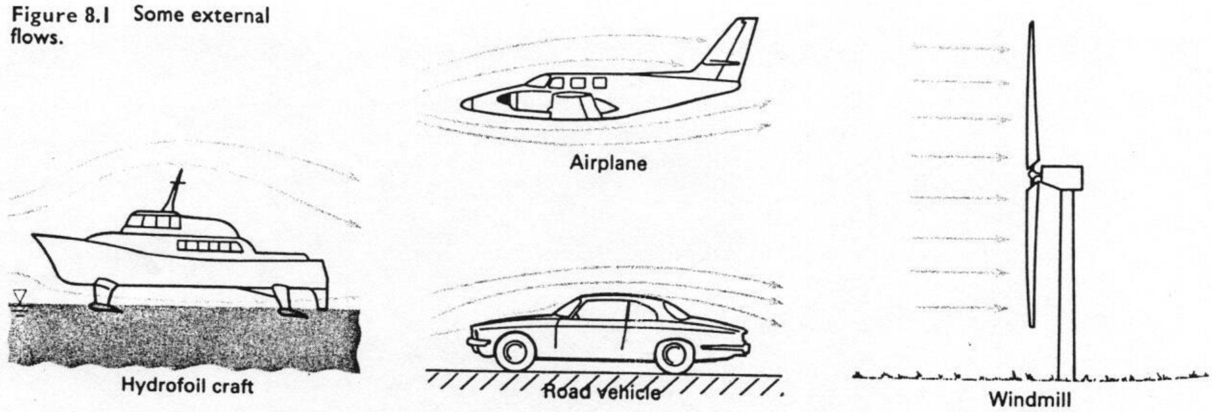
EXTERNAL INCOMPRESSIBLE VISCOUS FLOW

External flows are flows over bodies immersed in a unbounded fluid. The flow over semi-infinite plate, and the flow over a sphere are examples of external flows. A few external flows, that are of interest to engineers are shown in the figure

Figure 8.1 Some external flows.



BOUNDARY - LAYER CONCEPT

The concept of boundary layer was first introduced by Ludwig Prandtl, a German aerodynamicist, in 1904. Prior to Prandtl's historic breakthrough, the science of fluid mechanics had been developing in two different directions: Theoretical hydrodynamics, evolved from the solution of Euler's equations of motion for inviscid flows in 1755. Since the results of hydrodynamics contradicted many experimental observations, practicing engineers developed their own emprical art of hydraulics.

Although the Navier - Stokes equations, describing the motion of a viscous fluid developed by Navier 1827, and independently by Stokes in 1845, the mathematical difficulties in solving them, except for a few simple cases, prohibited the theoretical treatment of viscous flows.

Prandtl showed that many viscous flows can be analysed by dividing the flow field into two regions: one close to the solid boundaries, the other covering the rest of the flow field. Only in thin region adjacent to a solid boundary, (the boundary layer) the viscous effects are important. In this region, the viscous forces are much more significant than the inertia forces. The velocity of the fluid at the wall relative to the solid boundary is zero, and increases toward the main stream. Shear stresses in this zone are very high owing to the existence of extremely high velocity gradients at and near the solid boundaries. In the region outside the boundary layer the influence of viscosity negligible, so that the inertial forces dominate the viscous ones. Therefore, the flow in this region may be considered inviscid, and the potential flow theory may be used for analysing the flow.

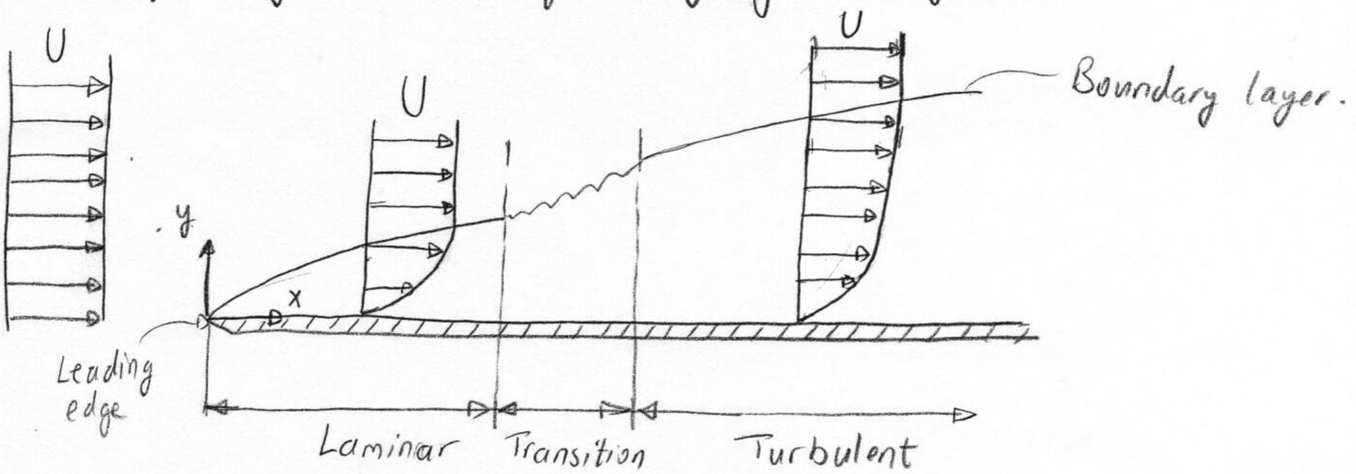


Fig. Boundary layer on a flat plate

laminar region begins at the leading edge and grows in thickness. A transition region is reached where the flow changes from laminar to turbulent. The transition depends on the Reynolds number which is defined as

$$Re = \frac{\rho U x}{\mu}$$

For calculation purposes, under typical flow conditions, transition usually is considered to occur at $Re = 500.000$. Therefore, if

$Re \leq 500.000$ flow is laminar

$Re > 500.000$ flow is turbulent.

BOUNDARY-LAYER THICKNESSES

a) <u>Boundary Layer Thickness</u>

The boundary layer thickness, δ, is usually defined as the distance from the solid boundary to the point where the velocity is within 1 percent of the freestream velocity.

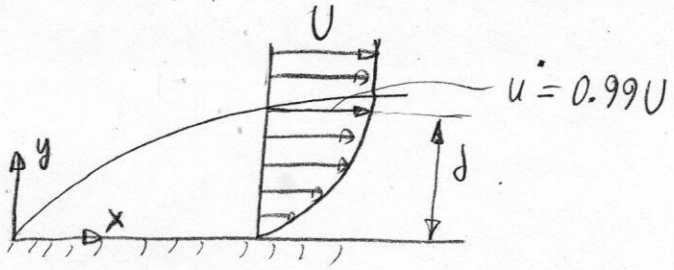
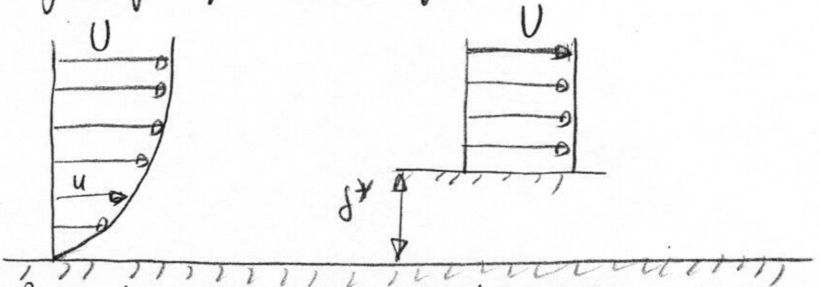


Fig: Definition of the boundary Layer thickness

$u = 0.99U$ at $y = \delta$

b) Boundary Layer Displacement Thickness

The boundary layer displacement thickness, δ^*, may be defined as the distance by which the solid boundary would have to be displaced to maintain the same mass flow rate in an imaginary frictionless flow.



a) Real flow b) ideal flow

The decrease in mass flow rate due to the influence of viscous forces is

$$\int_0^\infty \rho(U-u)w\,dy.$$

where w is the width of the surface in the direction perpendicular to flow.

Displacing the boundary by a distance δ^* (ideal flow case) would result in a mass flow deficiency of $\rho U \delta^* w$.

Thus

$$\rho U \delta^* w = \int_0^\infty \rho(U-u)\,w\,dy$$

For incompressible flow, $\rho =$ constant, and

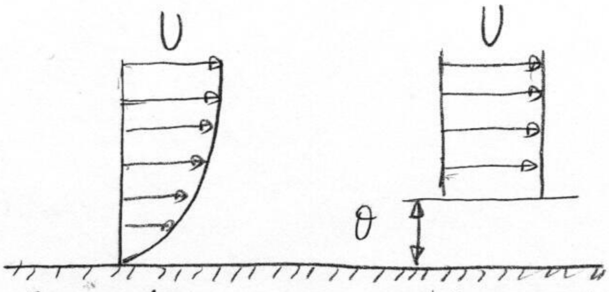
$$\delta^* = \int_0^\infty \left(1 - \frac{u}{U}\right) dy \cong \int_0^\delta \left(1 - \frac{u}{U}\right) dy$$

Since $u \approx U$ at $y = \delta$, the integrand is essentially zero for $y \geqslant \delta$.

c) Boundary Layer Momentum Thickness

Flow retardation within boundary layer also results in a reduction in momentum flux at a section compared with inviscid flow. The momentum deficiency through the boundary layer is

$$\int_0^{\infty} \rho u (U-u)\, w\, dy.$$



a) Real flow b) Ideal (inviscid) flow

If viscous forces were absent, it would be necessary to move the solid boundary outward to obtain a momentum deficiency. The momentum thickness is denoted by θ. The momentum deficiency would be $\rho U^2 \theta w$.

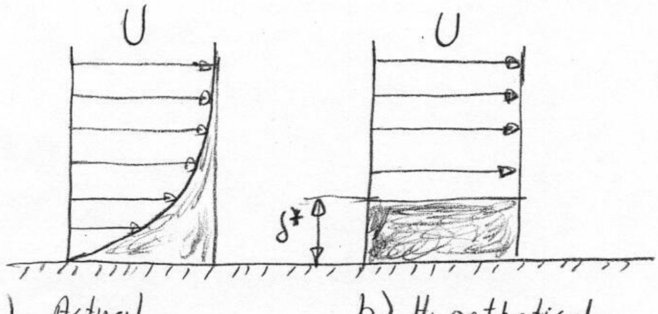
$$\rho U^2 \theta w = \int_0^{\infty} \rho u (U-u)\, w\, dy$$

If ρ = constant

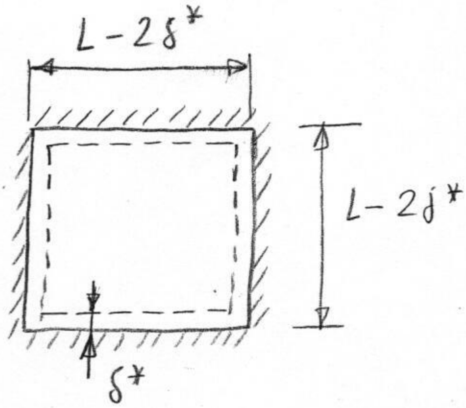
$$\theta = \int_0^{\infty} \frac{u}{U}\left(1-\frac{u}{U}\right) dy \approx \int_0^{\delta} \frac{u}{U}\left(1-\frac{u}{U}\right) dy$$

The integrand is essentially zero for $y \geq \delta$

<u>Example</u>: A laboratory wind tunnel has a test section that is 305 mm square. Boundary-layer velocity profiles are measured at two cross sections and displacement thicknesses are evaluated from the measured profiles. At section (1), where the freestream speed is $U_1 = 26$ m/sn, the displacement thickness $\delta_1^* = 1.5$ mm. At section (2), located downstream from section (1); $\delta_2^* = 2.1$ mm. Calculate the change in static pressure between section (1) and (2). Express the results as a freestream dynamic pressure at section (1). Assume standard atmosphere conditions.



a) Actual velocity profile   b) Hypothetical velocity profile

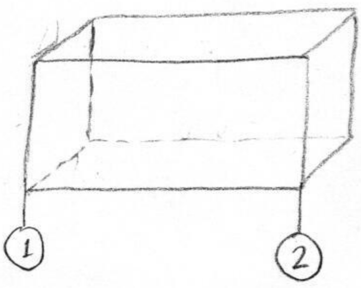


Cross section of wind tunnel.

$L = 305$ mm $\delta_1^* = 1.5$ mm $\delta_2^* = 2.1$ mm $U_1 = 26$ m/sn.

Apply the continuity and Bernoulli equations to freestream flow outside the boundary-layer displacement thickness, where viscous effects are negligible.

$$0 = \cancelto{0}{\frac{\partial}{\partial t}\int \rho\, d\forall} + \int_{CS} \rho \vec{V} \cdot d\vec{A}$$



$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + \cancel{g z_1} = \frac{p_2}{\rho} + \frac{V_2^2}{2} + \cancel{g z_2}$$

<u>Assumptions:</u>

1) Steady flow
2) Incompressible flow
3) Flow uniform at each section outside δ^*
4) Flow along a streamline between sections (1) and (2)
5) No frictional effects in freestream
6) Neglect elevation changes

From the Bernoulli equation we obtain

$$P_1 - P_2 = \frac{1}{2}\rho(V_2^2 - V_1^2) = \frac{1}{2}\rho(U_2^2 - U_1^2) = \frac{1}{2}\rho U_1^2\left[\left(\frac{U_2}{U_1}\right)^2 - 1\right]$$

or

$$\frac{P_1 - P_2}{\frac{1}{2}\rho U_1^2} = \left(\frac{U_2}{U_1}\right)^2 - 1$$

From continuity, $V_1 A_1 = U_1 A_1 = V_2 A_2 = U_2 A_2$, so $\dfrac{U_2}{U_1} = \dfrac{A_1}{A_2}$,

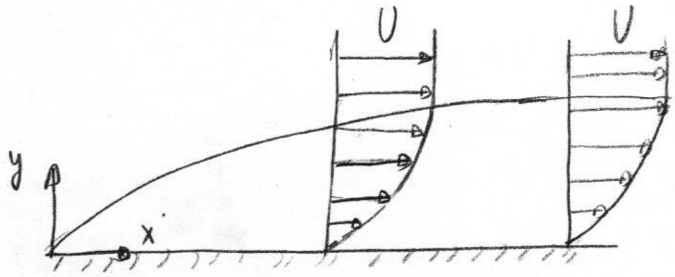
where $A = (L - 2\delta^*)^2$ is the effective flow area. Substituting gives

$$\frac{P_1 - P_2}{\frac{1}{2}\rho U_1^2} = \left(\frac{A_1}{A_2}\right)^2 - 1 = \left[\frac{(L - 2\delta_1^*)^2}{(L - 2\delta_2^*)^2}\right]^2 - 1$$

$$\frac{P_1 - P_2}{\frac{1}{2}\rho U_1^2} = \left[\frac{305 - 2*1.5}{305 - 2*2.1}\right]^4 - 1 = 0.0161 \text{ or } 1.61 \text{ percent}$$

LAMINAR FLAT-PLATE BOUNDARY LAYER - EXACT SOLUTION

Consider steady, two-dimensional and laminar flow of an incompressible fluid over a flat plate with zero pressure gradients under the absence of body forces. In this case, the governing equations of motion reduce to



$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}$$

with boundary conditions

at $y = 0$ $\quad u = 0 \quad v = 0$

at $y = \infty$ $\quad u = U, \quad \frac{du}{dy} = 0$

The dimensionless velocity profile, u/U, is similar for all values of x. Thus the solution is of the form

$$\frac{u}{U} = g(\eta) \qquad \text{where} \qquad \eta \propto \frac{y}{x^n}$$

Substituting these equations into the governing equations, one can deduce, for $n = 1/2$, the partial momentum equation becomes a ordinary differential equation, $\left(\eta \propto \frac{y}{x^{1/2}}\right)$.
In order to nondimensionalize η, the expression is multiplied by $\sqrt{\frac{U}{\nu}}$.
Therefore,

$$\eta = \frac{y}{\sqrt{\nu x / U}} = y\sqrt{\frac{U}{\nu x}}$$

Introducing the stream function, ψ , where

$$u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x}$$

satisfies the continuity equation. Defining a dimensionless stream function as

$$f(\eta) = \frac{\psi}{\sqrt{\nu x U}} \quad \text{or} \quad \psi = \sqrt{\nu x U}\, f(\eta) .$$

Then we can evaluate each of terms in momentum equation.

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial \eta} \cdot \frac{\partial \eta}{\partial y} = \sqrt{\nu x U}\, \frac{df}{d\eta} \cdot \sqrt{\frac{U}{\nu x}} = U \frac{df}{d\eta}$$

$$v = -\frac{\partial \psi}{\partial x} = -\left[\frac{\partial \psi}{\partial \eta} \frac{\partial \eta}{\partial x} + \frac{\partial \psi}{\partial x} \right]$$

$$v = -\left[\sqrt{\nu x U}\, \frac{df}{d\eta}\left(-\frac{1}{2}\eta \frac{1}{x}\right) + \frac{1}{2}\sqrt{\frac{\nu U}{x}}\, f \right]$$

$$v = \frac{1}{2}\sqrt{\frac{\nu U}{x}} \left[\eta \frac{df}{d\eta} - f \right]$$

By diffentiating the velocity components, it also can be shown that

$$\frac{\partial u}{\partial x} = -\frac{U}{2x} \eta \frac{d^2 f}{d\eta^2}$$

$$\frac{\partial u}{\partial y} = U \sqrt{U/\nu x}\, \frac{d^2 f}{d\eta^2}$$

and

$$\frac{\partial^2 u}{\partial y^2} = \frac{U^2}{\nu x}\frac{d^3 f}{d\eta^3}$$

Substituting these expressions into the momentum equation we obtain

$$2\frac{d^3 f}{d\eta^3} + f\frac{d^2 f}{d\eta^2} = 0$$

with boundary conditions

at $\eta = 0$ $\quad f = \frac{df}{d\eta} = 0 \quad (u=0,\ v=0)$

at $\eta \to \infty$ $\quad \frac{df}{d\eta} = 1 \quad (u = U)$

This nonlinear, third-order ordinary differential equation, which is called Blasius equation, can be solved numerically. The numerical values of f, $df/d\eta$, and $d^2f/d\eta^2$ is given in the Table.

Table 9.1 The Function $f(\eta)$ for the Laminar Boundary Layer along a Flat Plate at Zero Incidence

$\eta = y\sqrt{\frac{U}{\nu x}}$	f	$f' = \frac{u}{U}$	f''
0	0	0	0.3321
0.5	0.0415	0.1659	0.3309
1.0	0.1656	0.3298	0.3230
1.5	0.3701	0.4868	0.3026
2.0	0.6500	0.6298	0.2668
2.5	0.9963	0.7513	0.2174
3.0	1.3968	0.8460	0.1614
3.5	1.8377	0.9130	0.1078
4.0	2.3057	0.9555	0.0642
4.5	2.7901	0.9795	0.0340
5.0	3.2833	0.9915	0.0159
5.5	3.7806	0.9969	0.0066
6.0	4.2796	0.9990	0.0024
6.5	4.7793	0.9997	0.0008
7.0	5.2792	0.9999	0.0002
7.5	5.7792	1.0000	0.0001
8.0	6.2792	1.0000	0.0000

For $\eta = 5$ $\quad \frac{u}{U} = 0.992$ in the Table

Since

$$\eta = y\sqrt{\frac{U}{\nu x}} \Rightarrow y = \frac{\eta}{\sqrt{U/\nu x}}$$

$\therefore$

$$\delta \approx \frac{5.0}{\sqrt{U/\nu x}} = \frac{5.0\,x}{\sqrt{Re_x}} \qquad \text{where } Re_x = \frac{Ux}{\nu}$$

The wall shear stress may be expressed as

$$\tau_w = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0} = \mu U \sqrt{U/\nu x} \left.\frac{d^2 f}{d\eta^2}\right|_{\eta=0}$$

Then

$$\tau_w = 0.332\,U\sqrt{\rho\mu U/x} = \frac{0.332\,\rho U^2}{\sqrt{Re_x}}$$

and the wall shear stress coefficient, C_f, is given by

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho U^2} = \frac{0.664}{\sqrt{Re_x}}.$$

Example: Air with a density of 1.2 kg/m³ and kinematic viscosity of $1.5 * 10^{-5}$ m²/sn is flowing over a flat plate. Air approaches to the flat plate at a uniform velocity of 0.3 m/sn. The length and width of the flat plate are 4.5 m and 0.5 m respectively.

a) Determine the variation of the boundary layer thickness, the boundary layer displacement thickness, the boundary layer momentum thickness over the flat plate. Also plot their variation.

b) Determine the variation of the shear stress over the flat plate.

c) Determine the wall shear stress coefficient, Cf.

SOLUTION

One should first determine whether the flow over the flat plate laminar or turbulent.

$$Re = \frac{UL}{\nu} = \frac{0.3 * 4.5}{1.5 * 10^{-5}} = 9 * 10^4 < 5 * 10^5$$

Therefore the flow is laminar.

a)
$$\delta = \frac{5x}{\sqrt{Re_x}} = \frac{5x}{\sqrt{\frac{Ux}{\nu}}} = \frac{5x}{\sqrt{\frac{0.3x}{1.5*10^{-5}}}} = 0.0354\, x^{1/2} \text{ m.}$$

$$\delta^* = \int_0^\infty \left(1 - \frac{u}{U}\right) dy$$

But

$$\eta = y\sqrt{\frac{U}{\nu x}}, \quad \text{so} \quad y = \eta\sqrt{\frac{\nu x}{U}} \quad \text{and} \quad dy = d\eta\sqrt{\frac{\nu x}{U}}$$

and
$$\frac{u}{U} = \frac{df}{d\eta} = f'$$

Thus

$$\delta^* = \int_0^\infty (1-f')\sqrt{\frac{\nu x}{U}}\,d\eta = \sqrt{\frac{\nu x}{U}}\int_0^\infty (1-f')\,d\eta$$

By using the values given in the Table, the above integral may be evaluated to yield

$$\delta^* = 1.72\sqrt{\frac{\nu x}{U}} = \frac{1.72\,x}{\sqrt{Re_x}} = \frac{1.72x}{\sqrt{\frac{Ux}{\nu}}} = \frac{1.72x}{\sqrt{\frac{0.3x}{1.5\times10^{-5}}}}$$

$$\delta^* = 0.0122\,x^{1/2}$$

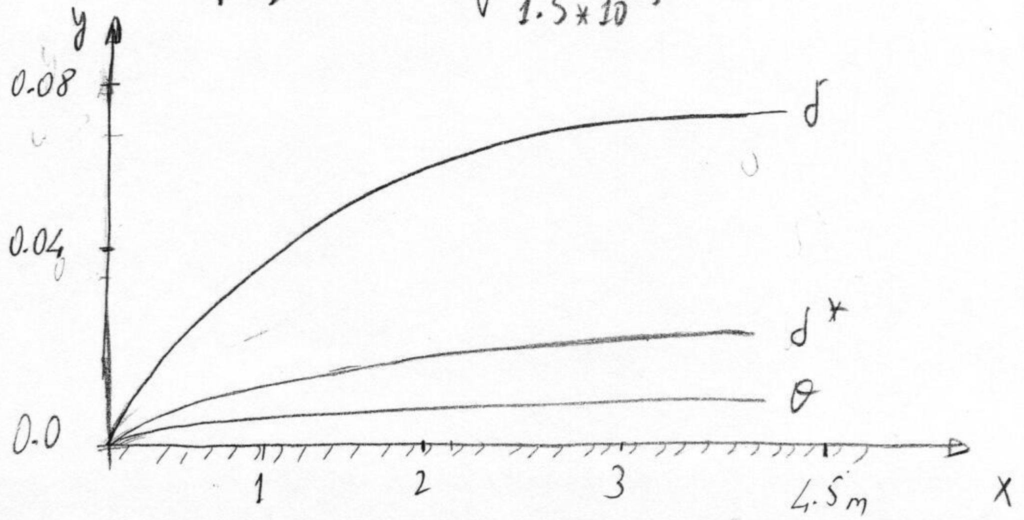
$$\theta = \int_0^\infty \frac{u}{U}\left(1-\frac{u}{U}\right)dy = \int_0^\infty f'(1-f')\sqrt{\frac{\nu x}{U}}\,d\eta = \sqrt{\frac{\nu x}{U}}\int_0 f'(1-f')\,d\eta$$

The above integral may be evaluated to yield

$$\theta = 0.664\sqrt{\frac{\nu x}{U}} = \frac{0.664\,x}{\sqrt{Re_x}}$$

or

$$\theta = \frac{0.664\,x}{\sqrt{\frac{Ux}{\nu}}} = \frac{0.664\,x}{\sqrt{\frac{0.3\,x}{1.5\times10^{-5}}}} = 0.0046\,x^{1/2}$$



b) The variation of the shear stress over the flat plate may be obtained

$$\tau_w = \frac{0.332 \rho U^2}{\sqrt{Re_x}} = \frac{0.332 \rho U^2}{\sqrt{\frac{Ux}{\nu}}}$$

$$\tau_w = \frac{0.332 * 1.2 * 0.3^2}{\sqrt{\frac{0.3 X}{1.5 * 10^{-5}}}} = \frac{2.535 * 10^{-4}}{\sqrt{X}}$$

c) The wall shear stress coefficient C_f

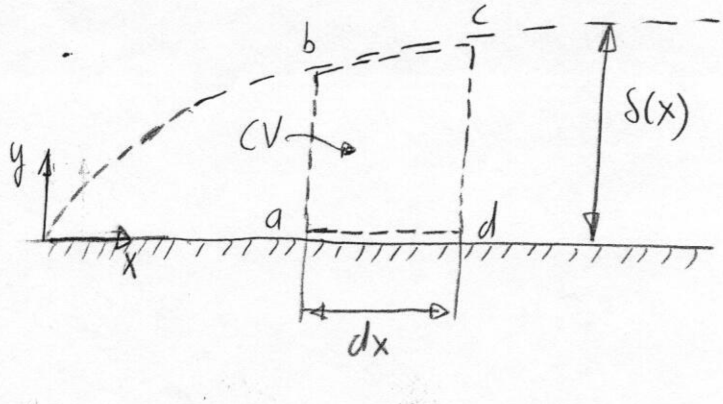
$$C_f = \frac{0.664}{\sqrt{Re_x}} = \frac{0.664}{\sqrt{\frac{Ux}{\nu}}}$$

$$C_f = \frac{0.664}{\sqrt{\frac{0.3 x}{1.5 * 10^{-5}}}} = \frac{4.695 * 10^{-3}}{\sqrt{x}}$$

MOMENTUM INTEGRAL EQUATION

In the previous section, an exact solution of the boundary layer equations is obtained for steady, laminar flow of an incompressible fluid over a flat plate with zero pressure gradient. However, it is not always possible to obtain the exact solutions of the boundary layer equations for more complicated flow fields. For this reason some approximate solution methods are devised for obtaining the boundary layer thickness. One of these approximate solution methods is the Von Karman integral momentum equation, which gives very good results for the boundary layer thickness, not only in the laminar flow range but also in the turbulent flow range.

To derive this equation, consider incompressible, steady, two-dimensional flow over a solid surface. For our analysis we choose a differential control volume, of length dx, width, w, and height $\delta(x)$, as shown in the figure.



Continuity Equation

Assumptions:

1) Steady flow
2) Two-dimensional flow

$$0 = \cancelto{0}{\frac{\partial}{\partial t}\int_{CV} \rho \, d\forall} + \int_{CS} \rho \vec{V} \cdot d\vec{A}$$

Then

$$0 = \int_{CS} \rho \vec{V} \cdot d\vec{A} = \dot{m}_{ab} + \dot{m}_{bc} + \dot{m}_{cd}$$

or

$$\dot{m}_{bc} = -\dot{m}_{ab} - \dot{m}_{cd}$$

Let us evaluate these terms for the differential control volume

Surface ab is located at x. Since the flow is two-dimensional, the mass flux through ab is

$$\dot{m}_{ab} = -\left\{ \int_0^{\delta} \rho u \, dy \right\} w$$

Surface cd is located at $x + dx$. Expanding $\dot{m}$ in Taylor series about location x, we obtain

$$\dot{m}_{x+dx} = \dot{m}_x + \left.\frac{\partial \dot{m}}{\partial x}\right]_x dx$$

and hence

$$\dot{m}_{cd} = \left\{ \int_0^{\delta} \rho u \, dy + \frac{\partial}{\partial x}\left[\int_0^{\delta} \rho u \, dy \right] dx \right\} w$$

Thus for surface bc we obtain

$$\dot{m}_{bc} = -\left\{ \frac{\partial}{\partial x}\left[\int_0^{\delta} \rho u \, dy \right] dx \right\} w$$

Momentum Equation

Apply the x component of the momentum equation to control volume abcd:

$$F_{Sx} + \cancel{F_{Bx}}^{\,0} = \cancel{\frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall}^{\,0} + \int_{CS} u\rho \vec{V}\cdot d\vec{A}$$

Assumption:

3) $F_{Bx} = 0$

Then

$$F_{Sx} = mf_{ab} + mf_{bc} + mf_{cd}$$

where mf represents the x component of momentum flux.
The x momentum flux through ab is

$$mf_{ab} = -\left\{\int_0^{\delta} u\rho u\, dy\right\} w$$

Surface cd is located at $x+dx$. Expanding the x momentum flux (mf) in a Taylor series about location x, we obtain

$$mf_{x+dx} = mf_x + \left.\frac{\partial mf}{\partial x}\right]_x dx$$

or

$$mf_{cd} = \left\{\int_0^{\delta} u\rho u\, dy + \frac{\partial}{\partial x}\left[\int_0^{\delta} u\rho u\, dy\right] dx\right\} w$$

Since the mass crossing surface bc has velocity component U in the x direction, the x momentum flux across bc is given by

$$mf_{bc} = U \dot{m}_{bc}$$

$$mf_{bc} = -U \left\{ \frac{\partial}{\partial x} \left[\int_0^\delta \rho u \, dy \right] dx \right\} w$$

Then, we can evaluate the net x momentum flux through the control surface as

$$\int_{CS} u \rho \vec{V} \cdot d\vec{A} = -\left\{ \int_0^\delta u \rho u \, dy \right\} w + \left\{ \int_0^\delta u \rho u \, dy \right\} w$$

$$+ \left\{ \frac{\partial}{\partial x} \left[\int_0^\delta u \rho u \, dy \right] dx \right\} w - U \left\{ \frac{\partial}{\partial x} \left[\int_0^\delta \rho u \, dy \right] dx \right\} w$$

Collecting terms, we find that

$$\int_{CS} u \rho \vec{V} \cdot d\vec{A} = \left\{ \frac{\partial}{\partial x} \left[\int_0^\delta u \rho u \, dy \right] dx - U \frac{\partial}{\partial x} \left[\int_0^\delta \rho u \, dy \right] dx \right\} w$$

Let us consider the surface forces acting on the control volume in the x direction.

a) Shear force (on surface ad)
b) Normal forces (on the other surfaces)

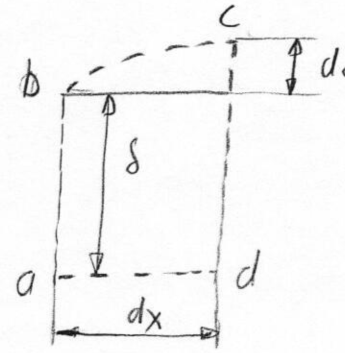


Fig. Differential control volume

If the pressure at x is p, then the force acting on surface ab is given by

$$F_{ab} = p w \delta$$

Because, the boundary layer is very thin, pressure variation in the y direction may be neglected. ($p = p(x)$)

The pressure at $x+dx$ can be found expanding in a Taylor series

$$p_{x+dx} = p + \left.\frac{dp}{dx}\right|_x dx$$

Then the force on surface cd is given by.

$$F_{cd} = -\left(p + \left.\frac{dp}{dx}\right|_x dx\right) w \left(\delta + d\delta\right)$$

The average pressure acting over surface bc is

$$\frac{p + p + \left.\frac{dp}{dx}\right|_x dx}{2} = p + \frac{1}{2}\left.\frac{dp}{dx}\right|_x dx$$

Then the x component of the normal force acting over bc is

$$F_{bc} = \left(p + \frac{1}{2}\left.\frac{dp}{dx}\right|_x dx\right) w \, d\delta$$

The shear force acting on ad is given by.

$$F_{ad} = -\left(\tau_w + \frac{1}{2} d\tau_w\right) w \, dx$$

Summing the x component ~~of each~~ force acting on the control volume, we obtain

$$F_{sx} = \left\{ -\frac{dp}{dx}\delta\, dx - \cancelto{0}{\frac{1}{2}\frac{dp}{dx}dx\, d\delta} - \tau_w dx - \cancelto{0}{\frac{1}{2}d\tau_w dx} \right\} w$$

where we note that $dx\, d\delta << \delta dx$ and $d\tau_w << \tau_w$, and so neglect the second and fourth terms.

Substituting the expressions into the momentum equation

$$F_{sx} = \int_{C.S} u \rho \vec{V} \cdot d\vec{A} \qquad \text{we obtain}$$

$$\left\{ -\frac{dp}{dx}\delta dx - \tau_w dx \right\} w = \left\{ \frac{\partial}{\partial x}\left[\int_0^\delta u\rho u\, dy\right] dx - U\frac{\partial}{\partial x}\left[\int_0^\delta \rho u\, dy\right] dx \right\} w$$

Dividing this equation by $w dx$ gives

$$-\delta\frac{dp}{dx} - \tau_w = \frac{\partial}{\partial x}\int_0^\delta u\rho u\, dy - U\frac{\partial}{\partial x}\int_0^\delta \rho u\, dy$$

This is a momentum integral equation.

The pressure gradient, dp/dx, can be determined by applying the Bernoulli equation to the inviscid flow outside the boundary layer

$$\frac{p}{\rho} + \frac{U^2}{2} = \text{constant}$$

differentiating it

$$\frac{dp}{dx} = -\rho U \frac{dU}{dx}$$

If we recognize that $\delta = \int_0^\delta dy$, then, the momentum integral equation can be written as

$$\tau_w = -\frac{\partial}{\partial x}\int_0^\delta u\rho u\,dy + U\frac{\partial}{\partial x}\int_0^\delta \rho u\,dy + \frac{dU}{dx}\int_0^\delta \rho U\,dy$$

Since

$$U\frac{\partial}{\partial x}\int_0^\delta \rho u\,dy = \frac{\partial}{\partial x}\int_0^\delta \rho u U\,dy - \frac{dU}{dx}\int_0^\delta \rho u\,dy$$

then

$$\tau_w = \frac{\partial}{\partial x}\int_0^\delta \rho u(U-u)\,dy + \frac{dU}{dx}\int_0^\delta \rho(U-u)\,dy$$

and

$$\tau_w = \frac{\partial}{\partial x}U^2\int_0^\delta \rho\frac{u}{U}\left(1-\frac{u}{U}\right)dy + U\frac{dU}{dx}\int_0^\delta \rho\left(1-\frac{u}{U}\right)dy$$

Using the definitions of displacement thickness, δ^*, and momentum thickness θ, then

$$\boxed{\frac{\tau_w}{\rho} = \frac{d}{dx}(U^2\theta) + \delta^* U\frac{dU}{dx}}$$

This is the Von Karman momentum integral equation. Where

$$\delta^* = \int_0^\delta\left(1-\frac{u}{U}\right)dy$$

and

$$\theta = \int_0^\delta \frac{u}{U}\left(1-\frac{u}{U}\right)dy.$$