ENM 202
OPERATIONS RESEARCH (I)
OR (I)
3

LECTURE NOTES

THE SIMPLEX METHOD

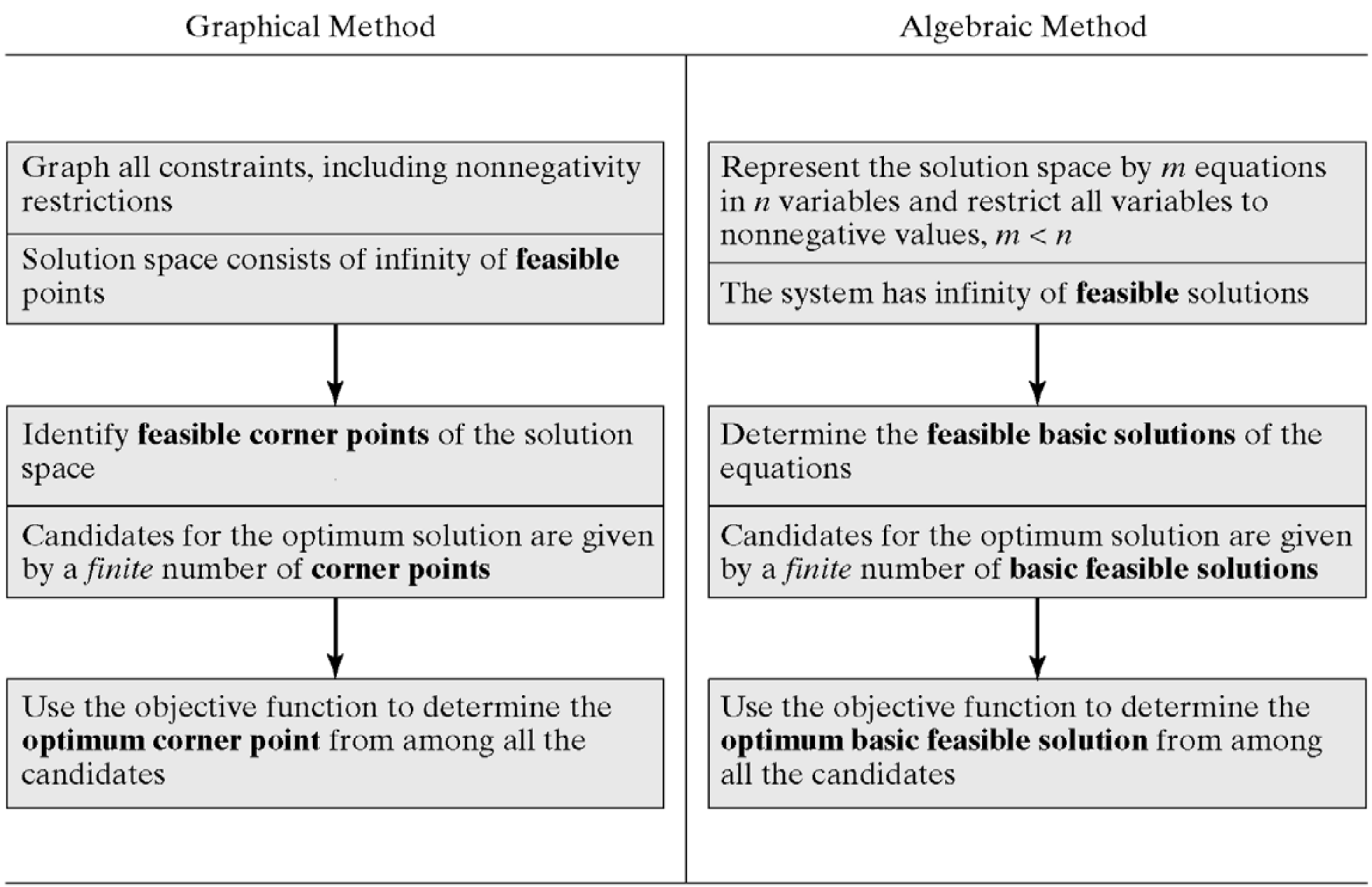


Figure 3.1
Transition from graphical to algebraic solution.

STANDARD FORM of an LP

- 1) All the constraints are equations with a nonnegative RHS
- 2) All the variables are nonnegative

Converting an LP to STANDARD FORM

- 1) A constraint of the type "≤" is converted to an equation by adding a **SLACK VARIABLE** to the left side of the constraint
- 2) A constraint of the type "≥" is converted to an equation by subtracting a **SURPLUS VARIABLE** from the left side of the constraint
- 3) If a RHS is negative, then it is multiplied by "-1" to make it nonnegative.
- 4) If there exist a variable (say x_i) which is unrestricted in sign (u.r.s), then express it in terms of two nonnegative variables :

$$x_i = x_i^+ - x_i^-$$

$$\text{where} \quad x_i^+ \geq 0, \qquad x_i^- \geq 0$$

Convert the following LP in standard form (or equation form)

- Example
- LP: Max $z= 2x_1+3x_2+5x_3$
- St
- $x_1+2x_2\leq 6$
- $3x_1+2x_2-3x_3\geq 5$
- $2x_1+3x_2-7x_3=-5$
- $x_1+x_2+4x_3=10$
- $2x_1-x_2-x_3\leq -8$
- $x_1,x_2,x_3\geq 0$

Convert the following LP in standard form (or equation form)

- Example
- LP: Max $z= 2x_1+3x_2+5x_3$
- st
- $x_1-2x_2- 3x_3 \leq 5$
- $6x_1-7x_2+9x_3 \leq -4$
- $x_1+ x_2+4x_3 =10$
- $x_1,x_3\geq 0$, x_2 u.r.s

General Representation of an LP Problem in Standard Form

- Maximize (or minimize) $Z = c_1x_1 + c_2x_2 + \ldots . + c_nx_n$

  Subject to

  $$
  \begin{aligned}
  &a_{11}x_1 + a_{12}x_2 + \ldots\ldots\ldots + a_{1n}x_n = b_1 \\
  &a_{21}x_1 + a_{22}x_2 + \ldots\ldots\ldots + a_{2n}x_n = b_2 \\
  &.. \quad .. \quad \ldots\ldots\ldots \quad .. \quad .. \\
  &a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots\ldots + a_{mn}x_n = b_m \\
  &x_1 \geq 0,\ x_2 \geq 0,\ \ldots\ldots,\ x_n \geq 0
  \end{aligned}
  $$

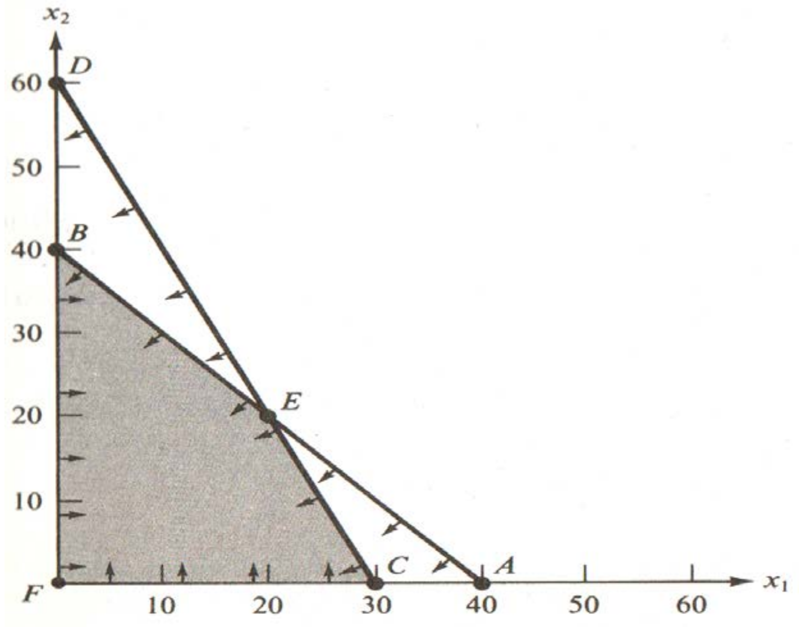
- We have n unknowns(decision variables), m equations(constraints)
- Subject to means:that the values of the decision variables $x_1, x_2, \ldots ., x_n$ must satisfy all constraints and all sign restrictions

Basic Solution

- Consider a system Ax=b of m linear equations in n variables(m<n),
- if we set n-m variables equal to zero and then solve m equations for the remaining m variables(n-(n-m)=m), the resulting solution, if unique, must corresponds to a corner point of the solution space. This is called a **BASIC SOLUTION**

Example

- LP: Max $Z=4x_1+3x_2$
- St
- $x_1+x_2\leq 40$ (1)
- $2x_1+x_2\leq 60$ (2)
- $x_1,x_2\geq 0$



- LP in standard form: Max $Z=4x_1+3x_2$
- St
- $x_1+x_2+S1=40$
- $2x_1+x_2+S2=60$
- $x_1, x_2, S1, S2 \geq 0$
- To find the corner points (basic solutions) we will set n-m=4-2 =2 variables equal to zero and then solve the system for the remaining m=2 variables

BV	NBV	Basic Solution	Assocaited corner point	Objective value Max $Z=4x_1+3x_2$
(x1,x2)	(s1,s2)	s1=s2=0 x1=x2=20	E	140*
(x1,s1)	(x2,s2)	x2=s2=0 x1=30,s1=10	C	120
(x1,s2)	(x2,s1)	x2=s1=0 x1=40, s2=-20	A	-
(x2,s1)	(x1,s2)	x1=s2=0 x2=60, s1=-20	D	-
(x2,s2)	(x1,s1)	x1=s1=0 x2=40, s2=20	B	120
(s1,s2)	(x1,x2)	x1=x2=0 s1=40,s2=60	F	**0**

Basic Feasible Solution

- Any basic solution to max (min) $Z=cx$ st $Ax=b$, $x\geq0$ in which all variables are nonnegative is **a basic feasible solution (b.f.s.)**
- B, C, E, F corner points correspond basic feasible solutions
- So corner points A and D are not extreme points. Because they do not belong to the feasible region. And also there exists negative variables in their associated basic solutions.

STEPS OF THE SIMPLEX METHOD

1) Convert the LP to standard form
2) Obtain a bfs from the standard form by setting n-m appropriate (nonbasic) variables at zero level
3) Determine whether the current bfs is optimal (by optimality condition)
4) If not optimal, then determine which NB variable should become a basic variable (by optimality condition) and which basic variable should become a NB variable (by feasibility condition) to find a new bfs with a better objective function value.
5) Use EROs (elemantary row operations) to find the new bfs with the better objective function value. Go to Step 3.

THE SIMPLEX METHOD

0) Convert the LP to the standard form. Represent Z function as an equation in which the right hand side equals to 0.

1) Determine a starting bfs (generally origin)

2) Select an entering variable using the **optimality condition*** Stop if there is no entering variable, last solution is optimal

3) Select a leaving variable using the **feasibility condition****

4) Determine the new bfs by using the appropriate Gauss Jordan row operations.

Go to Step 2

***OPTIMALITY CONDITION:** (The rule for selecting entering variable)

The entering variable in a maximization (minimization) problem is the NBV having the most negative (positive) coefficient in the Z row. Ties are broken arbitrarily.

The optimum is reached at the iteration where all the Z row coefficients of the NBVs are nonnegative (non-positive).

****FEASIBILITY CONDITION:** (The rule for selecting leaving variable)

For both the max (min) problem, the leaving variable is the BV associated with the smallest nonnegative ***ratio*** (with strictly positive denominator)

Minimum Ratio Test

- We compute the intercepts of all constraints with the nonegative direction of the axis which is represented by the entering variable

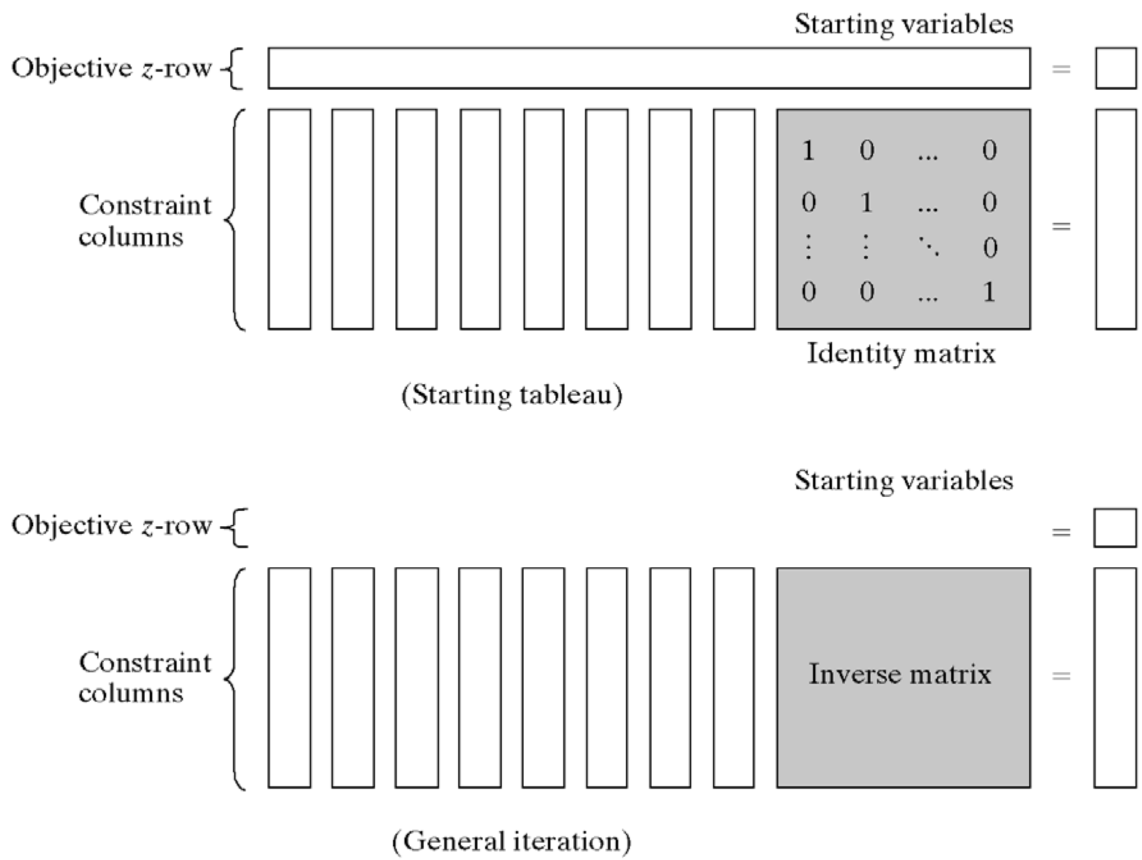


Figure 4.1

Schematic representation of the starting and general simplex tableaus.

Reddy Mikks Example

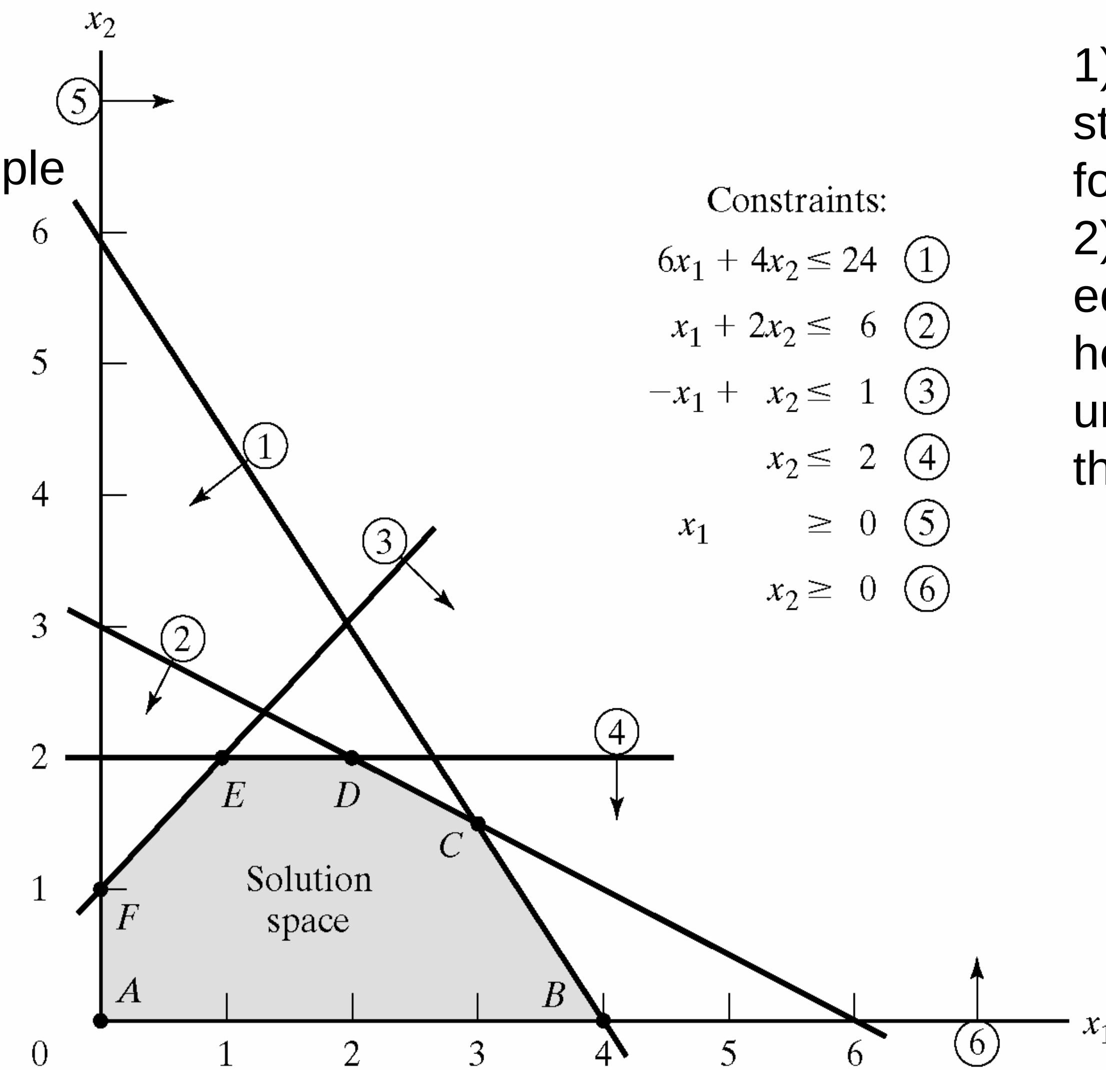


Figure 2.1
Feasible space of the Reddy Mikks model.

1) Write the standard form of this LP
2) How many equations and how many unknowns are there?

Reddy Mikks Example

LP: Max z= 5x1+4x2

s.t.

$6x_1 + 4x_2 \le 24$

$x_1 + 2x_2 \le 6$

$-x_1 + x_2 \le 1$

$x_2 \le 2$

$x_1, x_2 \ge 0$

LP in Standard Form:

Max z= 5x1+4x2+0S1+0S2+0S3+0S4

s.t.

$6x_1 + 4x_2 + S1 = 24$

$x_1 + 2x_2 + S2 = 6$

$-x_1 + x_2 + S3 = 1$

$x_2 + S4 = 2$

$x_1, x_2, S1, S2, S3, S4 \ge 0$

- n=6
- m=4

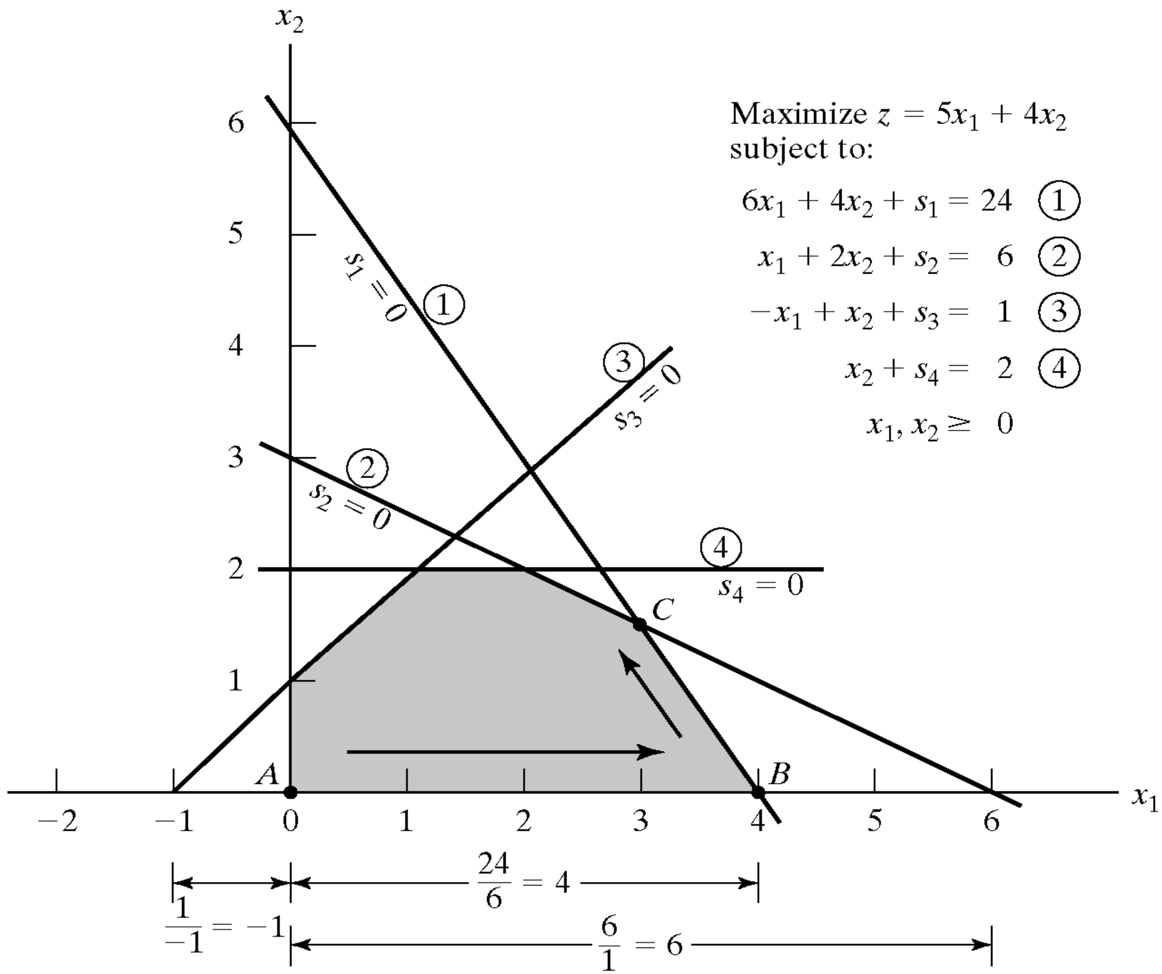


Figure 3.5
Graphical interpretation of the simplex method ratios in the Reddy Mikks model.

The Simplex Tableau offers additional information below:

- 1) The status of the resources
- 2) The unit worth of scarce resources
- 3) All the data required to carry out sensitivity analysis on the optimal solution

Reddy Mikks Example Optimal Simplex Tableau

BV	z	x1	x2	S1	S2	S3	S4	RHS
z	1	0	0	3/4	1/2	0	0	21
x1	0	1	0	1/4	-1/2	0	0	3
x2	0	0	1	-1/8	3/4	0	0	3/2
S3	0	0	0	3/8	-5/4	1	0	5/2
S4	0	0	0	1/8	-3/4	0	1	1/2

1) The status of the resources

Decision Variable	Optimum Value	Recommendation
x_1	3	Produce 3 tons of exterior paint daily
x_2	3/2	Produce 1.5 tons of interior paint daily
z	21	Daily profit is \$21 000

You can verify that the values S1=S2=0, S3=(5/2), S4=(1/2) are consistent with the given values of x_1 and x_2

	Resource	Slack Value	Status	Unit worth of a scarce resource
Constraint 1	Raw Material, M1	S1=0	Scarce	(3/4)=0.75
Constraint 2	Raw Material, M2	S2=0	Scarce	(1/2)=0.5
Constraint 3	Demand Limit 1	S3=(5/2)	Abundant	0
Constraint 4	Demand Limit 2	S4=(1/2)	Abundant	0

Constraint 1)	$6x1 + 4x2 \le 24$	
	$6(3)+4(3/2)=24$	S1=0
Constraint 2)	$x1 + 2x2 \le 6$	
	$(3)+2(3/2)=6$	S2=0
Constraint 3)	$-x1 + x2 \le 1$	
	$-(3)+(3/2)=-3/2$	S3=1+(3/2)=(5/2)
Constraint 4)	$x2 \le 2$	
	$(3/2)=3/2$	S4=2-(3/2)=(1/2)

Example for Minimization Case

LP: Min z= -x1-3x2

s.t.

$2x_1 + 3x_2 \leq 6$

$-x_1 + x_2 \leq 1$

$x_1, x_2 \geq 0$

LP in Standard Form:

Min z= -x1-3x2+0S1+0S2

s.t.

$2x_1 + 3x_2 + S1 = 6$

$-x_1 + x_2 + S2 = 1$

$x_1, x_2, S1, S2 \geq 0$

Z function is in equation form

Z+x1+3x2-0S1-0S2=0

Iteration	Basic	x1	x2	S1	S2	RHS	MRT
(0) starting	z	1	3	0	0	0	
	S1	2	3	1	0	6	6/3=2
x2 enters S2 leaves	S2	-1	1	0	1	1	1/1=1*
(1)	z	4	0	0	-3	-3	
x1 enters S1 leaves	S1	5	0	1	-3	3	3/5*
	x2	-1	1	0	1	1	ignore
(2) optimum	z	0	0	-4/5	-3/5	-27/5	
	x1	1	0	1/5	-3/5	3/5	
	x2	0	1	1/5	2/5	8/5	

Status of the Resources

	Resource	Slack Value	Status	Unit worth of a scarce Resource
Constraint 1	Resource 1	S1=0	Scarce	-4/5
Constraint 2	Resource 2	S2=0	Scarce	-3/5

Constraint 1) $2x1 + 3x2 \le 6$

$2(3/5)+3(8/5)=6$ S1=0

Constraint 2) $-x1 + x2 \le 1$

$-(3/5)+(8/5)=1$ S2=0