ENM 202
OPERATIONS RESEARCH (I)
OR (I)
8

LECTURE NOTES
“THE SIMPLEX METHOD IN TABLEAU FORM”
“MATRIX REPRESENTATION OF SIMPLEX METHOD”

Algebra of The Simplex Method

Matrix Representation of
Basis and Basis Inverse and Simplex Tableau

- Consider the standard LP problem:

(LP): maximize $z=cx$

subject to

$Ax=b$

$x\geq 0$

A is mxn matrix, b is a m-vector, x is a n-vector

Recall that a bfs to this problem corresponds to an extreme point of the feasible region and is characterized mathematically by partitioning matrix A into a nonsingular basis matrix B and the matrix of nonbasic columns N. That is:

$A=[B|N]$

Where

B is mxm nonsingular **(basis)** matrix

N is mx(n-m) matrix (the matrix of nonbasic columns)

Based on this partitioning, the linear system Ax=b can be rewritten to yield

$Bx_B+NX_N=b$

This simplfies to

$X_B+B^{-1}NX_N=B^{-1}b$

and solving for X_B in terms of X_N yields

MATRIX REPRESENTATION

(LP) maximize $Z=cx$

Subject to

$$Ax=b$$
$$x\geq 0$$

$$A=(B:N) \qquad (1)$$
$$Bx_B + Nx_N = b \qquad (2)$$

This simplifies to

$$X_B + B^{-1}Nx_N = B^{-1}b \qquad (3)$$

and solving for X_B in terms of X_N yields

$$X_B = B^{-1}b - B^{-1}Nx_N \qquad (4)$$

Now setting $X_N=0$, we see that (4) results in

$$\boxed{X_B = B^{-1}b}$$

The solution

$$x = \begin{pmatrix} x_B \\ x_N \end{pmatrix} = \begin{pmatrix} B^{-1}b \\ 0 \end{pmatrix}$$

is called a **basic solution**,

vector x_B called the **vector of basic variables** and

the vector x_N is called **the vector of nonbasic variables**.

If in addition, $x_B=B^{-1}b\geq 0$, then

$$x = \begin{pmatrix} B^{-1}b \\ 0 \end{pmatrix}$$

is called **a basic feasible solution**.

Now consider the objective function Z=cx.

Partitioning the cost vector c into basic and nonbasic components (i.e., $c=(c_B, c_N)$), the objective function can be written as

$$Z = c_B x_B + c_N x_N \qquad (5)$$

Now, substituting the expression for x_B defined in (4) into (5) yields

$$Z = c_B(B^{-1}b - B^{-1}Nx_N) + c_N x_N \qquad (6)$$

which can be rewritten as

$$Z = c_B B^{-1}b - (c_B B^{-1}N - c_N)x_N \qquad (7)$$

and setting x_N=0, we see that (7) results in

$$Z = c_B B^{-1}b$$

which is the objective value corresponding to <u>the current basic feasible solution</u>

**NOTE: From the strong duality theorem, we know that at the optimum point the primal and dual objective functions are equal to each other.

So, $z = yb = c_B B^{-1}b$

we can see that $y = c_B B^{-1}$

Therefore, the current extreme point solution can be represented in canonical form:

$$Z = c_B B^{-1} b - (c_B B^{-1} N - c_N) x_N \qquad (8)$$

$$x_B = B^{-1} b - B^{-1} N x_N \qquad (9)$$

with the current basic feasible solution given as

$$Z = c_B B^{-1} b \qquad (10)$$

$$X = \begin{pmatrix} x_B \\ x_N \end{pmatrix} = \begin{pmatrix} B^{-1} b \\ 0 \end{pmatrix} \geq 0 \qquad (11)$$

Now letting J denote the index set of the nonbasic variables, observe that (8-9) can be rewritten as follows:

$$Z = c_B B^{-1} b - \sum_{j \in J} (c_B B^{-1} a_j - c_j) x_j \qquad (12)$$

$$x_B = B^{-1} b - \sum_{j \in J} (B^{-1} a_j) x_j \qquad (13)$$

The central idea behind the simplex method is to move from an extreme point to an improving adjacent extreme point by interchanging a column of B and N.

The first question is when will such an exchange improve the objective function

$$\frac{\partial Z}{\partial x_j} = -\left(c_B B^{-1} a_j - c_j\right)$$

Observe that the coefficient $-\left(c_B B^{-1} a_j - c_j\right)$ of x_j represents the rate of change of Z with respect to the nonbasic variable x_j.

Thus, if $\partial Z / \partial x_j > 0$ then increasing the nonbasic variable x_j will increase Z. We know that the quantity $(c_B B^{-1} a_j - c_j)$ sometimes referred to as reduced cost and for convenience is usually denoted by $(z_j - c_j)$.

So, in a maximization problem, the basic feasible solution will be optimal if

$$\frac{\partial Z}{\partial x_j} = -(Z_j - c_j) = -(c_B B^{-1} a_j - c_j) \leq 0 \quad \textit{for all } j \in J$$

$$\textit{or equivalently, if}$$

$$z_j - c_j = c_B B^{-1} a_j - c_j \geq 0 \quad \textit{for all } j \in J$$

SIMPLEX METHOD IN TABLEAU FORM

Consider again the canonical form represented in (8)-(9).

$$Z = c_B B^{-1} b - (c_B B^{-1} N - c_N) x_N \qquad (8)$$

$$x_B = B^{-1} b - B^{-1} N x_N \qquad (9)$$

Now, rearranging terms so that all the variables are on the left-hand side of the equation, with the constants on the right-hand side, we have

$$Z + (c_B B^{-1} N - c_N) x_N = c_B B^{-1} b$$

$$x_B + B^{-1} N x_N = B^{-1} b$$

THE SIMPLEX TABLEAU

	Z	x_B	x_N	RHS
Z	1	0	$c_B B^{-1} N - c_N$	$c_B B^{-1} b$
x_B	0	I	$B^{-1} N$	$B^{-1} b$

In a more compact form:

	Z	x	RHS
Z	1	$c_B B^{-1} A - c$	$c_B B^{-1} b$
x_B	0	$B^{-1} A$	$B^{-1} b$

Example:

LP:

$$\text{Min } Z = 4x_1 + x_2$$

s.t.

$$\begin{aligned} 3x_1 + x_2 &= 3 \\ 4x_1 + 3x_2 &\geq 6 \\ x_1 + 2x_2 &\leq 4 \\ x_1, x_2 &\geq 0 \end{aligned}$$

LP in standard form with artificials:

$$\text{Min } Z = 4x_1 + x_2$$

s.t.

$$\begin{aligned} 3x_1 + x_2 \quad\quad\quad + R_1 \quad &= 3 \\ 4x_1 + 3x_2 - x_3 \quad\quad\quad + R_2 &= 6 \\ x_1 + 2x_2 \quad + x_4 \quad\quad &= 4 \\ x_1, x_2, x_3, x_4, R_1, R_2 &\geq 0 \end{aligned}$$

Iteration	Basic	x_1	x_2	x_3	R_1	R_2	x_4	RHS	MRT
(2) x_3 enters x_4 leaves	Z	0	0	1/5	8/5-M	-1/5-M	0	18/5	
	x_1	1	0	1/5	3/5	-1/5	0	3/5	3
	x_2	0	1	-3/5	-4/5	3/5	0	6/5	-
	X_4	0	0	1	1	-1	1	1	1
(3) OPTIMUM	Z	0	0	0	7/5-M	-M	-1/5	17/5	
	x_1	1	0	0	2/5	0	-1/5	2/5	
	x_2	0	1	0	-1/5	0	3/5	9/5	
	x_3	0	0	1	1	-1	1	1	

Dual:

Max $W = 3y_1 + 6y_2 + 4y_3$

s.t.

$3y_1 + 4y_2 + \quad y_3 \le 4$

$y_1 + 3y_2 + 2y_3 \le 1$

$x_1 + 2x_2 \le 4$

y_1: u.r.s., $y_2 \ge 0$, $y_3 \le 0$

Primal Optimal Solution	Dual Optimal Solution
$Z = 17/5$	$W = 17/5$
$x_1 = 2/5$	$y_1 = 7/5$
$x_2 = 9/5$	$y_2 = 0$
$X_3 = 1$	$y_3 = -1/5$

<u>Remember that</u>

Method I – Finding optimal values of the dual variables

$$\begin{pmatrix}\text{Optimal values}\\ \text{of dual variables}\end{pmatrix} = \begin{pmatrix}\text{Row vector of}\\ \text{original obj. coefficients}\\ \text{of optimal primal basic}\\ \text{variables}\end{pmatrix} * \begin{pmatrix}\text{Optimal}\\ \text{primal}\\ \text{inverse}\end{pmatrix}$$

$$y = c_B * B^{-1}$$

Method II – Finding the z-row coefficients (reduced cost)

$$\begin{pmatrix}\text{Optimal primal}\\ \text{z-row coefficient}\\ \text{of any variable } x_j\end{pmatrix} = \begin{pmatrix}\text{LHS of the}\\ \text{j}^{\text{th}}\text{ dual}\\ \text{constraint}\end{pmatrix} - \begin{pmatrix}\text{RHS of the}\\ \text{j}^{\text{th}}\text{ dual}\\ \text{constraint}\end{pmatrix}$$

$$z_j - c_j = \sum_{i=1}^{m} a_{ij} y_i - c_j \quad \text{or}$$

$$z_j - c_j = c_B B^{-1} a_j - c_j$$

Optimal Z value and x_B

$$c_B = \begin{bmatrix} 4 & 1 & 0 \end{bmatrix}, \quad B^{-1} = \begin{bmatrix} 2/5 & 0 & -1/5 \\ -1/5 & 0 & 3/5 \\ 1 & -1 & 1 \end{bmatrix}$$

$$Z = c_B B^{-1} b = \begin{bmatrix} 4 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2/5 & 0 & -1/5 \\ -1/5 & 0 & 3/5 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \\ 4 \end{bmatrix} = \begin{bmatrix} 7/5 & 0 & -1/5 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \\ 4 \end{bmatrix}$$

$$= 21/5 - 4/5 = 17/5$$

$$\left(\begin{array}{l} \text{Z value for iteration 2:} \\ = \begin{bmatrix} 4 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3/5 & -1/5 & 0 \\ -4/5 & 3/5 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \\ 4 \end{bmatrix} = \begin{bmatrix} 8/5 & -1/5 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \\ 4 \end{bmatrix} = 18/5 \end{array} \right)$$

$$x_B = B^{-1} b = \begin{bmatrix} 2/5 & 0 & -1/5 \\ -1/5 & 0 & 3/5 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \\ 4 \end{bmatrix} = \begin{bmatrix} 2/5 \\ 9/5 \\ 1 \end{bmatrix}$$

Finding Optimal Dual Solution

$$y = c_B B^{-1}$$

$$\Rightarrow \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} = \begin{bmatrix} 4 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2/5 & 0 & -1/5 \\ -1/5 & 0 & 3/5 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 7/5 & 0 & -1/5 \end{bmatrix}$$

$$\Rightarrow \text{Dual variable values in iteration 2:}$$

$$\begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} = \begin{bmatrix} 4 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3/5 & -1/5 & 0 \\ -4/5 & 3/5 & 0 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 8/5 & -1/5 & 0 \end{bmatrix}$$

Simplex Table Computations

Here, we will show how the entire simplex table at any iteration can be generated from the original problem data and the inverse associated with the corresponding iteration.

We can divide the computations into two types

1) Constraint columns

2) Objective z-row

Constraint Column Computations

$$\begin{pmatrix}\text{Constraint}\\ \text{column in}\\ \text{iteration } i\end{pmatrix} = \begin{pmatrix}\text{inverse in}\\ \text{iteration } i\end{pmatrix}\begin{pmatrix}\text{original}\\ \text{constraint}\\ \text{column}\end{pmatrix}$$

$$\alpha_j = B^{-1} a_j$$

Optimal inverse in the example $B^{-1} = \begin{bmatrix} 2/5 & 0 & -1/5 \\ -1/5 & 0 & 3/5 \\ 1 & -1 & 1 \end{bmatrix}$

$$A = \begin{array}{c} \begin{array}{cccccc} a_{x_1} & a_{x_2} & a_{x_3} & a_{R_1} & a_{R_2} & a_{x_4} \end{array} \\ \begin{bmatrix} 3 & 1 & 0 & 1 & 0 & 0 \\ 4 & 3 & -1 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$\Rightarrow$ Q. Find the x_1 column in optimal iteration

$$\alpha_1 = B^{-1}a_1 = \begin{bmatrix} 2/5 & 0 & -1/5 \\ -1/5 & 0 & 3/5 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ Q. Find the x_4 column in optimal iteration

$$\alpha_4 = B^{-1}a_4 = \begin{bmatrix} 2/5 & 0 & -1/5 \\ -1/5 & 0 & 3/5 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/5 \\ 3/5 \\ 1 \end{bmatrix}$$

$\Rightarrow$ Q. Find the R_1 column in iteration 2

$$\alpha_{R1} = B^{-1}a_{R1} = \begin{bmatrix} 3/5 & -1/5 & 0 \\ -4/5 & 3/5 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3/5 \\ -4/5 \\ 1 \end{bmatrix}$$

$\Rightarrow$ Q. Find the x_2 column in iteration 2

$$\alpha_{x2} = B^{-1}a_{x2} = \begin{bmatrix} 3/5 & -1/5 & 0 \\ -4/5 & 3/5 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Objective Z-Row Computations

For the optimal iteration

$$z_1 - c_1 = c_B B^{-1} a_1 - c_1 = \begin{bmatrix} 4 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2/5 & 0 & -1/5 \\ -1/5 & 0 & 3/5 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} - 4 = \begin{bmatrix} 7/5 & 0 & -1/5 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} - 4 = 0$$

$$z_2 - c_2 = c_B B^{-1} a_2 - c_2 = \begin{bmatrix} 7/5 & 0 & -1/5 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} - 1 = \frac{7}{5} - \frac{2}{5} - 1 = 0$$

$$z_3 - c_3 = c_B B^{-1} a_3 - c_3 = \begin{bmatrix} 7/5 & 0 & -1/5 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} - 0 = 0 - 0 = 0$$

$$z_4 - c_4 = c_B B^{-1} a_4 - c_4 = \begin{bmatrix} 7/5 & 0 & -1/5 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - 0 = -\frac{1}{5}$$

$$z_{R1} - c_{R1} = c_B B^{-1} a_{R1} - c_{R1} = \begin{bmatrix} 7/5 & 0 & -1/5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - M = \frac{7}{5} - M$$

$$z_{R2} - c_{R2} = c_B B^{-1} a_{R2} - c_{R2} = \begin{bmatrix} 7/5 & 0 & -1/5 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - M = -M$$

For iteration 2

$$z_2 - c_2 = c_B B^{-1} a_2 - c_2 = \begin{bmatrix} 4 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3/5 & -1/5 & 0 \\ -4/5 & 3/5 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} - 1 = \begin{bmatrix} 8/5 & -1/5 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} - 1$$

$$= \frac{8}{5} - \frac{3}{5} - 1 = 0$$

$$z_{R1} - c_{R1} = c_B B^{-1} a_{R1} - c_{R1} = \begin{bmatrix} 8/5 & -1/5 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - M = \frac{8}{5} - M$$

Example:

LP:

Max $Z = 5x_1 + 12x_2 + 4x_3$

s.t.

$x_1 + 2x_2 + x_3 \le 10$

$2x_1 - x_2 + 3x_3 = 8$

$x_1, x_2, x_3 \ge 0$

LP in standard form with artificials:

Max $Z = 5x_1 + 12x_2 + 4x_3 - MR$

s.t.

$x_1 + 2x_2 + x_3 + x_4 = 10$

$2x_1 - x_2 + 3x_3 + R = 8$

$x_1, x_2, x_3, x_4, R \ge 0$

Optimal Table

	x_1	x_2	x_3	x_4	R	RHS
Z						
x_2						
x_1						

If $B^{-1} = \begin{bmatrix} 2/5 & -1/5 \\ 1/5 & 2/5 \end{bmatrix}$ then fill the table

$$x_B = B^{-1}b = \begin{bmatrix} 2/5 & -1/5 \\ 1/5 & 2/5 \end{bmatrix}\begin{bmatrix} 10 \\ 8 \end{bmatrix} = \begin{bmatrix} 20/5 - 8/5 \\ 10/5 + 16/5 \end{bmatrix} = \begin{bmatrix} 12/5 \\ 26/5 \end{bmatrix}$$

$$Z = c_B B^{-1} b = \begin{bmatrix} 12 & 5 \end{bmatrix}\begin{bmatrix} 2/5 & -1/5 \\ 1/5 & 2/5 \end{bmatrix}\begin{bmatrix} 10 \\ 8 \end{bmatrix} = \frac{274}{5}$$

$$\alpha_3 = B^{-1}a_3 = \begin{bmatrix} 2/5 & -1/5 \\ 1/5 & 2/5 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -1/5 \\ 7/5 \end{bmatrix}$$

$$z_3 - c_3 = c_B B^{-1} a_3 - c_3 = \begin{bmatrix} 12 & 5 \end{bmatrix}\begin{bmatrix} 2/5 & -1/5 \\ 1/5 & 2/5 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} - 4 = \begin{bmatrix} 12 & 5 \end{bmatrix}\begin{bmatrix} -1/5 \\ 7/5 \end{bmatrix} - 4 = \frac{23}{5} - 4 = \frac{3}{5}$$

$$z_4 - c_4 = c_B B^{-1} a_4 - c_4 = \begin{bmatrix} 12 & 5 \end{bmatrix}\begin{bmatrix} 2/5 & -1/5 \\ 1/5 & 2/5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} - 0 = \begin{bmatrix} 12 & 5 \end{bmatrix}\begin{bmatrix} 2/5 \\ 1/5 \end{bmatrix} - 0 = \frac{29}{5} - 0 = \frac{29}{5}$$

$$z_R - c_R = c_B B^{-1} a_R - c_R = \begin{bmatrix} 12 & 5 \end{bmatrix}\begin{bmatrix} 2/5 & -1/5 \\ 1/5 & 2/5 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} - (-M) = -2/5 + M$$

	x_1	x_2	x_3	x_4	R	RHS
Z	0	0	3/5	29/5	-2/5+M	274/5
x_2	0	1	-1/5	2/5	-1/5	12/5
x_1	1	0	7/5	1/5	2/5	26/5

$Z_j - c_j$

α_j

B^{-1}

Z

x_B