**r**estricted in **s**ign (u.r.s))
  - Example (u.r.s) A decision variable may represent the cash balance of a firm. The firm may use the more money then it has owned.

Example (Giapetto Example –Winston OR)

Identify the parameters and the variables in the following model

- Max $Z=3x_1+2x_2$
- st
- $2x_1+x_2 \leq 100$
- $x_1+x_2 \leq 80$
- $x_1 \leq 40$
- $x_1 \geq 0,\ x_2 \geq 0$

Assumptions of Linear Programming (LP)

- 1) **Proportionality** requires the contribution of each decision variable in the objective function and its requirements in the constraints to be directly proportional to the value of the variable.
- Max $Z=3x_1+2x_2$
- st
- $2x_1+x_2 \leq 100$
- $x_1+x_2 \leq 80$
- $x_1 \leq 40$
- $x_1 \geq 0,\ x_2 \geq 0,$

- For example the contribution of x_1 in the objective function is "3"
- If we $x_1=4$ then its contribution to the objective function is $3\times4=12$
- If we $x_1=1$ then its contribution to the objective function is $3\times1=3$
- These values (12,3) are proportional to the "3"

Assumptions of LP

- 2) **Additivity** assumption is that the total contribution of all the variables in the objective function and their requirements in the constraints are the direct sum of the individual contribution or requirements of each variable.
- Max $Z=3x_1+2x_2$
- st
- $2x_1+x_2 \leq 100$
- $x_1+x_2 \quad \leq 80$
- $x_1 \qquad \leq 40$
- $x_1 \geq 0,\ x_2 \geq 0,$

- 3) **Divisibility** assumption requires that each decision variable be allowed to assume fractional values.
- In the example , the divisibility assumption implies that it is acceptable $x_1=1.5$ or $x_2=1.63$
- 4)**Certainty** assumption is that each ***parameter*** is known with certainty .

- Feasible Solution

A solution of the model is **feasible** if it satisfies all constraints.

- Optimal Solution

A solution is **optimal** if in addition to being feasible, it yields the best (max or min) value of the objective function.

- **Problem 1. (Giapetto example, Winston's OR Book, 2nd. Ed. pp 51 Example 1)**
- A company manufactures two types wooden toys: soldiers and trains. A soldiers sells for \$27 and uses \$10 worth of raw materials. Each soldier that is manufactured increases variable labor costs by \$14. A train sells for \$21 and uses \$9 worth of raw materials. Each train built increases the variable labor costs \$10. The manufacture of wooden soldiers and trains requires two types of skilled labor: A soldier requires 2 hours of finishing labor and 1 hour of carpentry labor. A train requires 1 hour of finishing labor and 1 hour of carpentry labor. Each week, the company can obtain all the needed raw material but only 100 finishing hours and 80 carpentry hours. Demand for trains is unlimited but at most 40 soldiers are bought each week. Company wishes to maximize weekly profit (revenues-costs). Formulate a mathematical model of company's situation that can be used to maximize its weekly profit.

Identify the elements of the decision making problem

1) Define Decision Variables
2) Define Constraints
3) Define Objective Function

Add sign restrictions

- **Problem 1. (Giapetto example, Winston's OR Book, 2nd. Ed. pp 51 Example 1)**
- A company manufactures two types wooden toys: soldiers and trains. A soldiers sells for \$27 and uses \$10 worth of raw materials. Each soldier that is manufactured increases variable labor costs by \$14. A train sells for \$21 and uses \$9 worth of raw materials. Each train built increases the variable labor costs \$10. The manufacture of wooden soldiers and trains requires two types of skilled labor: A soldier requires 2 hours of finishing labor and 1 hour of carpentry labor. A train requires 1 hour of finishing labor and 1 hour of carpentry labor. Each week, the company can obtain all the needed raw material but only 100 finishing hours and 80 carpentry hours. Demand for trains is unlimited but at most 40 soldiers are bought each week. Company wishes to maximize weekly profit (revenues-costs). Formulate a mathematical model of company's situation that can be used to maximize its weekly profit.

- Step 1. Define decision variables

In our example, we must decide how many toy soldiers and toy trains should be manufactured each week.

Hence,

x_1= The number of toy soldiers produced each week

x_2= The number of toy trains produced each week

Step 2. Construct the objective function: maximizing the weekly profit

	Selling Price ($)	Raw Material Cost ($)	Labor Cost ($)
A Toy Soldier	27	10	14
A Toy Train	21	9	10

Profit= Revenue-Cost

Profit A soldier= 27-10-14= $3

Profit A train =21- 9-10= $2

Weekly profit= (weekly contribution to profit from toy soldiers) +(weekly contribution to profit from toy trains)

$$=\left(\frac{cont.to\ profit}{soldier}\right)\left(\frac{soldiers}{week}\right)+\left(\frac{cont.to\ profit}{train}\right)\left(\frac{trains}{week}\right)$$

$$=(\quad 3 \quad)(\quad x_1 \quad)+(\quad 2 \quad)(\quad x_2 \quad)$$

Maximize z= = $3x_1 + 2x_2$

Step 3. Define constraints

	Finishing (hours/week)	Carpentry (hours/week)
A toy soldier	2	1
A toy train	1	1
Limitations	100	80

Constraint 1. Each week no more than 100 hours of finishing time

$$\left(\frac{Total\ finishing\ hours}{week}\right)=\left(\frac{finishing\ hours}{soldier}\right)\left(\frac{soldiers\ made}{week}\right)+\left(\frac{finishing\ hours}{train}\right)\left(\frac{trains\ made}{week}\right)=2x_1+x_2$$

$$2x_1+x_2\leq 100$$

Constraint 2. Each week no more than 80 hours of carpentry time

$$\frac{Total\ carpentry\ hours}{week}=\left(\frac{carpentry\ hours}{soldier}\right)\left(\frac{soldiers\ made}{week}\right)+\left(\frac{carpentry\ hours}{train}\right)\left(\frac{trains\ made}{week}\right)=x_1+x_2$$

$$x_1+x_2\leq 80$$

Constraint 3. Because of the limited demand, at most 40 toy soldiers should be produced each week

$$x_1\leq 40$$

Step 4. Add sign restrictions
$x_1 \geq 0$, $x_2 \geq 0$

- LP Model:

$$
\begin{aligned}
&\text{Max } z = 3x_1 + 2x_2 \\
&\quad \text{s.t.} \\
&\qquad 2x_1 + x_2 \leq 100 \\
&\qquad x_1 + x_2 \leq 80 \\
&\qquad x_1 \leq 40 \\
&\qquad x_1 \geq 0,\ x_2 \geq 0
\end{aligned}
$$

What if there is an additional requirement to produce four soldiers for each train?

Assume that there was requirement to produce four toy soldiers for each toy train.

- This is an additional constraint, which will ensure that for every train produced there will be four soldiers available. This condition can be expressed mathematically as follows:

- x2=(x1/4)

Homework. Problem 2. (Winston's OR Book, pp.67 Example 3)

- An auto company manufactures cars and trucks. Each vehicle must be processed in the paint shop and assembly shop. If the paint shop were only painting trucks, 40 trucks per day could be painted. If the paint shop were only painting cars, 60 cars per day could be painted. If the body shop were only producing cars, it could be process 50 cars per day. If the body shop were only producing trucks, it could be process 50 trucks per day. Each truck contributes $300 to profit, and each car contributes $200 to profit. Formulate a linear programming model to determine a daily production schedule that will maximize the company's profits.

Problem 3. Feed Mix Problem (Taha's OR Book 7th. Ed. pp.18 Example 2.2-2)

- A farm uses at least 800 lb of special feed daily. The special feed is a mixture of corn and soybean meal with the following compositions

Feedstuff	Protein	Fiber	Cost($/lb)
Corn	0.09	0.02	0.30
Soybean	0.60	0.06	0.90

- The dietary requirements of the special feed are at least 30% protein and at most 5% fiber. Farmer wishes to determine the daily minimum cost feed mix. Formulate the mathematical model.

Problem 4. Diet Problem (Winston 2nd Edition, pp. 73 Example 6)

My diet requires that all the food I ate come from one of the four “basic food groups” (chocolate cake, ice cream, soda, and cheesecake). At present, the following four foods are available for consumption: brownies, chocolate ice cream, cola, and pineapple cheesecake. Each brownie costs 50¢, each scoop of chocolate ice cream costs 20¢, each bottle of cola costs 30¢, and each pineapple cheesecake costs 80¢. Each day, I must ingest at least 500 calories, 6 oz of chocolate, 10 oz of sugar, and 8 oz of fat. The nutritional content per unit of each food is shown in Table. Formulate a linear programming model that can be used to satisfy my daily nutritional requirements at minimum cost.

	CALORIES	CHOCOLATE (ounces)	SUGAR (ounces)	FAT (ounces)
Brownie	400	3	2	2
Chocolate ice cream (1 scoop)	200	2	2	4
Cola (1 bottle)	150	0	4	1
Pineapple cheesecake (1 piece)	500	0	4	5

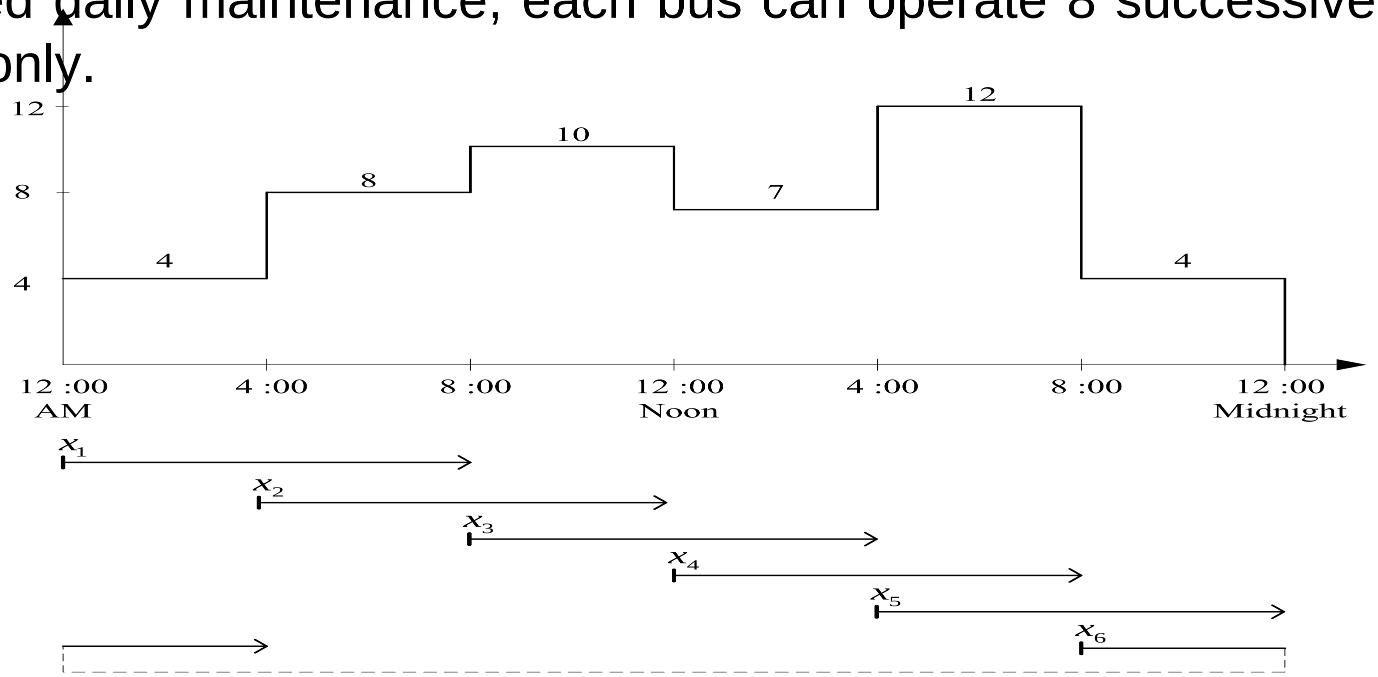
Problem 5. Work Scheduling Problem (Winston's OR Book 2nd. Edition, pp 77 Example 7)

A post office requires different numbers of full-time employees on different days of the week. The number of full-time employees required on each day is given in the Table below. Union rules state that each full-time employee must work five consecutive days and then receive two days off. For example, an employee who works on Monday to Friday must be off on Saturday and Sunday. The post office wants to meet its daily requirements using only full-time employees. Formulate an LP that the post office can use to minimize the number of full-time employees that must be hired.

Days	Number of full time Employees required
Monday	17
Tuesday	13
Wednesday	15
Thursday	19
Friday	14
Saturday	16
Sunday	11

Homework. Problem 6. Bus Scheduling Problem (Taha's OR Book 7th. Ed. pp.53 Example 2.5-3)

Progress City is studying the feasibility of introducing a mass-transit bus system that will alleviate the smog problem by reducing in-city driving. The study seeks the minimum number of buses that can handle the transportation needs. After gathering necessary information, the city engineer noticed that the minimum number of buses needed fluctuated with the time of the day and that the required number of buses could be approximated by constant values over successive 4 hour intervals. Figure 2.8 summarizes the engineer's findings. To carry out the required daily maintenance, each bus can operate 8 successive hours a day only.



- **Problem 7. Bank Loan Policy (Adopted from Taha's OR Book 7th. Ed. pp 47 Example 2.1)**
- Thriften Bank is in the process of devising a loan policy. The Bank has $12 million available for these loans. The titles of these loans, with their respective yearly interest rates charged to customers, are presented in the table below. Competition with other financial institutions requires the bank to allocate at least 40% of the total loans to farm and commercial loans. To assist the housing industry in the region, home loans must equal at least 50% of personal, car and home loan. Formulate the LP model for the situation which seeks to determine the amount of loan in each category to maximize the total yield of the Bank.

	Type of loan	**Interest Rate**
1	personal	**0.140**
2	car	**0.130**
3	home	**0.120**
4	farm	**0.125**
5	commercial	**0.100**

- **Homework-Problem 8. (Production Planning Problem,)**
- A company manufactures four variants of the same product and in the final part of the manufacturing process there are assembly, polishing and packing operations. Four each variant the time required for these operations is shown below (in minutes) as is the profit per unit sold.
- a) Given the current state of the labor force the company estimate that, each year, they have 100 000 minutes of assembly time, 50 000 minutes of polishing time and 60 000 minutes of packing time available. How many of each variant should the company make per year and what is the associated profit. **Formulate the LP model.**
- b) Suppose now that the company is free to decide how much time to devote to each of the three operations within the total allowable time of 210 000 minute. How many of each variant should the company make per year and what is the associated profit. Formulate the LP model.

Variant	Assembly	Polish	Pack	Profit
1	2	3	2	1.5
2	4	2	3	2.5
3	3	3	2	3.0
4	7	4	5	4.5

- **Problem 9. Advertising-Marketing Management Problem (Winston's OR Book 2nd Ed. pp 63 Example 2)**
- Dorian Auto manufactures luxury **cars** and **trucks**. The company believes that its most likely customers are high-income women and high-income men. To reach these groups Dorian Auto has embarked on an ambitious TV advertising campaign and has decided to purchase 1-minute commercial spots on two types of programs: comedy shows and football games. Each comedy commercial is seen by 7 million high-income women and 2 million high-income men. Each football commercial is seen by 2 million high-income women and 12 million high-income men. A 1-minute comedy ad costs $50,000 and a 1-minute football ad costs $100,000. Dorian would like the commercials to be seen by at least 28 million high-income women and 24 million high-income men. Use linear programming to determine how Dorian Auto can meet its advertising requirements at minimum costs.

Problem 10. (Taha's OR Book 7th. Ed. pp 12 Example 2.1-1.)

Reddy Mikks Company produces both interior and exterior paints from two raw materials, M1 and M2. The following table provides the basic data of the problem:

	Tons of raw material per ton of		Maximum daily availability (tons)
	Exterior paint	Interior paint	
Raw material, M1	6	4	24
Raw material, M2	1	2	6
Profit per ton ($1000)	5	4	

A market survey indicates that the daily demand for interior paint cannot exceed that of exterior paint by more than 1 ton. Also, the maximum daily demand of interior paint is 2 tons.

Reddy Mikks wants to determine the optimum (best) product mix of interior and exterior paints that maximizes the total daily profit.

GRAPHICAL SOLUTION

- **The graphical solution procedure can be used when the LP problem has only two decision variables.**

- The procedure includes two steps:

- 1) Determination of the feasible region (solution space) of the LP model
  - The **feasible region** for an LP is the set of all points that satisfies all the LP's constraints and sign restrictions.So, graph all constraints and sign restrictions.

- 2) Determination of the optimum solution from among all feasible points in the solution space.
  - For a maximization (minimization) problem optimal solution to an LP is **<u>a point</u>** in the solution space with the largest (the smallest) objective function value.
  - The term **<u>point</u>** to mean a specification of the value for each decision variable.

Remember The Following Definitions

- <u>*Feasible Solution*</u>

*A solution of the model is **feasible** if it satisfies all constraints and sign restrictions.*

- <u>*Optimal Solution*</u>

*A solution is **optimal** if in addition to being feasible, it yields the best (max or min) value of the objective function.*

The graphical solution of Maximization Problems

- ***REMEMBER* THE VERBAL DEFINITION**
- ***Problem 1. (Giapetto Example, Maximization Problem)***
- *A company manufactures two types wooden toys: soldiers and trains. A soldiers sells for $27 and uses $10 worth of raw materials. Each soldier that is manufactured increases variable labor costs by $14. A train sells for $21 and uses $9 worth of raw materials. Each train built increases the variable labor costs $10. The manufacture of wooden soldiers and trains requires two types of skilled labor: A soldier requires 2 hours of finishing labor and 1 hour of carpentry labor. A train requires 1 hour of finishing labor and 1 hour of carpentry labor. Each week, the company can obtain all the needed raw material but only 100 finishing hours and 80 carpentry hours. Demand for trains is unlimited but at most 40 soldiers are bought each week. Company wishes to maximize weekly profit (revenues-costs). Formulate a mathematical model of company's situation that can be used to maximize its weekly profit.*

The graphical solution of Maximization Problems-Example 1

- LP: Max $z=3x_1+2x_2$

  st.

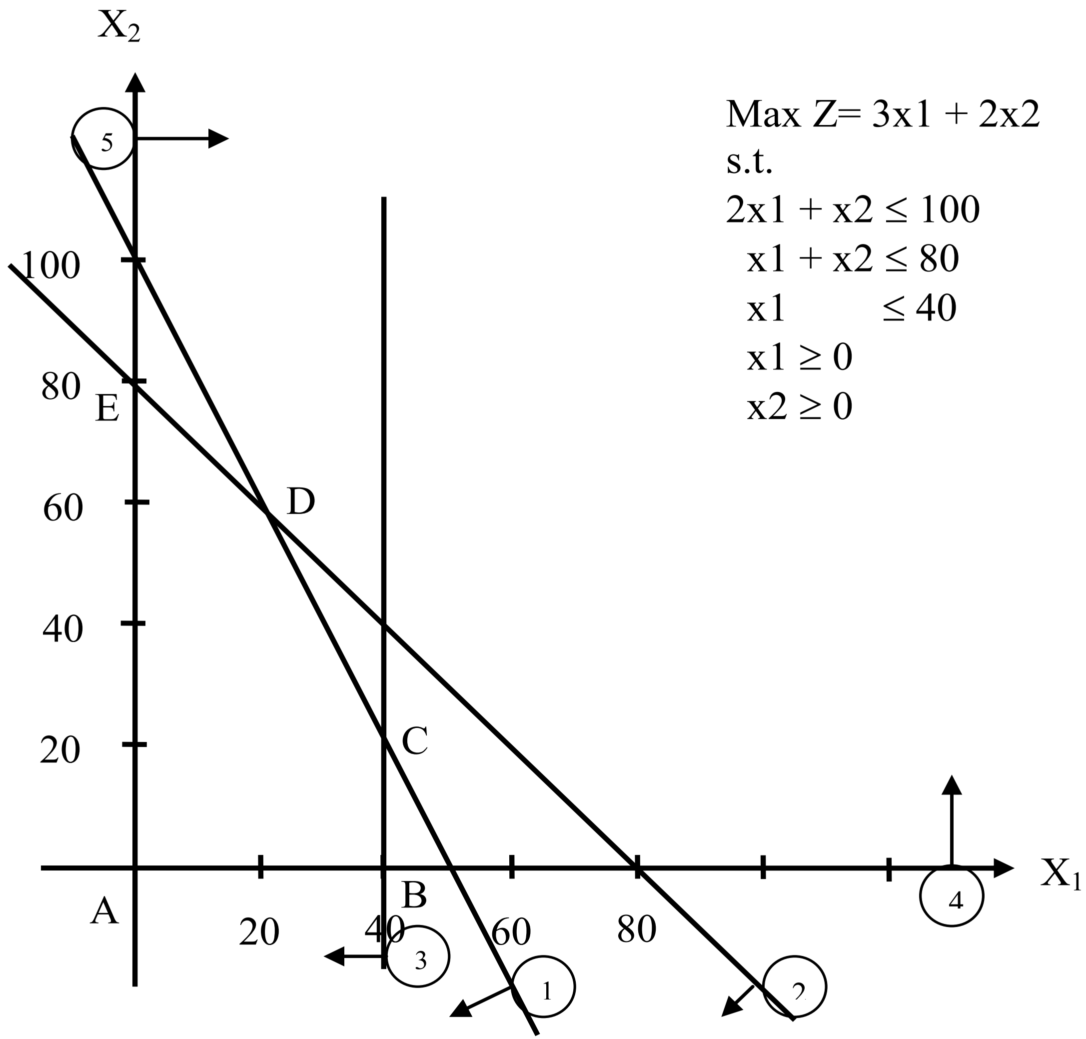
  $2x_1+x_2 \leq 100$

  $x_1+x_2 \leq 80$

  $x_1 \leq 40$

  $x_1 \geq 0,\ x_2 \geq 0,$
- Consider the non-negativity constraints first,
- Consider the other three constraints,
- What is the effect of the inequality in a constraint?
- Identify the direction in which the profit function $z=3x_1+2x_2$ increases,
- Draw iso-profit lines for maximization problems

(Draw iso-cost lines for minimization problems)



$$\text{Max } Z = 3x1 + 2x2$$

s.t.

$$2x1 + x2 \le 100 \qquad (1)$$

$$x1 + x2 \le 80 \qquad (2)$$

$$x1 \le 40 \qquad (3)$$

$$x1 \ge 0 \qquad (4)$$

$$x2 \ge 0 \qquad (5)$$

Graphical Solution

Max Z= 3x1 + 2x2
s.t.

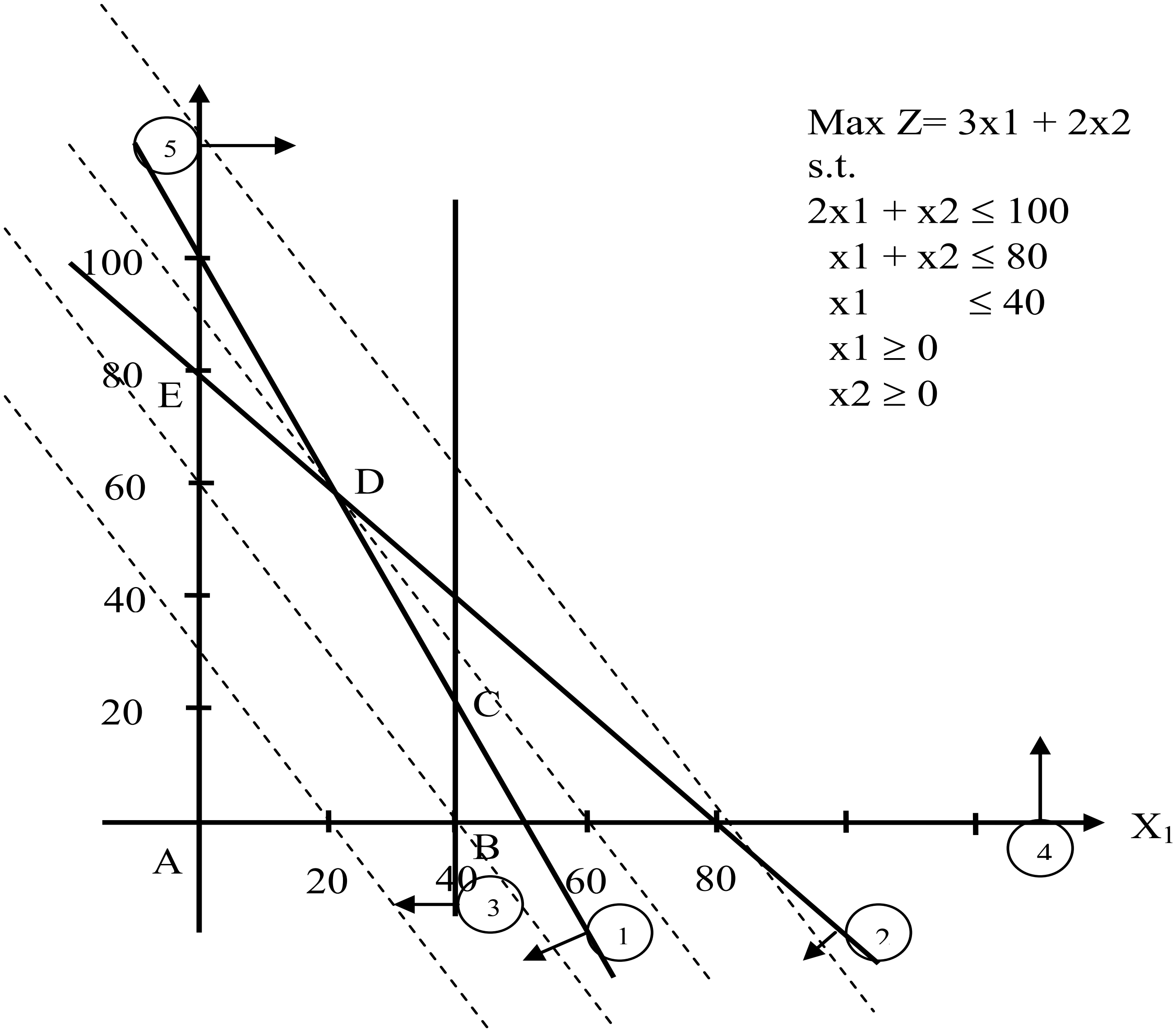
$2x_1 + x_2 \leq 100$ (1)

$x_1 + x_2 \leq 80$ (2)

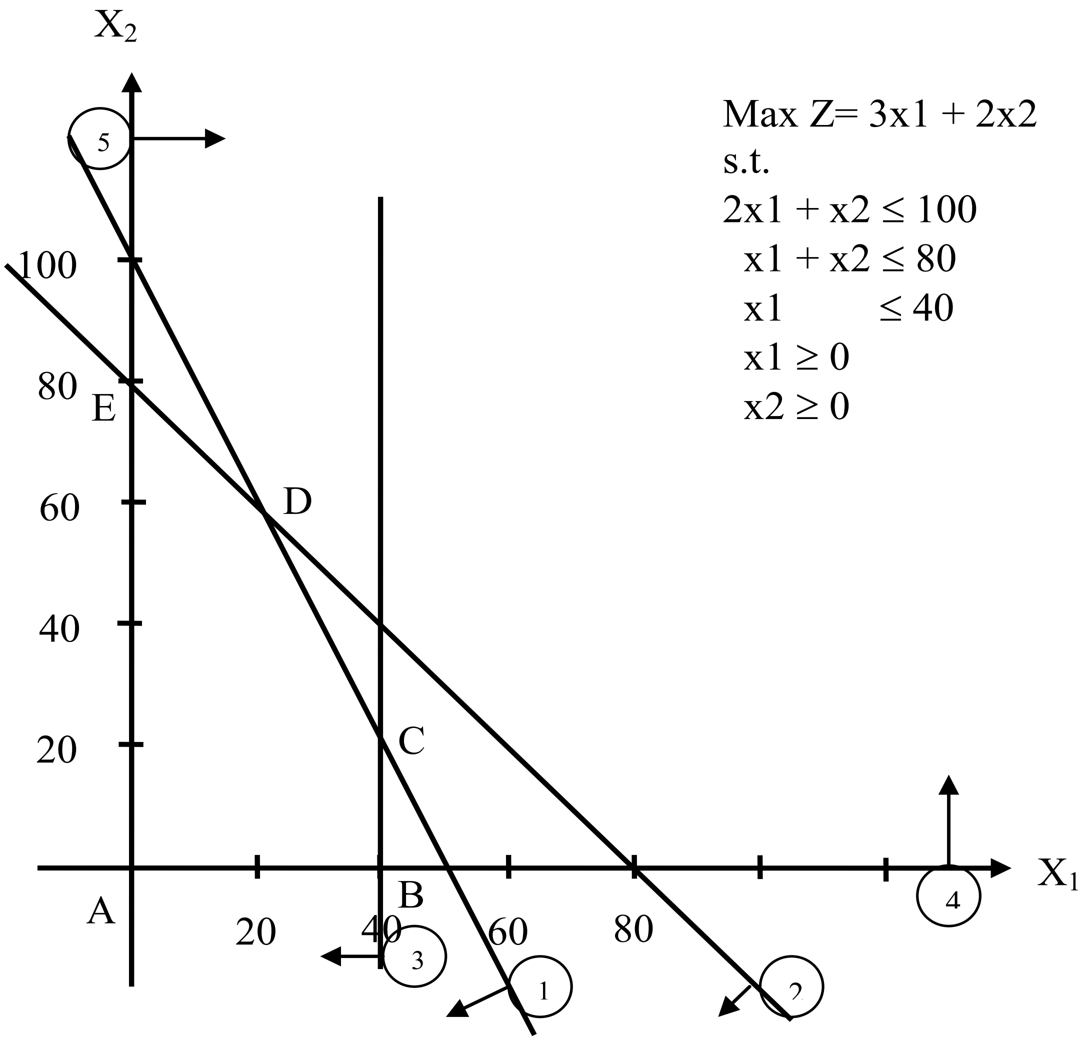
$x_1 \leq 40$ (3)

$x_1 \geq 0$ (4)

$x_2 \geq 0$ (5)



Graphical Solution



Max Z= 3x1 + 2x2

s.t.

2x1 + x2	≤ 100	(1)
x1 + x2	≤ 80	(2)
x1	≤ 40	(3)
x1	≥ 0	(4)
x2	≥ 0	(5)

Graphical Solution

•**Example 2.1-1. (Reddy Mikks Company, Taha’s OR Book pp 12)**

Reddy Mikks produces both interior and exterior paints from two raw materials, M1 and M2. The following table provides the basic data of the problem:

	Tons of raw material per ton of		Maximum daily availability (tons)
	Exterior paint	Interior paint	
Raw material, M1	6	4	24
Raw material, M2	1	2	6
Profit per ton ($1000)	5	4	

A market survey indicates that the daily demand for interior paint cannot exceed that of exterior paint by more than 1 ton. Also, the maximum daily demand of interior paint is 2 tons.

Reddy Mikks wants to determine the optimum (best) product mix of interior and exterior paints that maximizes the total daily profit.

- Max $Z=5x_1+4x_2$
- st
- $6x_1 + 4x_2 \le 24$
- $x_1 + 2x_2 \le 6$
- $-x_1 + x_2 \le 1$
- $x_2 \le 2$
- $x_1 \ge 0, x_2 \ge 0,$

- Consider the non-negativity constraints first,
- Consider the other four constraints,
- What is the effect of the inequality in each constraint?
- Identify the direction in which the profit function $Z=5x_1+4x_2$ increases,
- Draw iso-profit lines

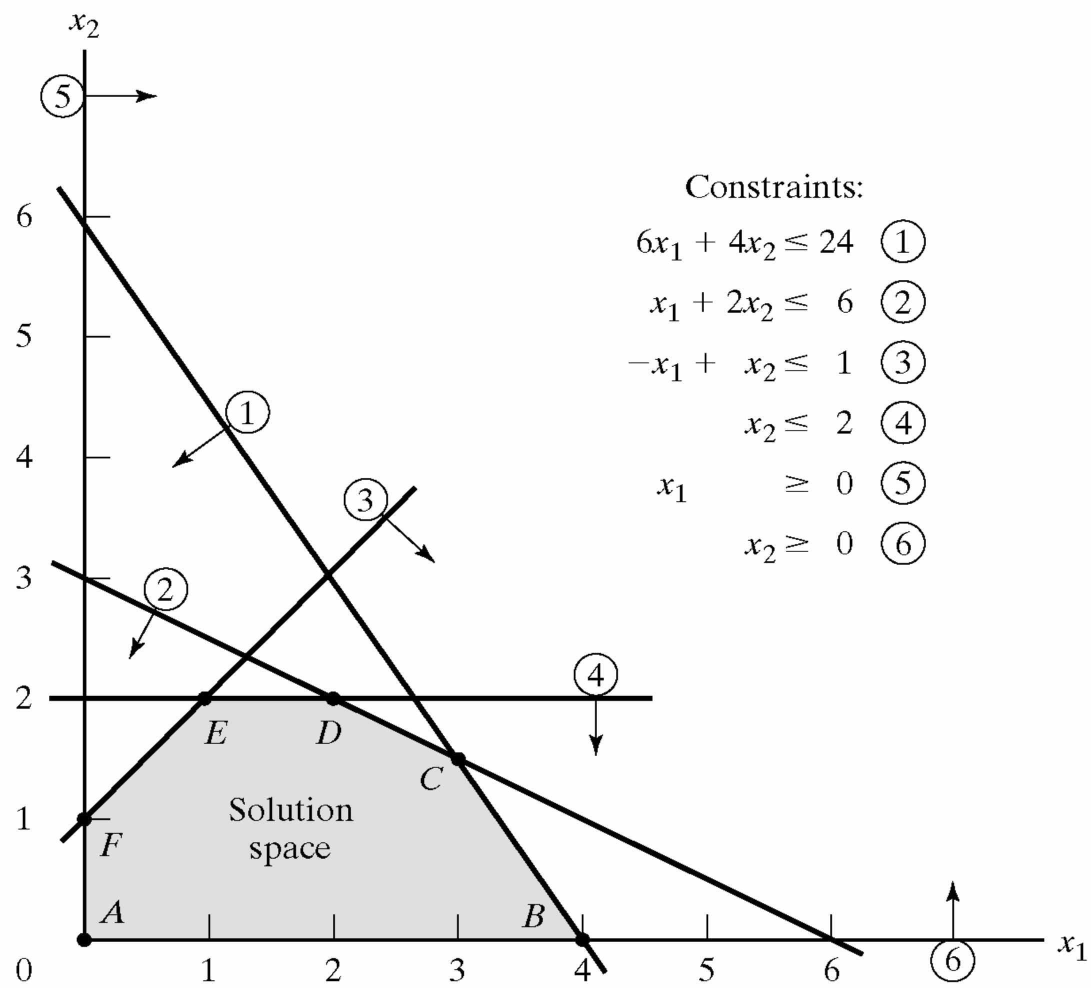


Figure 2.1

Feasible space of the Reddy Mikks model.

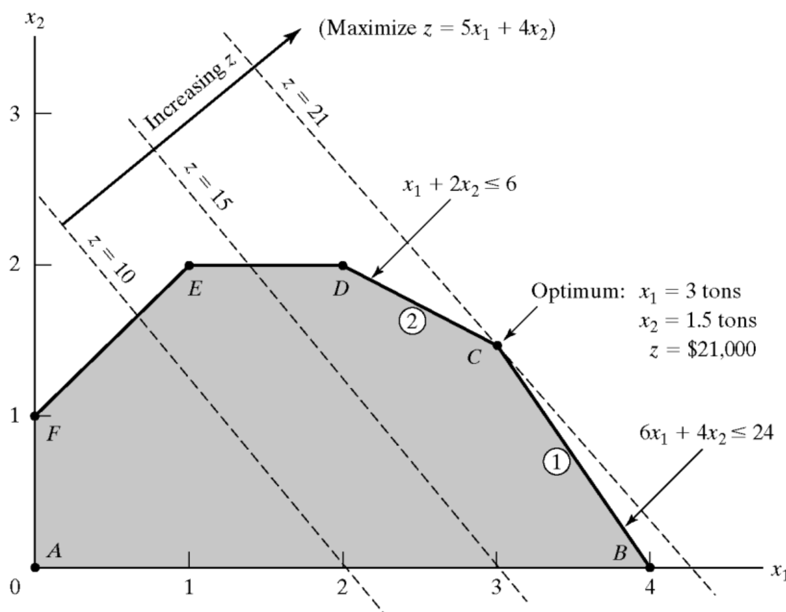


Figure 2.2
Optimum solution of the Reddy Mikks model.

Special Note

- It is not accidental that the optimum solution occurs at a corner point of the solution space where two lines intersect.

- Indeed, If we change the slope of the objective function, we will discover that the optimum solution always occurs at a **corner point**.

- This observation is the key to the development of the general SIMPLEX ALGORITHM.

Convex Sets

- A set of points S is a convex set if the line segment joining any pair of points in S is wholly contained in S.

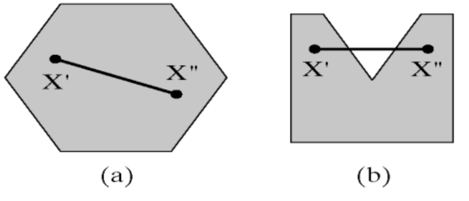


Figure 7.1
Examples of a convex and a nonconvex set.

CORNER POINT(Extreme Point)

- For any **convex set** S, a point P in S is an extreme point if each line segment, that lies completely in S and contains the point P, has P as an endpoint of the line segment.
- Extreme points are also called Corner Points because if the set S is a polygon, the extreme points of S will be the vertices, or corners of the polygon.

The feasible region (solution space) of the Reddy Mikks example

- The feasible region (solution space) of the Reddy Mikks example is a **<u>convex set</u>**. This is not accidental.

- The feasible region for any LP has only a finite number of extreme points. Also, the optimal solution is an extreme point in any LP.

- This is very **<u>important result</u>**, because it reduces the set of points that yield an optimal solution from the entire feasible region(which is generally contains an infinite number of points) to the set of extreme points.

- Graphical solution

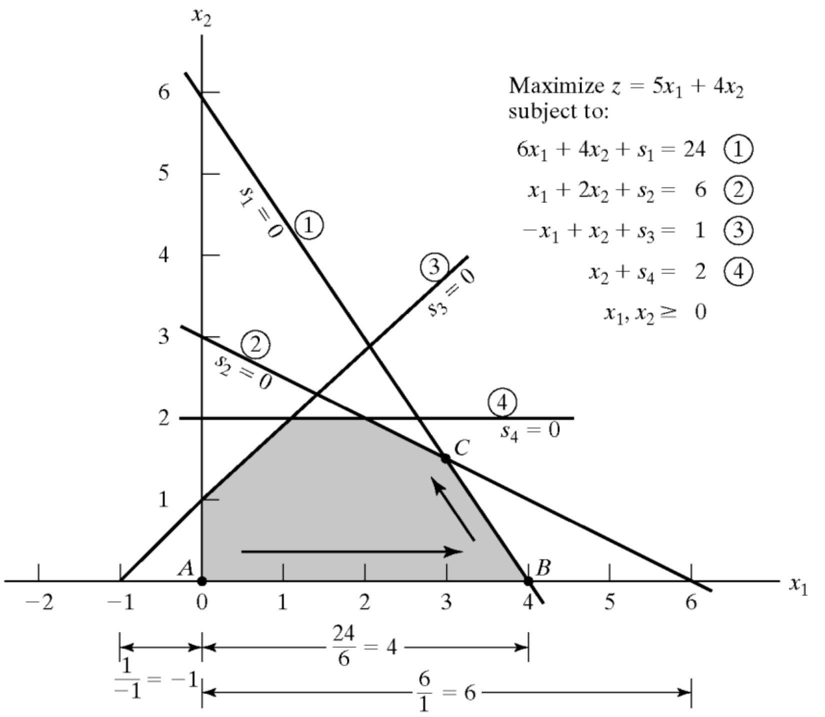


Figure 3.5
Graphical interpretation of the simplex method ratios in the Reddy Mikks model.

BINDING and NONBINDING CONSTRAINTS

- **BINDING CONSTRAINTS:** A constraint is binding if the LHS and the RHS of the constraint <u>**are equal**</u> when the optimal values of the decision variables are substituted into the constraint.
  - A binding constraint must pass through the optimum point. If it does not, it is nonbinding
  - If a constraint is binding, we may regard it as a SCARCE RESOURCE since it has been completely used.
- **NONBINDING CONSTRAINTS**: A constraint is nonbinding if the LHS and the RHS of the constraint <u>**are NOT equal**</u> when the optimal values of the decision variables are substituted into the constraint.
  - On the other hand a nonbinding constraint represents an ABUNDANT RESOURCE

LET’S LOOK AT REDDY MIKKS EXAMPLE

- (Reddy Mikks Example – Taha OR)

- Max $Z=5x_1+4x_2$
- St
- $6x_1+4x_2 \leq 24$ (1)
- $x_1+2x_2 \leq 6$ (2)
- $-x_1+ x_2 \leq 1$ (3)
- $x_2 \leq 2$ (4)
- $x_1 \geq 0$, $x_2 \geq 0$, (5,6)

Optimal values of decision variable $x_1^* = 3$,
$x_2^* = 1{,}5$. If we substituted these values into the constraints

		LHS	RHS
(1)	6(3)+4(1,5)	24	24
(2)	3 + 2(1,5)	6	6
(3)	-3 + 1,5	-1,5	1
(4)	1,5	1,5	2

The graphical solution of Minimization Problems

- **Problem 2. (Diet Problem Taha's OR book pp 19)**
- A farm uses at least 800 lb of special feed daily. The special feed is a mixture of corn and soybean meal with the following compositions

Feedstuff	Protein	Fiber	Cost($/lb)
Corn	0.09	0.02	0.30
Soybean	0.60	0.06	0.90

- The dietary requirements of the special feed are at least 30% protein and at most 5% fiber. Farmer wishes to determine the daily minimum cost feed mix. Formulate the mathematical model.

- x_1=lb of corn in the daily mix
- x_2=lb of soybean meal in the daily mix
- Min $Z=0.3x_1+0.9x_2$
- St
- $x_1+ \quad x_2 \quad \geq 800$ (1)
- $0.09x_1+0.60x_2 \quad \geq 0.3(x_1+x_2)$ (2)
- $0.02x_1+0.06x_2 \quad \leq 0.05(x_1+x_2)$ (3)
- $x_1\geq0,\ x_2\geq0,$ (4,5)

- Mathematical Model of the minimization problem
- Min $Z=0.3x_1+0.9x_2$
- St
- $x_1 + x_2 \geq 800$ (1)
- $0.21x_1-0.30x_2 \leq 0$ (2)
- $0.03x_1-0.01x_2 \leq 0$ (3)
- x1≥0, x2≥0, (4,5)

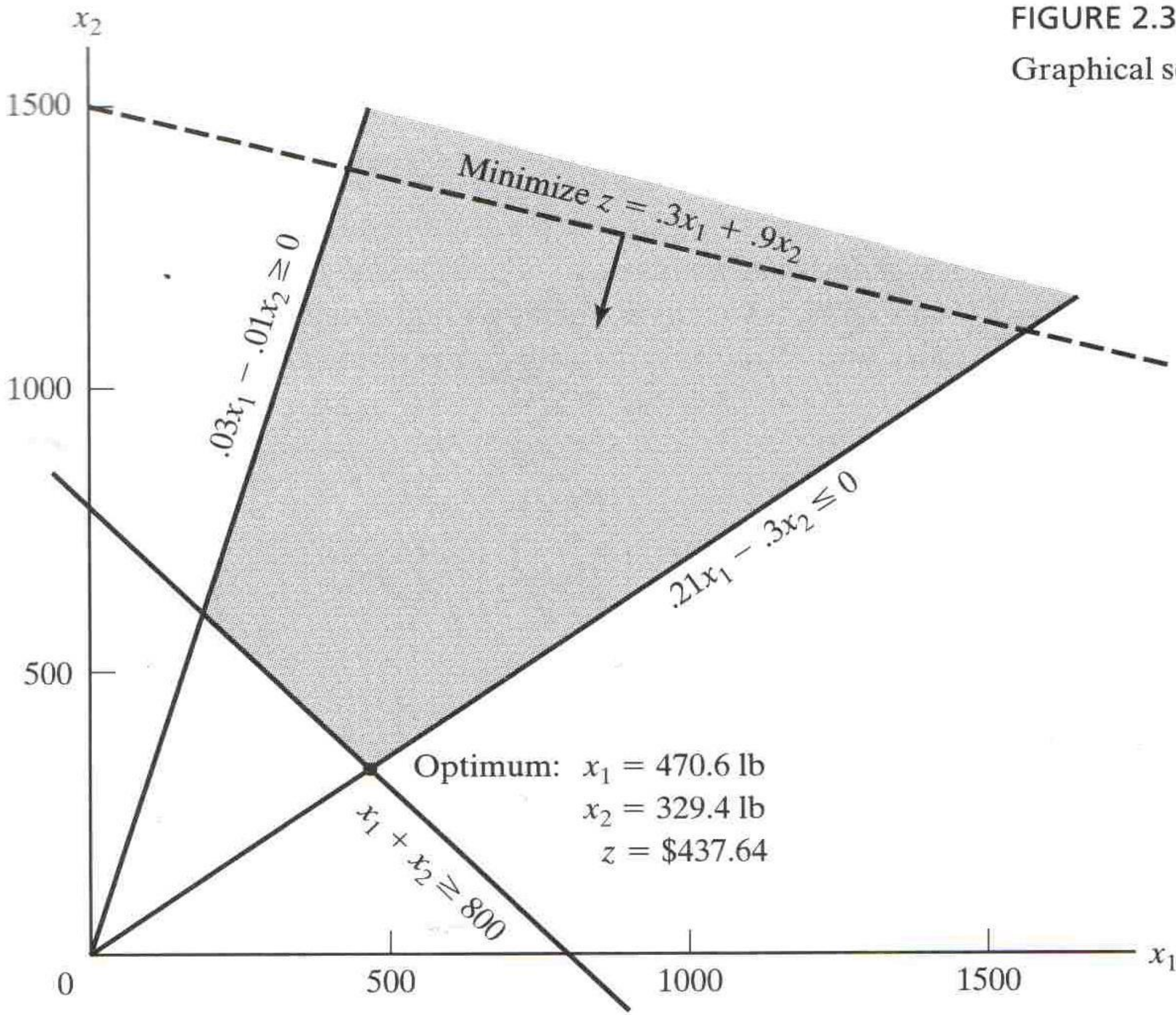


FIGURE 2.3

Graphical solution of the diet model

Solution Cases:

- 1. Unique Optimal Solution Case
- 2. Alternative Optimal Solution Case
- 3. Infeasible Solution Case
- 4. Unbounded Solution Case
- 5. Degenerate Optimal Solution Case