AC STEADY-STATE ANALYSIS

LEARNING GOALS

SINUSOIDALS
Review basic facts about sinusoidal signals

SINUSOIDAL AND COMPLEX FORCING FUNCTIONS
Behavior of circuits with sinusoidal independent sources and modeling of sinusoids in terms of complex exponentials

PHASORS
Representation of complex exponentials as vectors. It facilitates steady-state analysis of circuits.

IMPEDANCE AND ADMITANCE
Generalization of the familiar concepts of resistance and conductance to describe AC steady state circuit operation

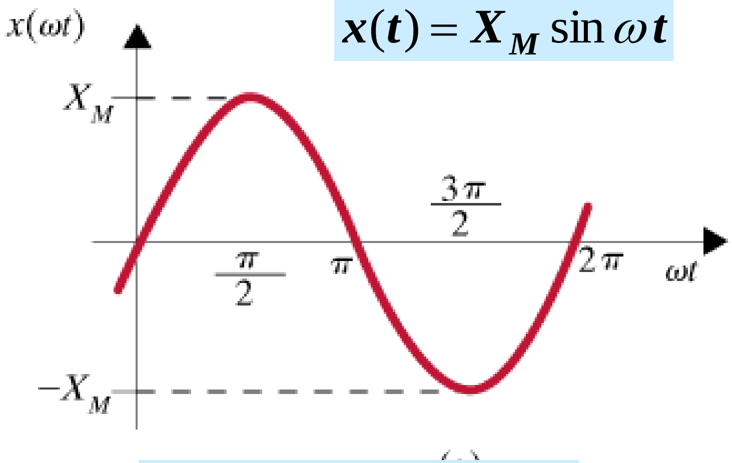
PHASOR DIAGRAMS
Representation of AC voltages and currents as complex vectors

BASIC AC ANALYSIS USING KIRCHHOFF LAWS

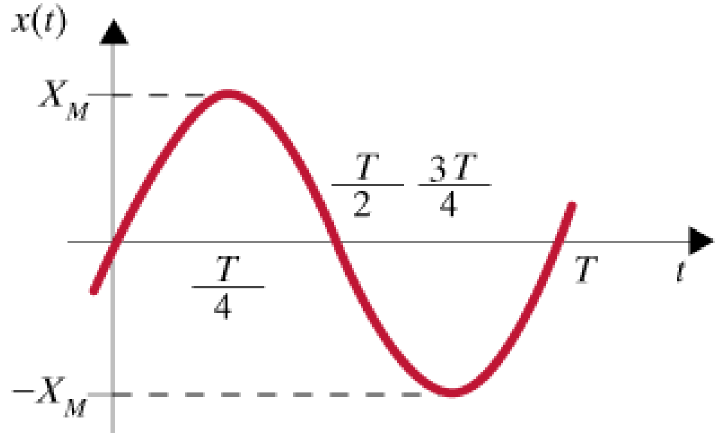
ANALYSIS TECHNIQUES
Extension of node, loop, Thevenin and other techniques

SINUSOIDAL SIGNALS

$x(t) = X_M \sin \omega t$



Adimensional plot



As function of time

$X_M =$ amplitude or maximum value

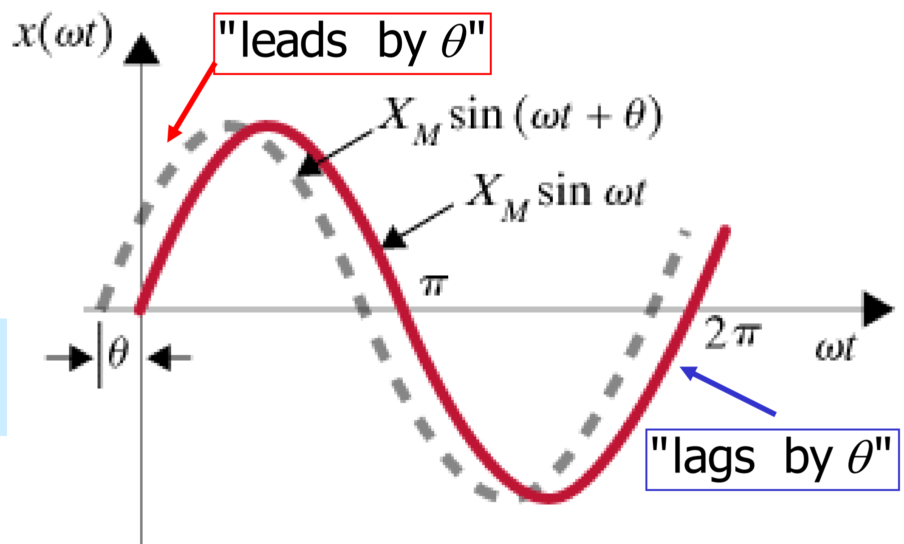
$\omega =$ angular frequency (rads/sec)

$\omega t =$ argument (radians)

$T = \frac{2\pi}{\omega} =$ Period $\Rightarrow x(t) = x(t+T), \forall t$

$f = \frac{1}{T} = \frac{\omega}{2\pi} =$ frequency in Hertz (cycle/sec)

$\omega = 2\pi f$



BASIC TRIGONOMETRY (RECALL FROM HIGH SCHOOL

ESSENTIAL IDENTITIES

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$

$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$

$$\sin(-\alpha)=-\sin\alpha$$

$$\cos(-\alpha)=\cos\alpha$$

SOME DERIVED IDENTITIES

$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$$

$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$$

$$\sin\alpha\cos\beta=\frac{1}{2}\sin(\alpha+\beta)+\frac{1}{2}\sin(\alpha-\beta)$$

$$\cos\alpha\cos\beta=\frac{1}{2}\cos(\alpha+\beta)+\frac{1}{2}\cos(\alpha-\beta)$$

RADIANS AND DEGREES

2π radians = 360 degrees

$$\theta(\text{rads})=\frac{180}{\pi}\theta\ (\text{degrees})$$

ACCEPTED EE CONVENTION

$$\sin(\omega \boldsymbol{t}+\frac{\pi}{2})=\sin(\omega \boldsymbol{t}+90°)$$

APPLICATIONS

$$\cos\omega \boldsymbol{t}=\sin(\omega \boldsymbol{t}+\frac{\pi}{2})$$

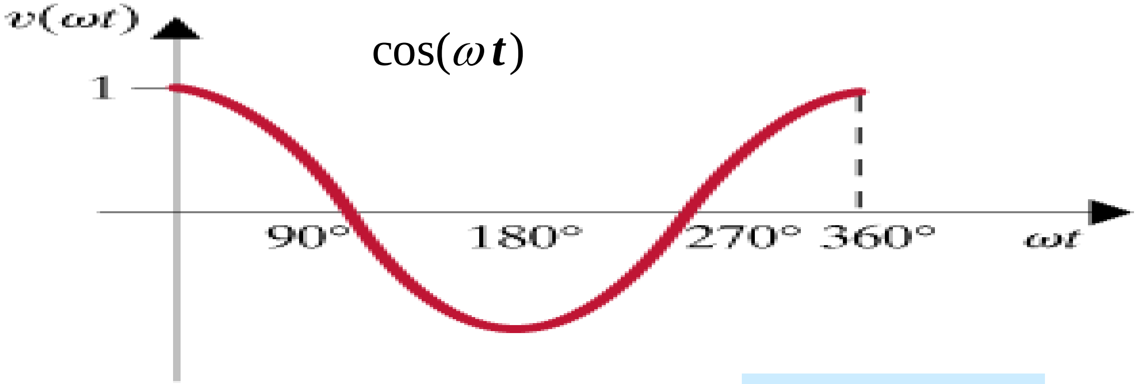
$$\sin\omega \boldsymbol{t}=\cos(\omega \boldsymbol{t}-\frac{\pi}{2})$$

$$\cos\omega \boldsymbol{t}=-\cos(\omega \boldsymbol{t}\pm\pi)$$

$$\sin\omega \boldsymbol{t}=-\sin(\omega \boldsymbol{t}\pm\pi)$$

EXAMPLE 1

$\cos(\omega t)$

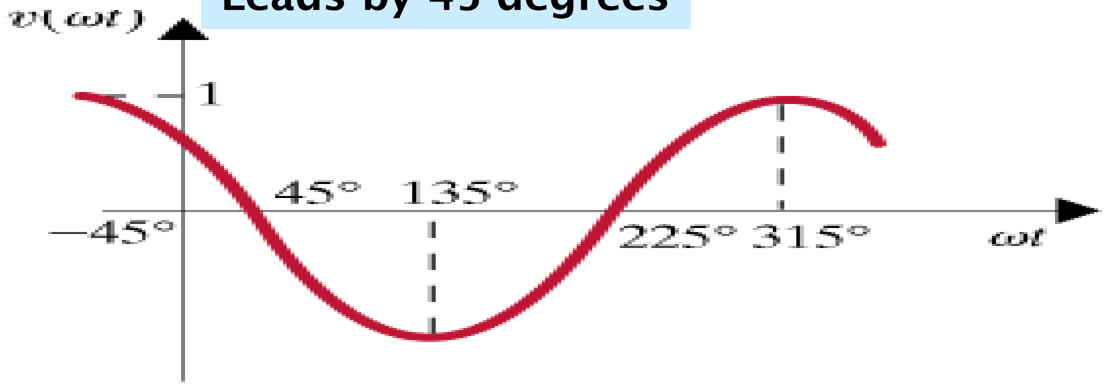


Leads by 45 degrees

Lags by 315

$\cos(\omega t + 45°)$

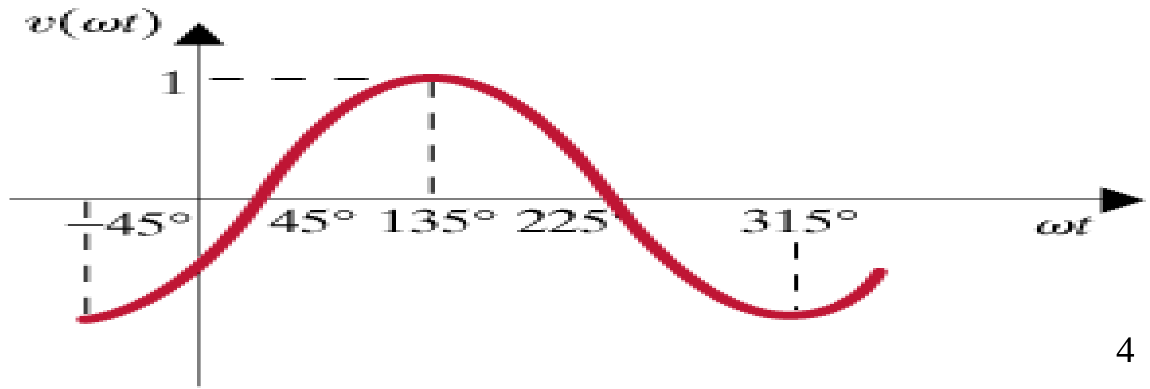
$\cos(\omega t + 45 - 360)$



$-\cos(\omega t + 45°)$

$\cos(\omega t + 45 \pm 180)$

Leads by 225 or lags by 135



(c)

EXAMPLE 2

$v_1(t) = 12\sin(1000t + 60°),\ v_2(t) = -6\cos(1000t + 30°)$

FIND FREQUENCY AND PHASE ANGLE BETWEEN VOLTAGES

Frequency in radians per second is the factor of the time variable $\omega = 1000\sec^{-1}$

$$f(Hz) = \frac{\omega}{2\pi} = 159.2Hz$$

To find phase angle we must express both sinusoids using the same trigonometric function; either sine or cosine with positive amplitude

take care of minus sign with $\cos(\alpha) = -\cos(\alpha \pm 180°)$

$$-6\cos(1000t + 30°) = 6\cos(1000t + 30° + 180°)$$

Change sine into cosine with $\cos(\alpha) = \sin(\alpha + 90°)$

$$6\cos(1000t + 210°) = 6\sin(1000t + 210° + 90°)$$

We like to have the phase shifts less than 180 in absolute value

$$6\sin(1000t + 300°) = 6\sin(1000t - 60°)$$

$v_1(t) = 12\sin(1000t + 60°)$

$v_2(t) = 6\sin(1000t - 60°)$

$(1000t + 60°) - (1000t - 60°) = 120°$

v_1 leads v_2 by 120°

$(1000t - 60°) - (1000t + 60°) = -120°$

v_2 lags v_1 by 120°

EXAMPLE 3

$$i_1(t) = 2\sin(377t + 45°)$$

$$i_2(t) = 0.5\cos(377t + 10°)$$

$$i_3(t) = -0.25\sin(377t + 60°)$$

i_1 leads i_2 by______?

i_1 leads i_3 by______?

$$\cos\alpha = \sin(\alpha + 90°)$$

$$0.5\cos(377t + 10°) = 0.5\sin(377t + 10° + 90°)$$

$$(377t + 45°) - (377 + 100°) = -55°$$

i_1 leads i_2 by $-55°$

$$\sin\alpha = -\sin(\alpha \pm 180°)$$

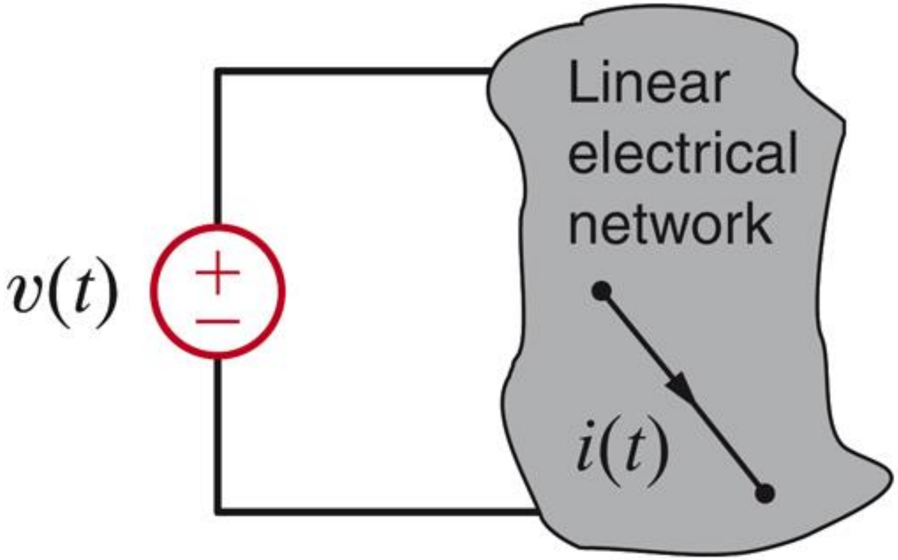
$$-0.25\sin(377t + 60°) = 0.25\sin(377t + 60° - 180°)$$

$$(377t + 45°) - (377t - 120°) = 165°$$

i_1 leads i_3 by $165°$

SINUSOIDAL AND COMPLEX FORCING FUNCTIONS

Example 4

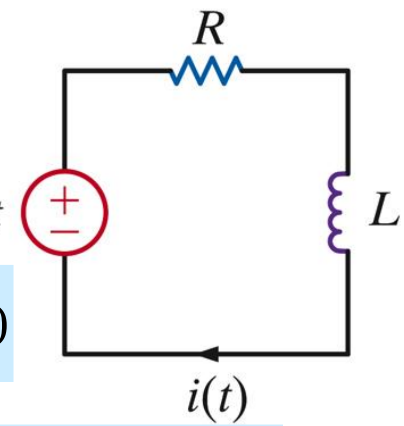


If the independent sources are sinusoids of the same frequency then for any variable in the linear circuit the steady state response will be sinusoidal and of the same frequency

$$v(t) = A\sin(\omega t + \theta) \Rightarrow i_{SS}(t) = B\sin(\omega t + \phi)$$

To determine the steady state solution we only need to determine the parameters B, ϕ

$v(t) = V_M \cos \omega t$



KVL: $L\dfrac{di}{dt}(t) + Ri(t) = v(t)$

In steady state $i(t) = A\cos(\omega t + \phi)$, or

$$i(t) = A_1 \cos\omega t + A_2 \sin \omega t \qquad */R$$

$$\frac{di}{dt}(t) = -A_1\omega \sin \omega t + A_2\omega \cos\omega t \qquad */L$$

$$(-L\omega A_1 + RA_2)\sin \omega t + (L\omega A_2 + RA_1)\cos\omega t =$$
$$= V_M \cos\omega t$$

$$-L\omega A_1 + RA_2 = 0$$
$$L\omega A_2 + RA_1 = V_M$$

algebraic problem

$$A_1 = \frac{RV_M}{R^2 + (\omega L)^2}, \quad A_2 = \frac{\omega L V_M}{R^2 + (\omega L)^2}$$

Determining the steady state solution can be accomplished with only algebraic tools!

FURTHER ANALYSIS OF THE SOLUTION

The solution is $i(t) = A_1 \cos\omega t + A_2 \sin\omega t$

The applied voltage is $v(t) = V_M \cos\omega t$

For comparison purposes one can write $i(t) = A\cos(\omega t + \phi)$

$$A_1 = A\cos\phi, \quad A_2 = -A\sin\phi$$

$$A = \sqrt{A_1^2 + A_2^2}, \quad \tan\phi = -\frac{A_2}{A_1}$$

$$A_1 = \frac{RV_M}{R^2 + (\omega L)^2}, \quad A_2 = \frac{\omega L V_M}{R^2 + (\omega L)^2}$$

$$A = \frac{V_M}{\sqrt{R^2 + (\omega L)^2}}, \quad \varphi = \tan^{-1}\frac{\omega L}{R}$$

$$i(t) = \frac{V_M}{\sqrt{R^2 + (\omega L)^2}} \cos(\omega t - \tan^{-1}\frac{\omega L}{R})$$

For $L \neq 0$ the current ALWAYS lags the voltage

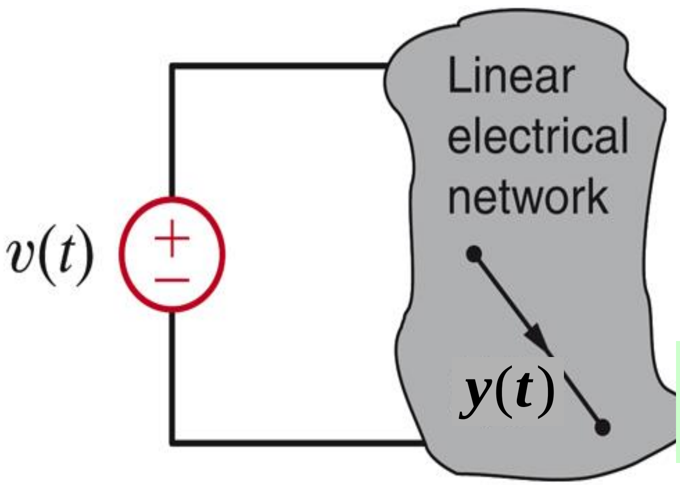
If $R = 0$ (pure inductor) the current lags the voltage by 90°

SOLVING A SIMPLE ONE LOOP CIRCUIT CAN BE VERY LABORIOUS IF ONE USES SINUSOIDAL EXCITATIONS

TO MAKE ANALYSIS SIMPLER ONE RELATES SINUSOIDAL SIGNALS TO COMPLEX NUMBERS. THE ANALYSIS OF STEADY STATE WILL BE CONVERTED TO SOLVING SYSTEMS OF ALGEBRAIC EQUATIONS ...

... WITH COMPLEX VARIABLES

ESSENTIAL IDENTITY: $\boldsymbol{e}^{j\theta} = \cos\theta + \boldsymbol{j}\sin\theta$ (Euler identity)



$$v(t) = V_M \cos\omega t \rightarrow y(t) = A\cos(\omega t + \theta)$$

$$v(t) = V_M \sin\omega t \rightarrow y(t) = A\sin(\omega t + \theta)$$ $*/\,\boldsymbol{j}$ (and add)

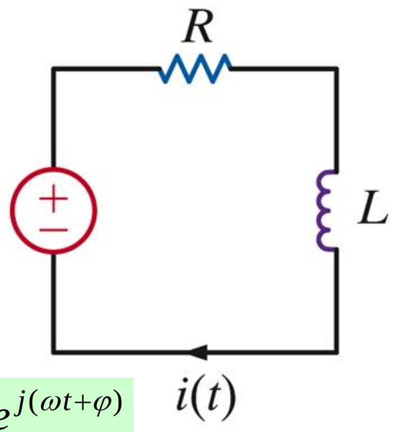
$$V_M e^{j\omega t} \rightarrow Ae^{j(\omega t + \theta)} = Ae^{j\theta}e^{j\omega t}$$

If everybody knows the frequency of the sinusoid then one can skip the term *exp(jwt)*

$$\boldsymbol{V_M \rightarrow Ae^{j\theta}}$$

Example 5

$v(t) = V_M \cos \omega t$



$$\boldsymbol{v(t) = V_M e^{j\omega t}}$$

Assume $i(t) = I_M e^{j(\omega t+\varphi)}$

$$\text{KVL:}\quad \boldsymbol{L\frac{di}{dt}(t) + Ri(t) = v(t)}$$

$$\frac{di}{dt}(t) = j\omega I_M e^{j(\omega t+\varphi)}$$

$$L\frac{di}{dt}(t) + Ri(t) = j\omega L I_M e^{j(\omega t+\varphi)} + R I_M e^{j(\omega t+\varphi)}$$

$$= (j\omega L + R) I_M e^{j(\omega t+\varphi)}$$

$$= (j\omega L + R) I_M e^{j\varphi} e^{j\omega t}$$

$$\boldsymbol{(j\omega L + R) I_M e^{j\phi} e^{j\omega t} = V_M e^{j\omega t}}$$

$$\boldsymbol{I_M e^{j\phi} = \frac{V_M}{j\omega L + R}} \quad \boldsymbol{* / \frac{R - j\omega L}{R - j\omega L}}$$

$$\boldsymbol{I_M e^{j\phi} = \frac{V_M (R - j\omega L)}{R^2 + (\omega L)^2}}$$

$$\boldsymbol{R - j\omega L = \sqrt{R^2 + (\omega L)^2}\, e^{-\tan^{-1}\frac{\omega L}{R}}}$$

$$\boldsymbol{I_M e^{j\phi} = \frac{V_M}{\sqrt{R^2 + (\omega L)^2}}\, e^{-\tan^{-1}\frac{\omega L}{R}}}$$

$$\boldsymbol{I_M = \frac{V_M}{\sqrt{R^2 + (\omega L)^2}}, \quad \phi = -\tan^{-1}\frac{\omega L}{R}}$$

$$\boldsymbol{v(t) = V_M \cos\omega t = \mathrm{Re}\{V_M e^{j\omega t}\}}$$

$$\boldsymbol{\Rightarrow i(t) = \mathrm{Re}\{I_M e^{(j\omega t - \phi)}\} = I_M \cos(\omega t - \phi)}$$

$C \leftrightarrow P$

$$x + jy = re^{j\theta}$$

$$r = \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1}\frac{y}{x}$$

$$x = r\cos\theta, \quad y = r\sin\theta$$

PHASORS

ESSENTIAL CONDITION
ALL INDEPENDENT SOURCES ARE SINUSOIDS OF THE SAME FREQUENCY

BECAUSE OF SOURCE SUPERPOSITION ONE CAN CONSIDER A SINGLE SOURCE

$$u(t) = U_M \cos(\omega t + \theta)$$

THE STEADY STATE RESPONSE OF ANY CIRCUIT VARIABLE WILL BE OF THE FORM

$$y(t) = Y_M \cos(\omega t + \phi)$$

SHORTCUT 1

$$u(t) = U_M e^{j(\omega t + \theta)} \Rightarrow y(t) = Y_M e^{j(\omega t + \phi)}$$

$$\mathrm{Re}\{U_M e^{j(\omega t + \theta)}\} \Rightarrow \mathrm{Re}\{Y_M e^{j(\omega t + \phi)}\}$$

NEW IDEA: $U_M e^{j(\omega t + \theta)} = U_M e^{j\theta} e^{j\omega t}$ $\quad u = U_M e^{j\theta} \Rightarrow y = Y_M e^{j\phi}$

SHORTCUT IN NOTATION

INSTEAD OF WRITING $u = U_M e^{j\theta}$ WE WRITE $u = U_M \angle\theta$

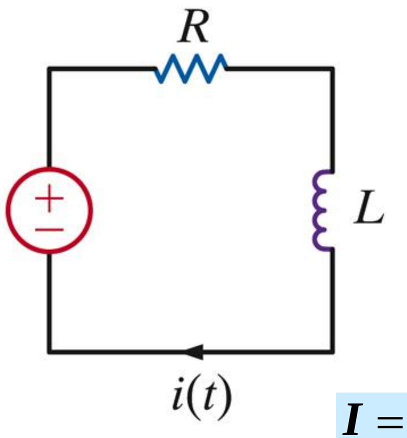
... AND WE ACCEPT ANGLES IN DEGREES

$U_M \angle\theta$ IS THE PHASOR REPRESENTATION FOR $U_M \cos(\omega t + \theta)$

$$u(t) = U_M \cos(\omega t + \theta) \rightarrow U = U_M \angle\theta \Rightarrow Y = Y_M \angle\phi \rightarrow y(t) = Y_M \cos(\omega t + \phi)$$

SHORTCUT 2: DEVELOP EFFICIENT TOOLS TO DETERMINE THE PHASOR OF THE RESPONSE GIVEN THE INPUT PHASOR(S)

Example 6



$v(t) = V_M \cos \omega t$

$$\mathbf{V} = V_M \angle 0$$
$$\mathbf{v} = \mathbf{V} e^{j\omega t}$$

$$\mathbf{I} = I_M \angle \phi$$
$$\mathbf{i} = \mathbf{I} e^{j\omega t}$$

$$L\frac{di}{dt}(t) + Ri(t) = v$$
$$L(j\omega \mathbf{I} e^{j\omega t}) + R\mathbf{I} e^{j\omega t} = \mathbf{V} e^{j\omega t}$$

In terms of phasors one has

$$j\omega L\mathbf{I} + R\mathbf{I} = \mathbf{V}$$
$$\mathbf{I} = \frac{\mathbf{V}}{R + j\omega L}$$

The phasor can be obtained using only complex algebra

We will develop a phasor representation for the circuit that will eliminate the need of writing the differential equation

Learning Extensions

It is essential to be able to move from sinusoids to phasor representation

$$A\cos(\omega t \pm \theta) \leftrightarrow A\angle \pm\theta$$
$$A\sin(\omega t \pm \theta) \leftrightarrow A\angle \pm\theta - 90°$$

$$v(t) = 12\cos(377t - 425°) \leftrightarrow 12\angle -425°$$
$$y(t) = 18\sin(2513t + 4.2°) \leftrightarrow 18\angle -85.8°$$

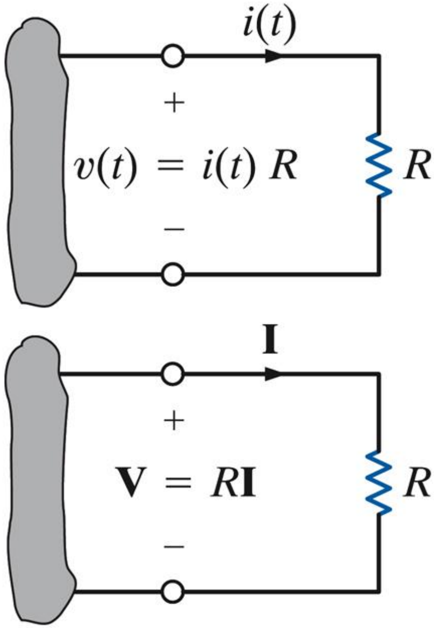
Given $f = 400Hz$

$$\mathbf{V}_1 = 10\angle 20° \leftrightarrow v_1(t) = 10\cos(800\pi t + 20°)$$
$$\mathbf{V}_2 = 12\angle -60° \leftrightarrow v_2(t) = 12\cos(800\pi t - 60°)$$

Phasors can be combined using the rules of complex algebra

$$(V_1\angle\theta_1)(V_2\angle\theta_2) = V_1 V_2 \angle(\theta_1 + \theta_2)$$
$$\frac{V_1\angle\theta_1}{V_2\angle\theta_2} = \frac{V_1}{V_2}\angle(\theta_1 - \theta_2)$$

PHASOR RELATIONSHIPS FOR CIRCUIT ELEMENTS



RESISTORS

$$v(t) = Ri(t)$$

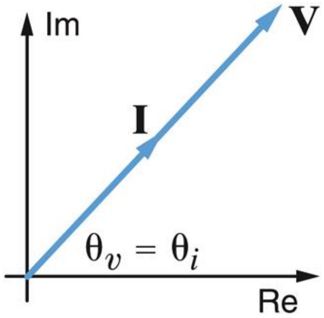
$$V_M e^{j(\omega t+\theta)} = RI_M e^{j(\omega t+\theta)}$$

$$V_M e^{j\theta} = RI_M e^{j\theta}$$

$$\mathbf{V} = R\mathbf{I}$$

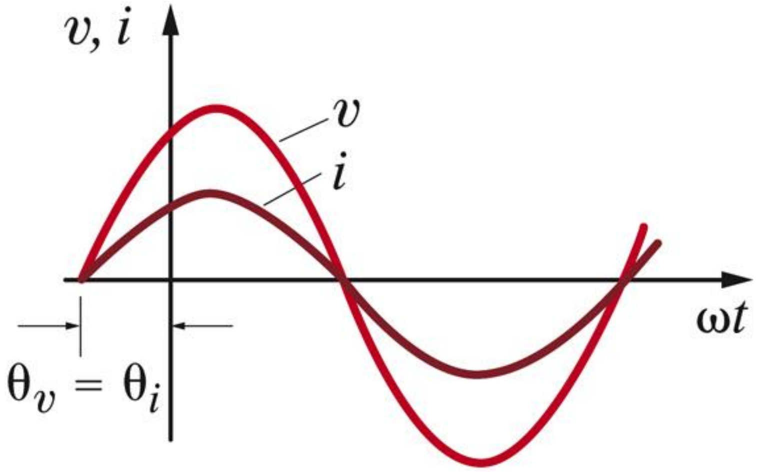
Phasor representation for a resistor

Phasors are complex numbers. The resistor model has a geometric interpretation

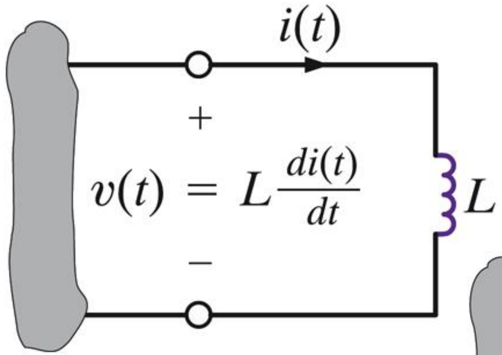


The voltage and current phasors are colinear

In terms of the sinusoidal signals this geometric representation implies that the two sinusoids are "in phase"



INDUCTORS

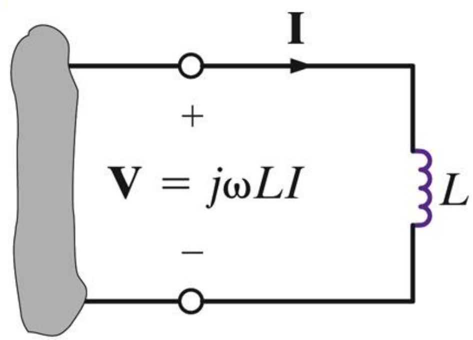


$$V_M e^{j(\omega t+\theta)} = L\frac{d}{dt}\left(I_M e^{j(\omega t+\phi)}\right)$$

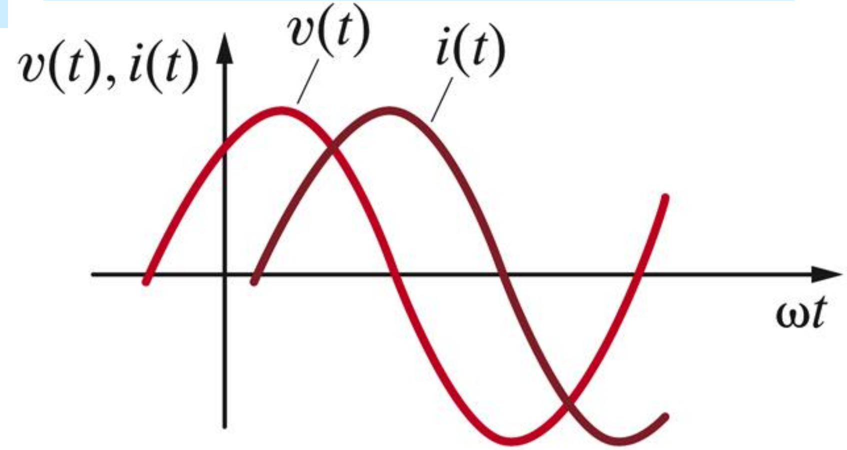
$$= j\omega L I_M e^{j(\omega t+\phi)}$$

$$V_M e^{j\theta} = j\omega L I_M e^{j\phi}$$

$$\boldsymbol{V} = \boldsymbol{j\omega L I}$$



Relationship between sinusoids

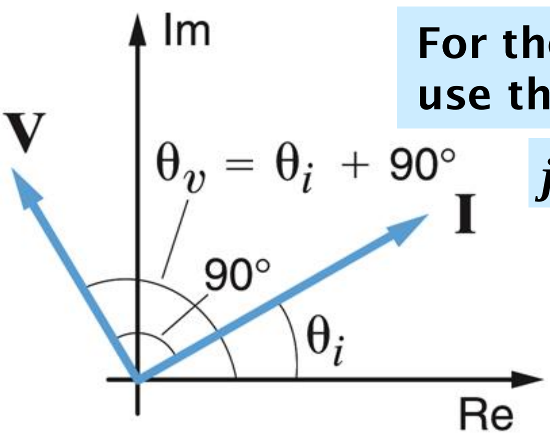


The relationship between phasors is algebraic

For the geometric view use the result

$$\boldsymbol{j} = 1\angle 90° = \boldsymbol{e}^{j90°}$$

$$\boldsymbol{V} = \omega \boldsymbol{LI}\angle 90°$$



The voltage leads the current by 90 deg
The current lags the voltage by 90 deg

Learning Example

$\boldsymbol{L} = 20\boldsymbol{mH}, \boldsymbol{v(t)} = 12\cos(377\boldsymbol{t} + 20°)$. Find $\boldsymbol{i(t)}$

$$\omega = 377$$

$$\boldsymbol{V} = 12\angle 20°$$

$$\boldsymbol{I} = \frac{\boldsymbol{V}}{\boldsymbol{j\omega L}}$$

$$\boldsymbol{I} = \frac{12\angle 20°}{\omega \boldsymbol{L}\angle 90°}(\boldsymbol{A})$$

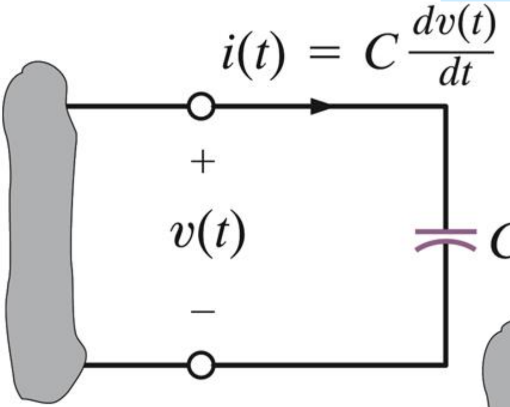
$$\boldsymbol{I} = \frac{12}{377\times 20\times 10^{-3}}\angle -70°(\boldsymbol{A})$$

$$i(t) = \frac{12}{377\times 20\times 10^{-3}}\cos(377t - 70°)$$

$$i(t) = 1.592\cos(377t - 70°), A$$

CAPACITORS

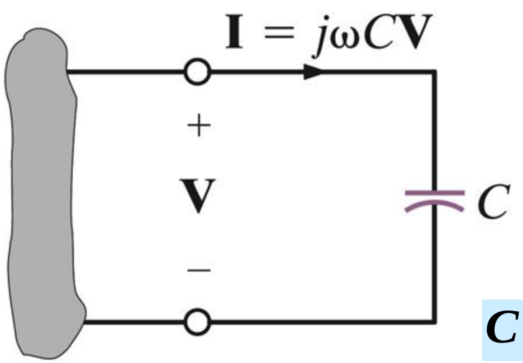
$$I_M e^{j(\omega t+\phi)} = C\frac{d}{dt}(V_M e^{j(\omega t+\theta)})$$



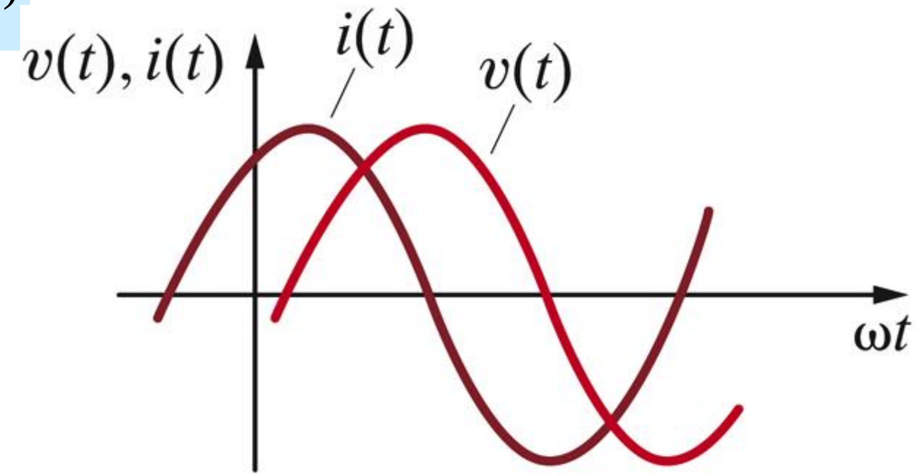
$$I = j\omega CV$$

$$I_M e^{j\phi} = j\omega C V_M e^{j\theta}$$

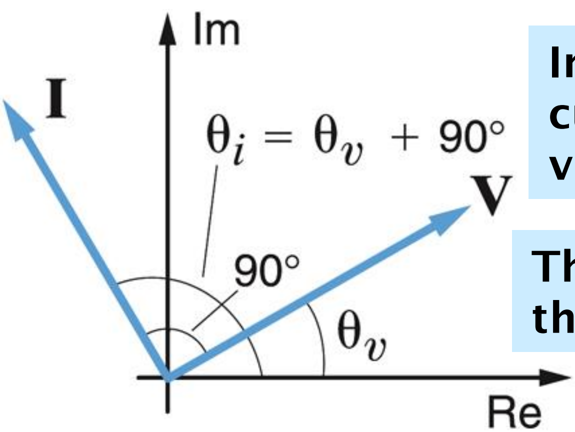
$$I = \omega CV\angle 90°$$



Relationship between sinusoids



The relationship between phasors is algebraic



In a capacitor the current leads the voltage by 90 deg

The voltage lags the current by 90 deg

Example 7

$C = 100\mu F, v(t) = 100\cos(314t + 15°)$. Find $i(t)$

$\omega = 314$

$V = 100\angle 15°$

$I = j\omega CV$

$$I = \omega C \times 1\angle 90° \times 100\angle 15°$$

$$I = 314 \times 100 \times 10^{-6} \times 100\angle 105°(A)$$

$$i(t) = 3.14\cos(314t + 105°)(A)$$

Example 8

$L = 0.05H,\ I = 4\angle -30°(A),\ f = 60Hz$

Find the voltage across the inductor

$\omega = 2\pi f = 120\pi$

$V = j\omega LI$

$V = 120\pi \times 0.05 \times 1\angle 90° \times 4\angle -30°$

$V = 24\pi\angle 60°$

$v(t) = 24\pi\cos(120\pi + 60°)$

Example 9

$C = 150\mu F,\ I = 3.6\angle -145°,\ f = 60Hz$

Find the voltage across the inductor

$\omega = 2\pi f = 120\pi$

$$I = j\omega CV \Rightarrow V = \frac{I}{j\omega C}$$

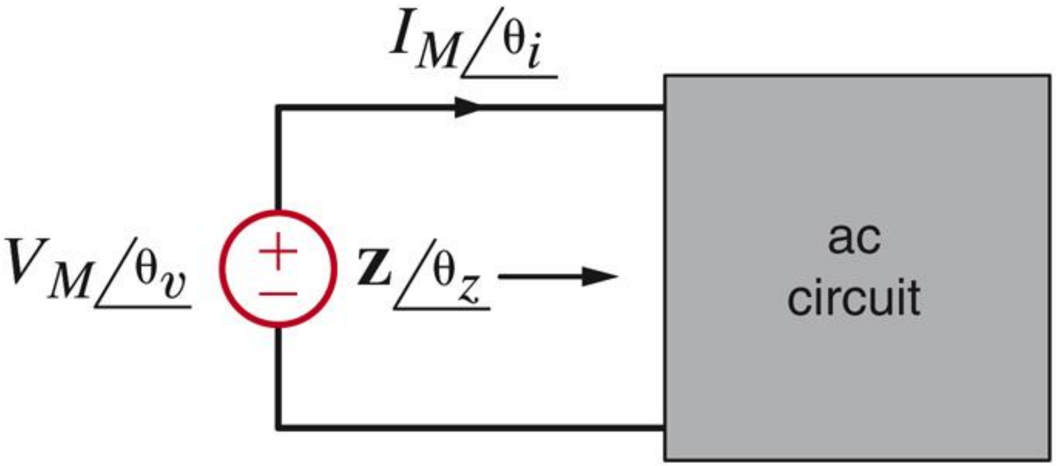
$$V = \frac{3.6\angle -145°}{120\pi \times 150 \times 10^{-6} \times 1\angle 90°}$$

$$V = \frac{200}{\pi}\angle -235°$$

$$v(t) = \frac{200}{\pi}\cos(120\pi t - 235°)$$

IMPEDANCE AND ADMITTANCE

For each of the passive components the relationship between the voltage phasor and the current phasor is algebraic. We now generalize for an arbitrary 2-terminal element



$$Z(\omega) = R(\omega) + jX(\omega)$$

$R(\omega)$=Resistive component

$X(\omega)$=Reactive component

$$|Z| = \sqrt{R^2 + X^2}$$

$$\theta_z = \tan^{-1}\frac{X}{R}$$

(INPUT) IMPEDANCE

$$Z = \frac{V}{I} = \frac{V_M\angle\theta_v}{I_M\angle\theta_i} = \frac{V_M}{I_M}\angle(\theta_v - \theta_i) = |Z|\angle\theta_z$$

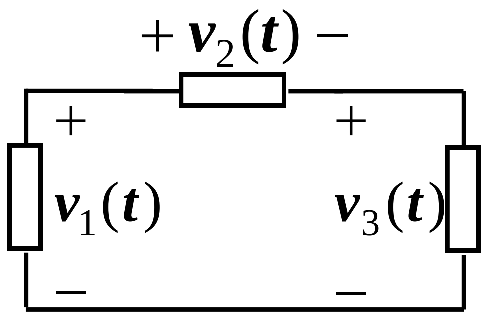
(DRIVING POINT IMPEDANCE)

The units of impedance are OHMS

Element	Phasor Eq.	Impedance
R	$V = RI$	$Z = R$
L	$V = j\omega LI$	$Z = j\omega L$
C	$V = \frac{1}{j\omega C}I$	$Z = \frac{1}{j\omega C}$

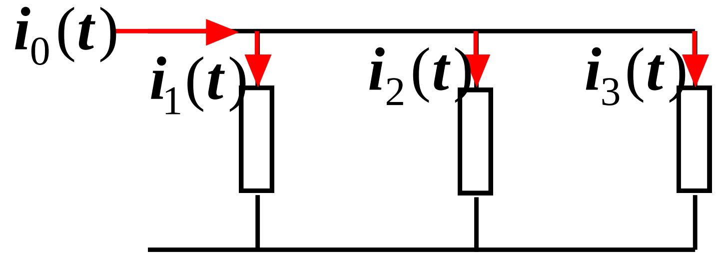
Impedance is NOT a phasor but a complex number that can be written in polar or Cartesian form. In general its value depends on the frequency

KVL AND KCL HOLD FOR PHASOR REPRESENTATIONS



KVL: $v_1(t)+v_2(t)+v_3(t)=0$

$$v_i(t)=V_{Mi}e^{j(\omega t+\theta_i)},\ i=1,2,3$$



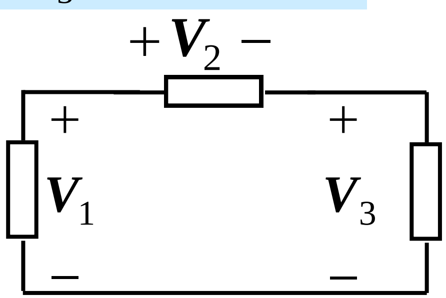
KCL: $-i_0(t)+i_1(t)+i_2(t)+i_3(t)=0$

$$i_k(t)=I_{Mk}e^{j(\omega t+\phi_k)},\ k=0,1,2,3$$

KVL: $(V_{M1}e^{j\theta_1}+V_{M2}e^{j\theta_2}+V_{M3}e^{j\theta_3})e^{j\omega t}=0$

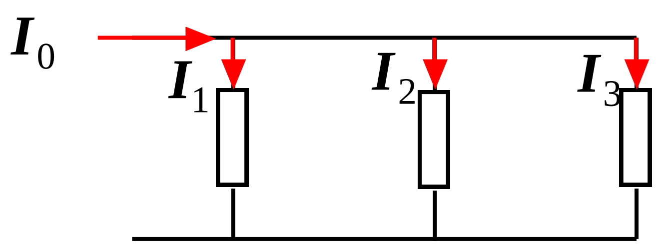
$$V_{M1}\angle\theta_1+V_{M2}\angle\theta_2+V_{M3}\angle\theta_3=0$$

$V_1+V_2+V_3=0$ Phasors!



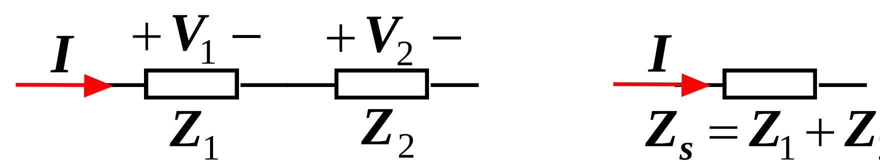
In a similar way, one shows ...

$$-I_0+I_1+I_2+I_3=0$$



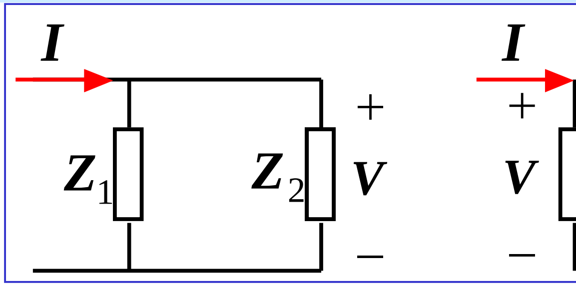
The components will be represented by their impedances and the relationships will be entirely algebraic!!

SPECIAL APPLICATION:
IMPEDANCES CAN BE COMBINED USING THE SAME RULES DEVELOPED FOR RESISTORS



$$Z_s = Z_1 + Z_2$$

$$Z_s = \sum_k Z_k$$



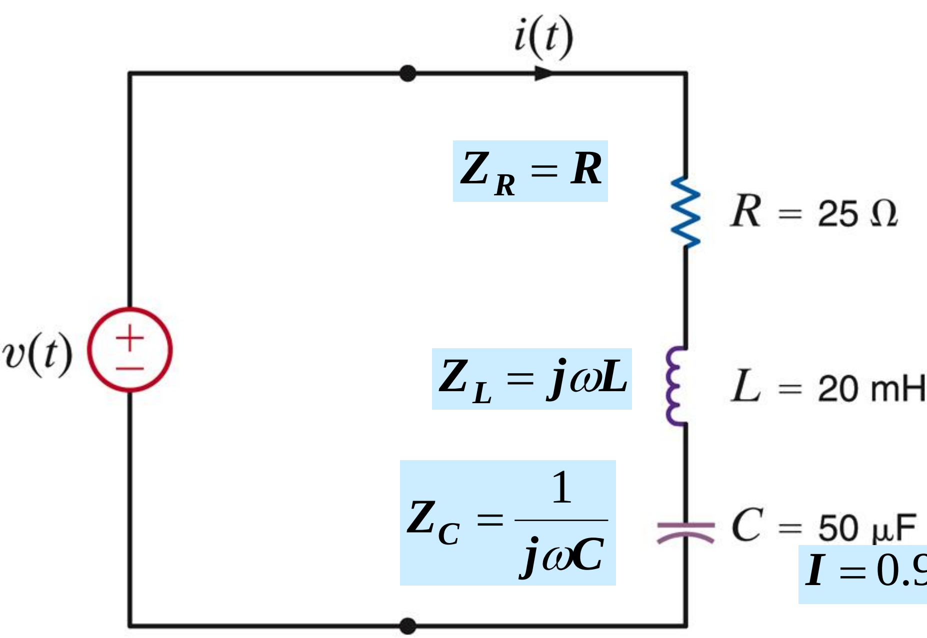
$$Z_p = \frac{Z_1 Z_2}{Z_1 + Z_2}$$

$$\frac{1}{Z_p} = \sum_k \frac{1}{Z_k}$$

EXAMPLE 1

$f = 60Hz, v(t) = 50\cos(\omega t + 30°)$

Compute equivalent impedance and current



$$Z_R = R$$

$$Z_L = j\omega L$$

$$Z_C = \frac{1}{j\omega C}$$

$$\omega = 120\pi, V = 50\angle 30°, \; Z_R = 25\Omega$$

$$Z_L = j120\pi \times 20 \times 10^{-3}\Omega, \; Z_C = \frac{1}{j120\pi \times 50 \times 10^{-6}}$$

$$Z_L = j7.54\Omega, \;\; Z_C = -j53.05\Omega$$

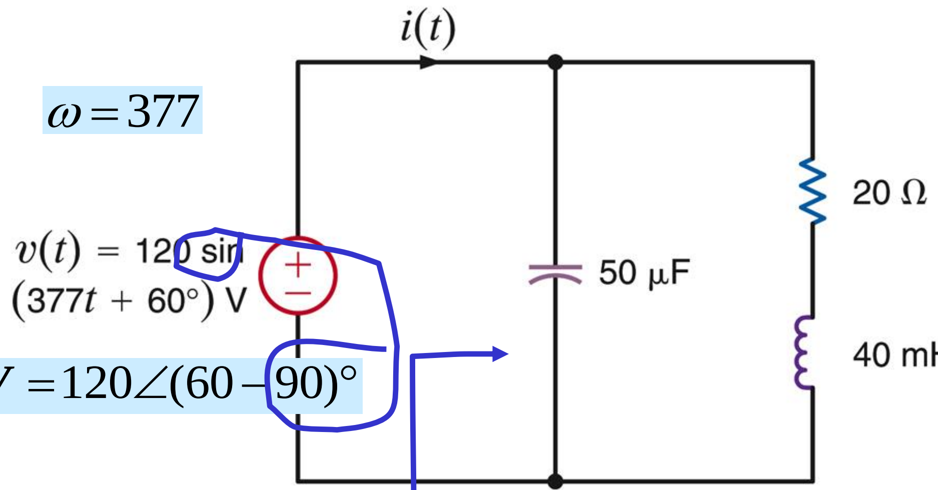
$$Z_s = Z_R + Z_L + Z_C = 25 - j45.51\Omega$$

$$I = \frac{V}{Z_s} = \frac{50\angle 30°}{25 - j45.51}(A) = \frac{50\angle 30°}{51.93\angle -61.22°}(A)$$

$$I = 0.96\angle 91.22°(A) \Rightarrow i(t) = 0.96\cos(120\pi t + 91.22°)(A)$$

EXAMPLE 2

FIND $\boldsymbol{i}(\boldsymbol{t})$



$\omega = 377$

$\boldsymbol{V} = 120\angle(60-90)°$

$\boldsymbol{Z_R} = 20\Omega$

$\boldsymbol{Z_L} = \boldsymbol{j}377 \times 40 \times 10^{-3} = \boldsymbol{j}15.08\Omega$

$$\boldsymbol{Z_C} = \frac{\boldsymbol{j}}{377 \times 50 \times 10^{-6}} = -\boldsymbol{j}53.05$$

$\boldsymbol{Z_{eq}} = \boldsymbol{Z_C} \,||\, (\boldsymbol{Z_R} + \boldsymbol{Z_L})$

$$\boldsymbol{Z_{eq} = -j53.05 \,||\, (20 + j15.08)}$$

$\boldsymbol{Z_{eq}} = 30.5616 + \boldsymbol{j}4.9714 = 30.963\angle 9.239°$

$$\boldsymbol{I} = \frac{\boldsymbol{V}}{\boldsymbol{Z_{eq}}} = \frac{120\angle -30°}{30.963\angle 9.239°} = 3.876\angle -39.924°(\boldsymbol{A})$$

(COMPLEX) ADMITTANCE

$$Y = \frac{1}{Z} = G + jB \text{ (Siemens)}$$

$G =$ conductance

$B =$ Suceptance

$$\frac{1}{Z} = \frac{1}{R + jX} \times \frac{R - jX}{R - jX} = \frac{R - jX}{R^2 + X^2}$$

$$G = \frac{R}{R^2 + X^2}$$

$$B = \frac{-X}{R^2 + X^2}$$

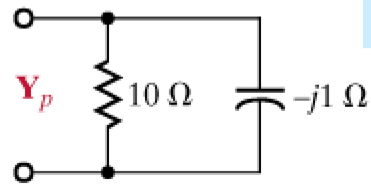
Element	Phasor Eq.	Impedance	Admittance
R	$V = RI$	$Z = R$	$Y = \frac{1}{R} = G$
L	$V = j\omega LI$	$Z = j\omega L$	$Y = \frac{1}{j\omega L}$
C	$V = \frac{1}{j\omega C} I$	$Z = \frac{1}{j\omega C}$	$Y = j\omega C$

Parallel Combination of Admittances

$$Y_p = \sum_k Y_k$$

$Y_R = 0.1S$

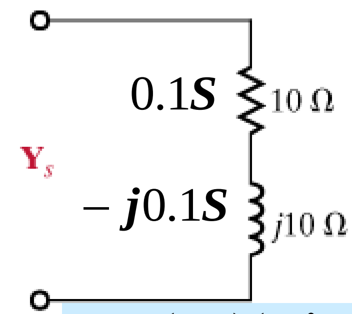
$$Y_C = \frac{1}{-j1} = j1(S)$$



$Y_p = 0.1 + j1(S)$

Series Combination of Admittances

$$\frac{1}{Y_s} = \sum_k \frac{1}{Y_k}$$



$$\frac{1}{Y_s} = \frac{1}{0.1} + \frac{1}{-j0.1} = 10 + j10$$

$$Y_s = \frac{(0.1)(-j0.1)}{0.1 - j0.1} \times \frac{0.1 + j0.1}{0.1 + j0.1}$$

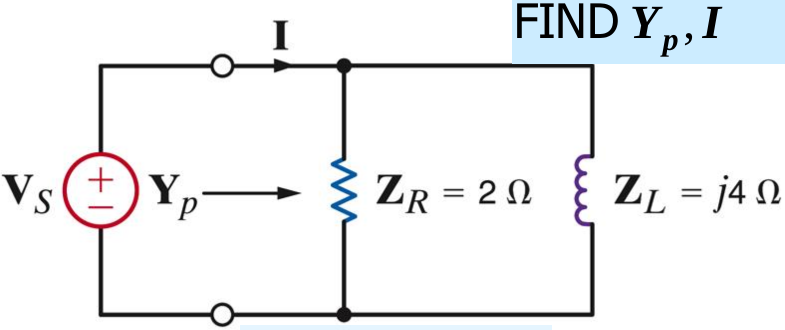
$$Y_s = \frac{1}{10 + j10} = \frac{10 - j10}{200}$$

$$Y_s = 0.05 - j0.05S$$

EXAMPLE 3

$$V_S = 60\angle 45°(V)$$

FIND Y_p, I



$$Y_p = Y_R + Y_L$$
$$= 0.5 - j0.25$$

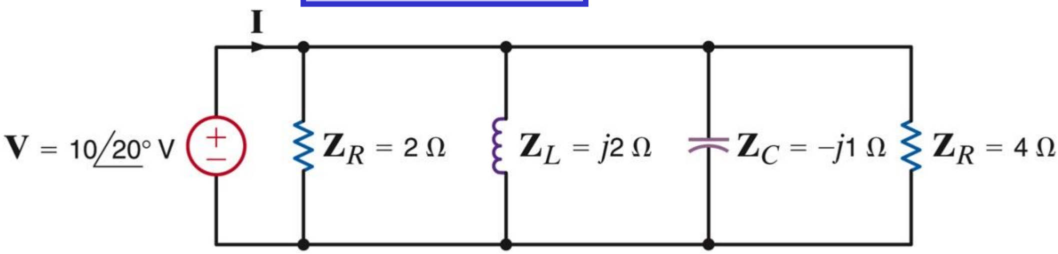
$$Z_p = \frac{2\times j4}{2+j4} \quad Y_p = \frac{2+j4}{j8} = 0.5 - j0.25(S)$$

$$I = Y_p V = (0.5 - j0.25)\times 60\angle 45°(A)$$

$$I = 0.559\angle -26.565° \times 60\angle 45°(A)$$

$$I = 33.54\angle 18.435°(A)$$

EXAMPLE 4



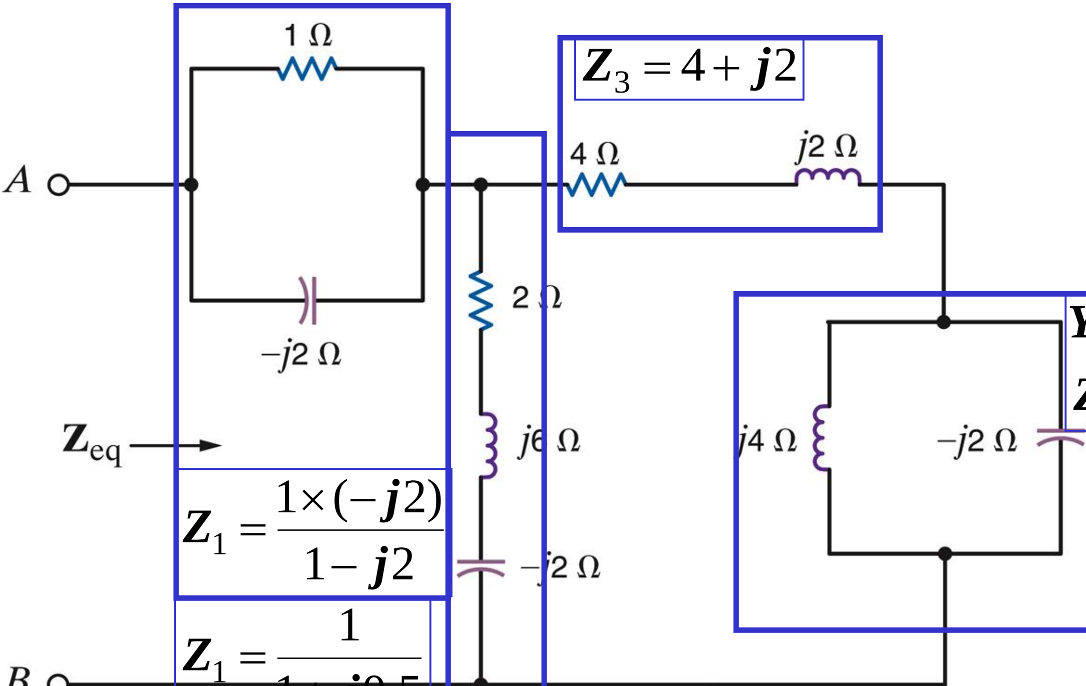
$$Y_p = 0.5 - j0.5 + j1 + 0.25 = 0.75 + j0.5(S)$$

$$Y_p = 0.9014\angle 33.69°(S)$$

$$I = Y_p V = 0.9014\angle 33.69° \times 10\angle 20°$$
$$I = 9.014\angle 53.79°(A)$$

EXAMPLE 5

SERIES-PARALLEL REDUCTIONS



$$\boldsymbol{Z}_3 = 4 + \boldsymbol{j}2$$

$$\boldsymbol{Y}_2 = \frac{1}{2+\boldsymbol{j}4} = \frac{2-\boldsymbol{j}4}{(2)^2+(4)^2}$$

$$\boldsymbol{Y}_{34} = \frac{1}{4-\boldsymbol{j}2} = \frac{4+\boldsymbol{j}2}{20}$$

$$\boldsymbol{Y}_4 = -\boldsymbol{j}0.25 + \boldsymbol{j}0.5 = \boldsymbol{j}0.25$$

$$\boldsymbol{Z}_4 = 1/\boldsymbol{Y}_4 = -\boldsymbol{j}4$$

$$\boldsymbol{Z}_4 = \frac{\boldsymbol{j}4\times(-\boldsymbol{j}2)}{\boldsymbol{j}4-\boldsymbol{j}2} = \frac{8}{\boldsymbol{j}2}$$

$$\boldsymbol{Z}_1 = \frac{1\times(-\boldsymbol{j}2)}{1-\boldsymbol{j}2}$$

$$\boldsymbol{Z}_1 = \frac{1}{1+\boldsymbol{j}0.5}$$

$$\boldsymbol{Z}_1 = \frac{1-\boldsymbol{j}0.5}{1+(0.5)^2}$$

$$\boldsymbol{Z}_1 = 0.8 - \boldsymbol{j}0.4(\Omega)$$

$$\boldsymbol{Y}_2 = 0.1 - \boldsymbol{j}0.2(\boldsymbol{S})$$

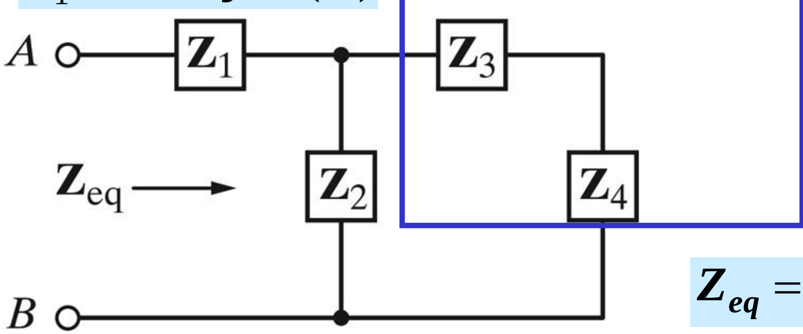
$$\boldsymbol{Y}_{34} = 0.2 + \boldsymbol{j}0.1$$

$$\boldsymbol{Y}_{234} = 0.3 - \boldsymbol{j}0.1(\boldsymbol{S})$$

$$\boldsymbol{Z}_2 = 2 + \boldsymbol{j}6 - \boldsymbol{j}2 = 2 + \boldsymbol{j}4$$

$$\boldsymbol{Z}_{34} = 4 - \boldsymbol{j}2$$

$$\boldsymbol{Z}_{234} = \frac{1}{\boldsymbol{Y}_{234}} = \frac{1}{0.3-\boldsymbol{j}0.1} = \frac{0.3+\boldsymbol{j}0.1}{0.1}$$

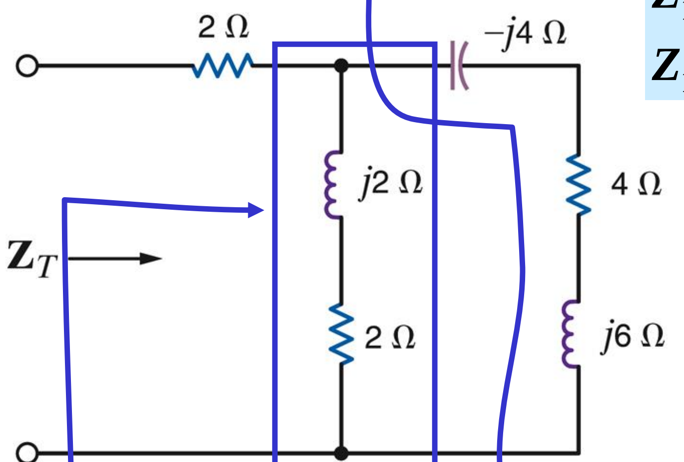


$$\boldsymbol{Z}_{234} = \frac{\boldsymbol{Z}_2\boldsymbol{Z}_{34}}{\boldsymbol{Z}_2+\boldsymbol{Z}_{34}} = 3 + \boldsymbol{j}1$$

$$\boldsymbol{Z}_{eq} = \boldsymbol{Z}_1 + \boldsymbol{Z}_{234} = 3.8 + \boldsymbol{j}0.6\Omega = 3.847\angle 8.973°$$

EXAMPLE 6

FIND THE IMPEDANCE Z_T



$$Z_1 = 4 + j6 - j4$$
$$Z_1 = 4 + j2$$

$$(R \to P) Z_1 = 4.472\angle 26.565°$$
$$Y_1 = 0.224\angle -26.565°$$
$$(P \to R) Y_1 = 0.200 - j0.100$$

$$Z_2 = 2 + j2$$
$$(R \to P) Z_2 = 2.828\angle 45°$$
$$Y_2 = 0.354\angle -45°$$
$$(P \to R) Y_2 = 0.250 - j0.250$$

$$Y_{12} = Y_1 + Y_2$$
$$Z_{12} = \frac{1}{Y_{12}}$$

$$Y_1 = \frac{1}{4 + j2} = \frac{4 - j2}{(4)^2 + (2)^2}$$
$$Y_2 = \frac{1}{2 + j2} = \frac{2 - j2}{(2)^2 + (2)^2}$$
$$Z_{12} = \frac{1}{Y_{12}} = \frac{1}{0.45 - j0.35} = \frac{0.45 + j0.35}{0.325}$$

$$Y_{12} = Y_1 + Y_2 = 0.45 - j0.35$$
$$(R \to P) Y_{12} = 0.570\angle -37.875°$$
$$Z_{12} = 1.754\angle 37.875°$$
$$(P \to R) Z_{12} = 1.384 + j1.077$$

$$Z_T = 2 + (1.384 + j1077) = 3.383 + j1.077$$

PHASOR DIAGRAMS

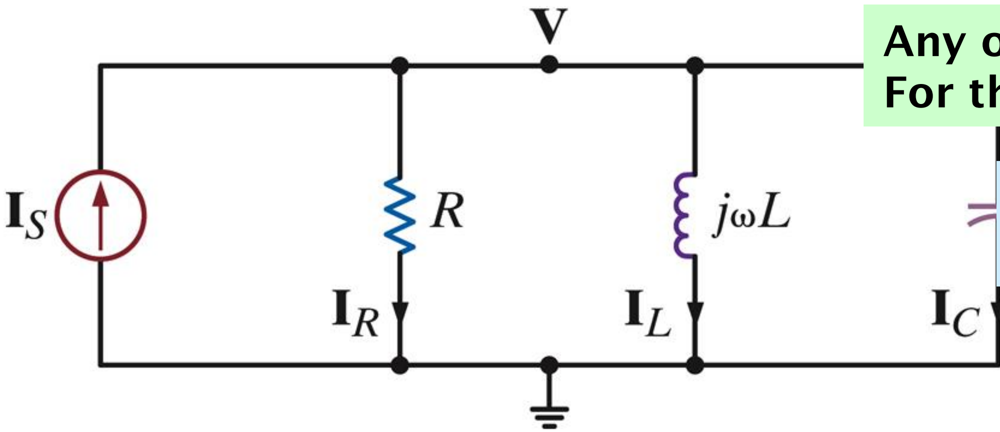
Display all relevant phasors on a common reference frame

Very useful to visualize phase relationships among variables.
Especially if some variable, like the frequency, can change

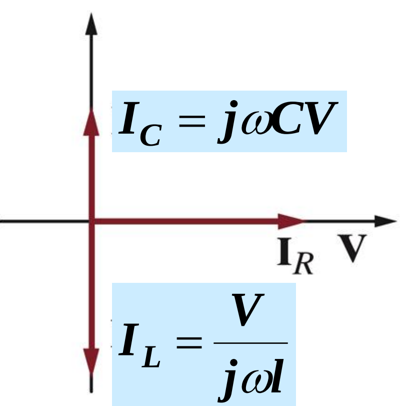
EXAMPLE 1

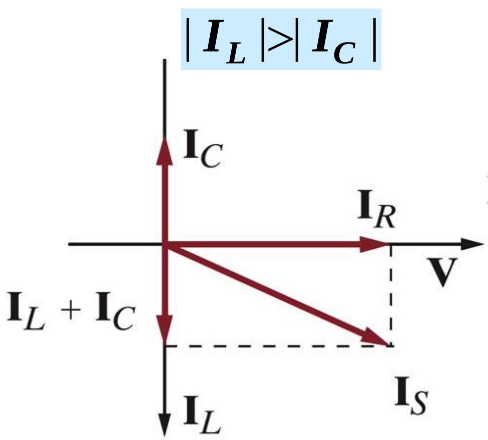
SKETCH THE PHASOR DIAGRAM FOR THE CIRCUIT

Any one variable can be chosen as reference.
For this case select the voltage V

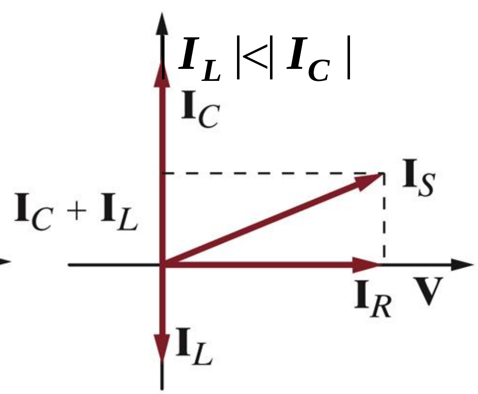


KCL: $I_S = \frac{V}{R} + \frac{V}{j\omega L} + j\omega CV$

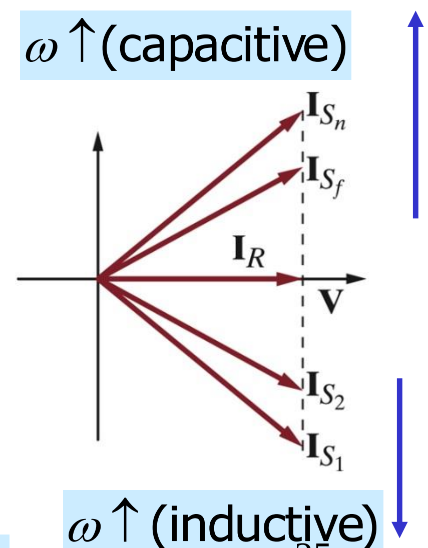




INDUCTIVE CASE



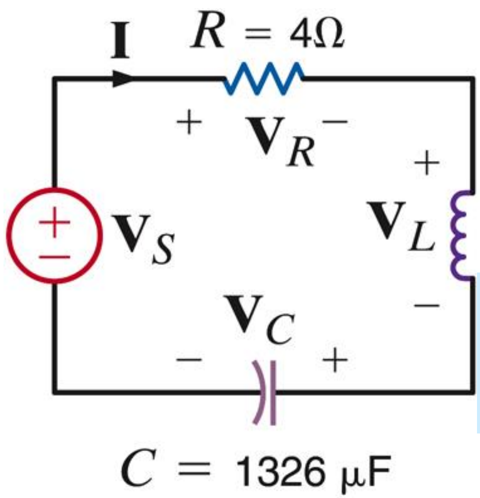
CAPACITIVE CASE



EXAMPLE 2

DO THE PHASOR DIAGRAM FOR THE CIRCUIT

$\omega = 377(s^{-1})$



$Z_L = j377x15.92x10^{-3}$

$Z_L = j6\Omega = 6\angle 90^0\,\Omega$

$V_R = RI$

$V_L = j\omega L I$

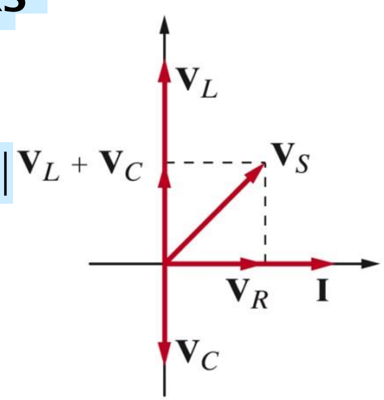
$V_C = \frac{1}{j\omega C} I$

$V_S = V_R + V_L + V_C$

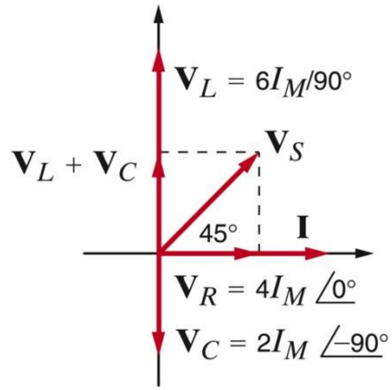
It is convenient to select the current as reference

1. DRAW ALL THE PHASORS

$|V_L| > |V_C|$



2. PUT KNOWN NUMERICAL VALUES



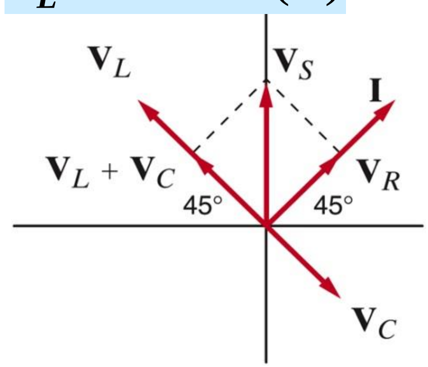
$|V_L - V_C| = |V_R|$

$$Z_C = \frac{1}{j377x1326x10^{-6}}$$

$Z_C = -j2 = 2\angle -90^0\,\Omega$

DIAGRAM WITH REFERENCE $V_S = 12\sqrt{2}\angle 90°$

$V_L = 18\angle 135°(V)$



Read values from diagram!

$\therefore I = 3\angle 45°(A)$

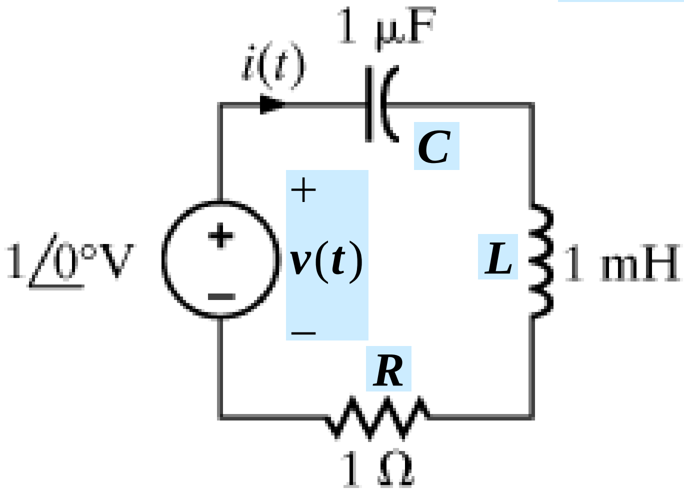
$V_R = 12\angle 45°(V)$

(Pythagoras)

$V_C = 6\angle -45°$

EXAMPLE 3

FIND THE FREQUENCY AT WHICH $v(t)$ AND $i(t)$ ARE IN PHASE

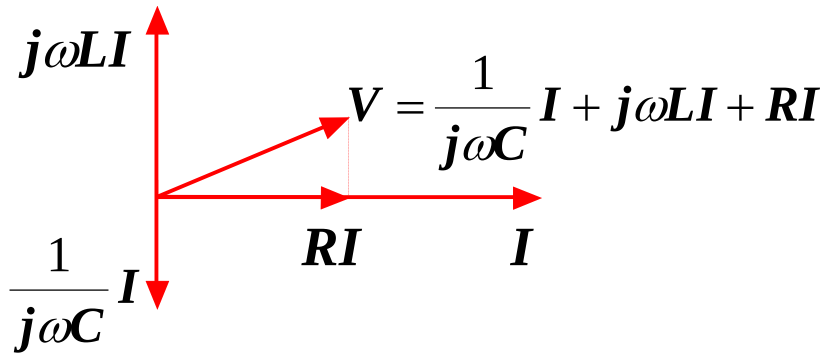


i.e., the phasors for $i(t), v(t)$ are co-linear

$$V = \frac{1}{j\omega C} I + j\omega L I + RI$$

Notice that I was chosen as reference

PHASOR DIAGRAM



V **and** I **are co-linear iff** $j\omega L + \frac{1}{j\omega C} = 0$

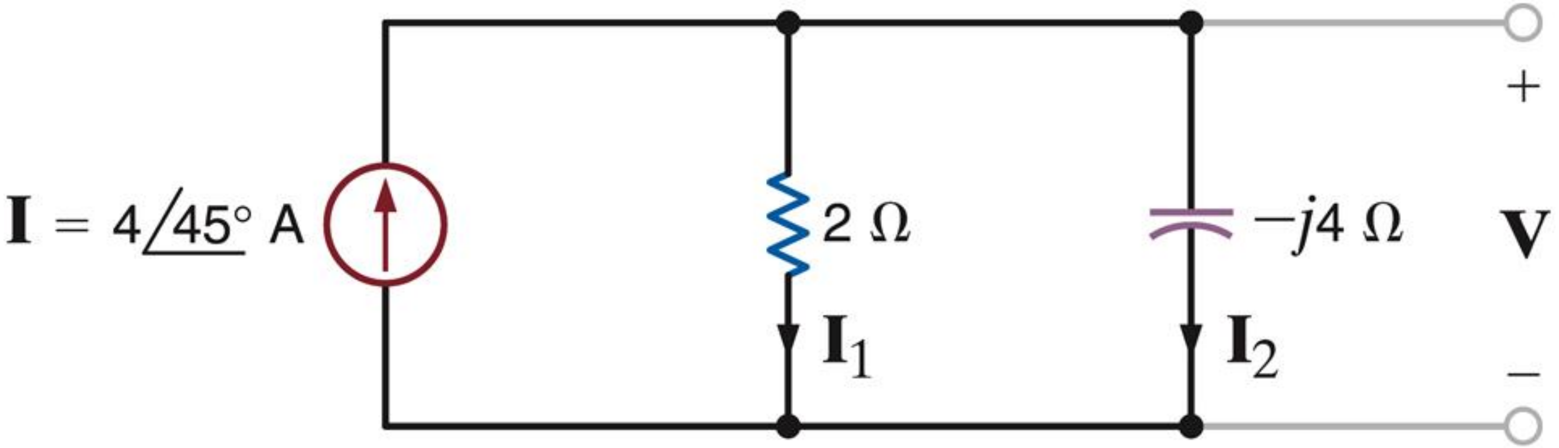
$$\Rightarrow \omega^2 = \frac{1}{LC}$$

$$\omega^2 = \frac{1}{10^{-3} \times 10^{-6}} = 10^9 \Rightarrow \omega = 3.162 \times 10^4 (rad/s)$$

$$f = \frac{\omega}{2\pi} = 5.033 \times 10^3 \, Hz$$

EXAMPLE 4

Draw a phasor diagram illustrating all voltages and currents



$$I_1 = \frac{-j4}{2-j4} I = \frac{4\angle -90^\circ}{4.472\angle -63.435^\circ} 4\angle 45^\circ$$

Current divider

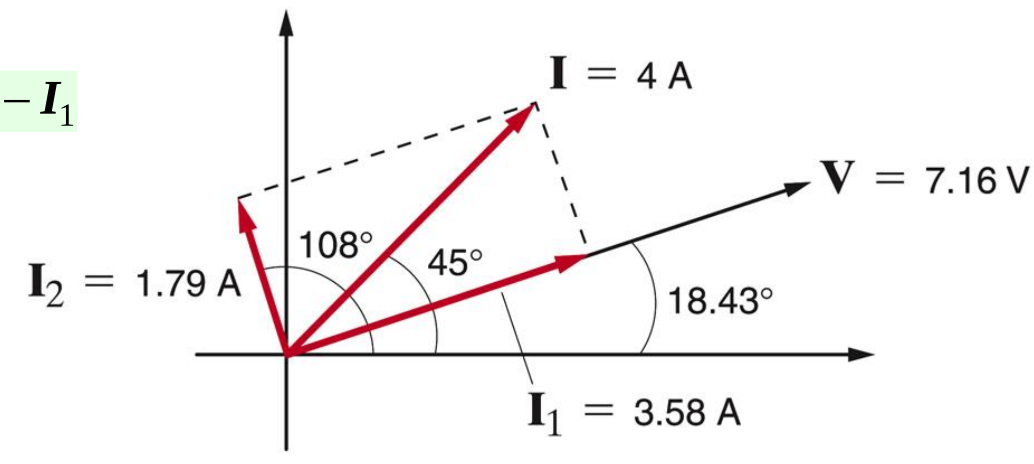
$$I_1 = 3.578\angle 18.435^\circ (A)$$

$$I_2 = \frac{2}{2-j4} I = \frac{2\angle 0^\circ}{4.472\angle -63.435^\circ} 4\angle 45^\circ$$

$I_2 = 1.789\angle 108.435^\circ$ Simpler than $I_2 = I - I_1$

$$V = 2I_1 = 7.156\angle 18.435^\circ (V)$$

DRAW PHASORS. ALL ARE KNOWN. NO NEED TO SELECT A REFERENCE



BASIC ANALYSIS USING KIRCHHOFF'S LAWS

PROBLEM SOLVING STRATEGY

For relatively simple circuits use

Ohm's law for AC analysis; i.e., $V = IZ$

The rules for combining Z and Y

KCL and KVL

Current and voltage divider

For more complex circuits use

Node analysis

Loop analysis

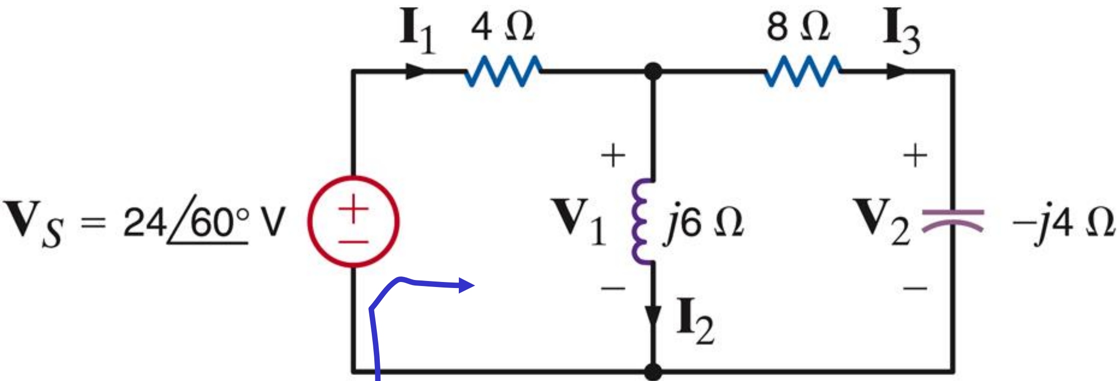
Superposition

Source transformation

Thevenin's and Norton's theorems

EXAMPLE 1

COMPUTE ALL THE VOLTAGES AND CURRENTS



Compute I_1

Use current divider for I_2, I_3

Ohm's law for V_1, V_2

$$Z_{eq} = 4 + (j6 \parallel 8 - j4)$$

$$V_1 = 6\angle 90° I_2 \qquad V_2 = 4\angle -90° I_3$$

$$V_1 = 16.26\angle 78.42°(V)$$

$$V_2 = 7.28\angle 15°(V)$$

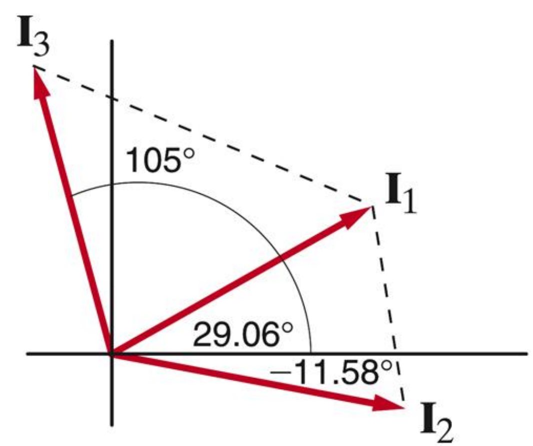
$$Z_{eq} = 4 + \frac{24 + j48}{8 + j2} = \frac{32 + j8 + 24 + j48}{8 + j2}$$

$$Z_{eq} = \frac{56 + j56}{8 + j2} = \frac{79.196\angle 45°}{8.246\angle 14.036°} = 9.604\angle 30.964°(\Omega)$$

$$I_1 = \frac{V_S}{Z_{eq}} = \frac{24\angle 60°}{9.604\angle 30.964°} = 2.498\angle 29.036°(A)$$

$$I_3 = \frac{j6}{8 + j2} I_1 = \frac{6\angle 90°}{8.246\angle 14.036} 2.498\angle 29.036(A)$$

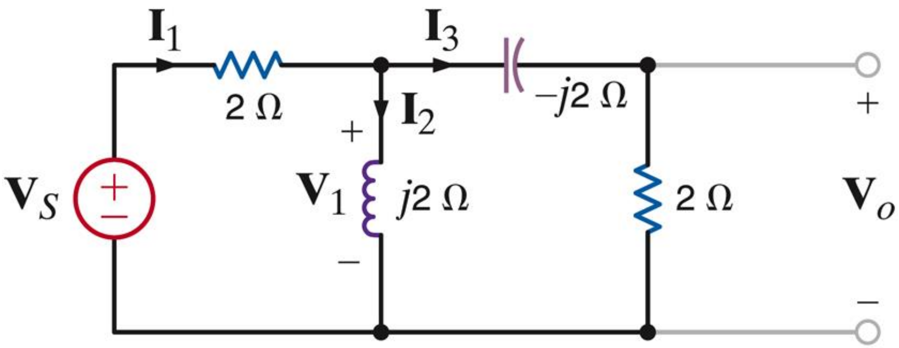
$$I_2 = \frac{8 - j4}{8 + j2} I_1 = \frac{8.944\angle -26.565°}{8.246\angle 14.036°} 2.498\angle 29.036°(A)$$



$$I_1 = 2.5\angle 29.06° \qquad I_2 = 2.71\angle -11.58° \qquad I_3 = 1.82\angle 105°$$

EXAMPLE 2

IF $V_O = 8\angle 45°$, COMPUTE V_S



THE PLAN...

COMPUTE I_3

COMPUTE V_1

COMPUTE I_2, I_1

COMPUTE V_S

$$\boldsymbol{I}_3 = \frac{\boldsymbol{V}_O}{2}(\boldsymbol{A}) = 4\angle 45°(\boldsymbol{A})$$

$$\boldsymbol{V}_1 = (2 - \boldsymbol{j}2)\boldsymbol{I}_3 = \sqrt{8}\angle -45° \times 4\angle 45°$$

$$\boldsymbol{V}_1 = 11.314\angle 0°(\boldsymbol{V})$$

$$\boldsymbol{I}_2 = \frac{\boldsymbol{V}_1}{\boldsymbol{j}2} = \frac{11.314\angle 0°}{2\angle 90°} = 5.657\angle -90°(\boldsymbol{A})$$

$$\boldsymbol{I}_1 = \boldsymbol{I}_2 + \boldsymbol{I}_3 = 5.657\angle -90° + 4\angle 45°$$

$$\boldsymbol{I}_1 = -\boldsymbol{j}5.657 + (2.828 + \boldsymbol{j}2.828)(\boldsymbol{A})$$

$$\boldsymbol{I}_1 = 2.828 - \boldsymbol{j}2.829(\boldsymbol{A})$$

$$\boldsymbol{V}_S = 2\boldsymbol{I}_1 + \boldsymbol{V}_1 = 2(2.828 - \boldsymbol{j}2.829) + 11.314\angle 0°$$

$$\boldsymbol{V}_S = 16.97 - \boldsymbol{j}5.658(\boldsymbol{V})$$

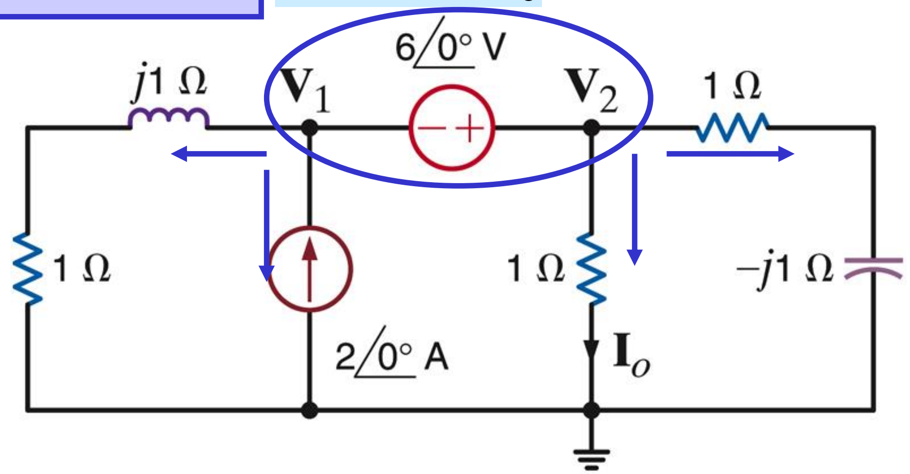
$$\boldsymbol{V}_S = 17.888\angle -18.439°$$

ANALYSIS TECHNIQUES

PURPOSE: TO REVIEW ALL CIRCUIT ANALYSIS TOOLS DEVELOPED FOR RESISTIVE CIRCUITS; I.E., NODE AND LOOP ANALYSIS, SOURCE SUPERPOSITION, SOURCE TRANSFORMATION, THEVENIN'S AND NORTON'S THEOREMS.

EXAMPLE 3

COMPUTE I_0



1. NODE ANALYSIS

$$\frac{V_1}{1+j1} - 2\angle 0° + \frac{V_2}{1} + \frac{V_2}{1-j1} = 0$$

$$V_1 - V_2 = -6\angle 0°$$

$$I_0 = \frac{V_2}{1}(A)$$

$$\frac{V_2 - 6\angle 0°}{1+j1} - 2\angle 0° + V_2 + \frac{V_2}{1-j1} = 0$$

$$V_2\left[\frac{1}{1+j1} + 1 + \frac{1}{1-j1}\right] = 2 + \frac{6}{1+j1}$$

$$V_2 \frac{(1-j1)+(1+j1)(1-j1)+(1+j1)}{(1+j1)(1-j1)} = \frac{2(1+j1)+6}{1+j1}$$

$$V_2 \frac{4}{1-j} = 8 + j2$$

$$V_2 = \frac{(4+j)(1-j)}{2} = \frac{5-j3}{2}$$

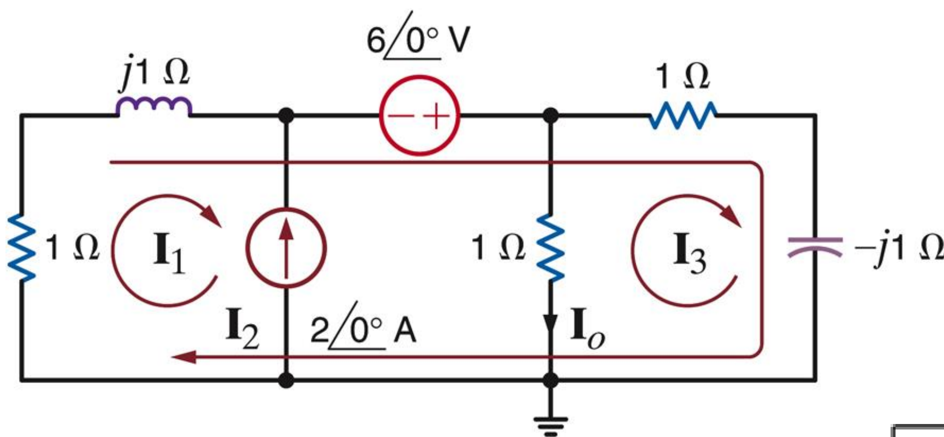
$$I_0 = \left(\frac{5}{2} - j\frac{3}{2}\right)(A)$$

$$I_0 = 2.92\angle -30.96°$$

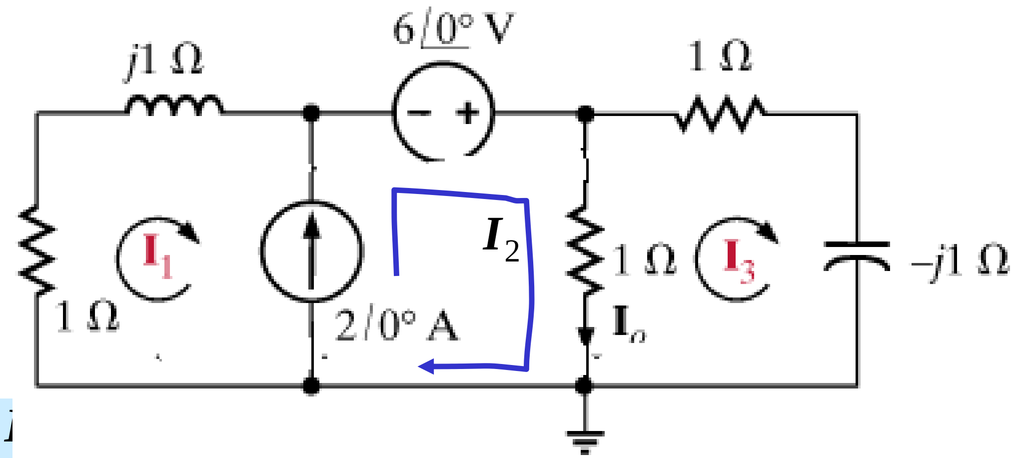
NEXT: LOOP ANALYSIS

EXAMPLE 3

LOOP ANALYSIS



ONE COULD ALSO USE THE SUPERMESH TECHNIQUE



SOURCE IS NOT SHARED AND Io IS DEFINED BY ONE LOOP CURRENT

$I_0 = -I_3$

LOOP 1: $I_1 = -2\angle 0°$

LOOP 2: $(1+j)(I_1+I_2) - 6\angle 0° + (1-j)(I_2 + I$

LOOP 3: $(1-j)(I_2+I_3) + I_3 = 0$

MUST FIND I_3

$2I_2 + (1-j)I_3 = 6 - (1+j)(-2)$ $\quad$ /* $(1-j)$

$(1-j)I_2 + (2-j)I_3 = 0$ $\quad$ /* (-2)

$$\left((1-j)^2 - 2(2-j)\right)I_3 = (1-j)(8+2j)$$

$$I_3 = \frac{10-6j}{-4}$$

$$I_0 = -I_3 = -\left(-\frac{5}{2} + \frac{3}{2}j\right) = \frac{5}{2} - \frac{3}{2}j\ A$$

CONSTRAINT: $I_1 - I_2 = -2\angle 0°$

SUPERMESH: $(1+j)I_1 + 6\angle 0° + (I_2 - I_3) = 0$

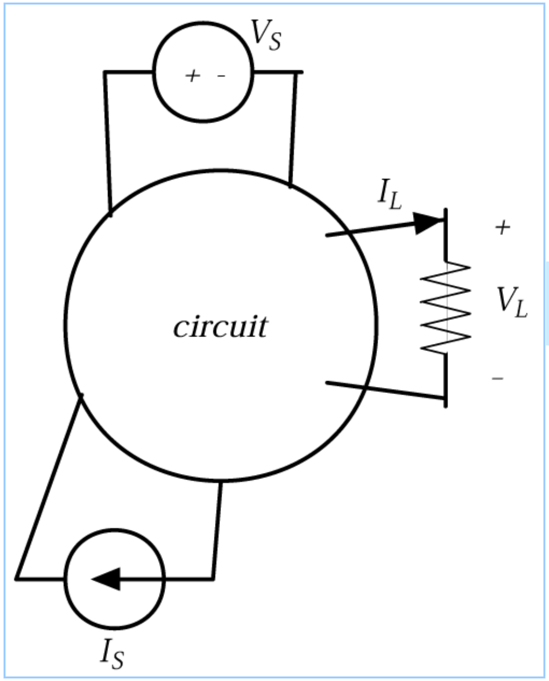
MESH 3: $(I_3 - I_2) + (1-j)I_3 = 0$

$I_0 = I_2 - I_3$

NEXT: SOURCE SUPERPOSITION

SOURCE SUPERPOSITION

Circuit with voltage source set to zero (SHORT CIRCUITED)



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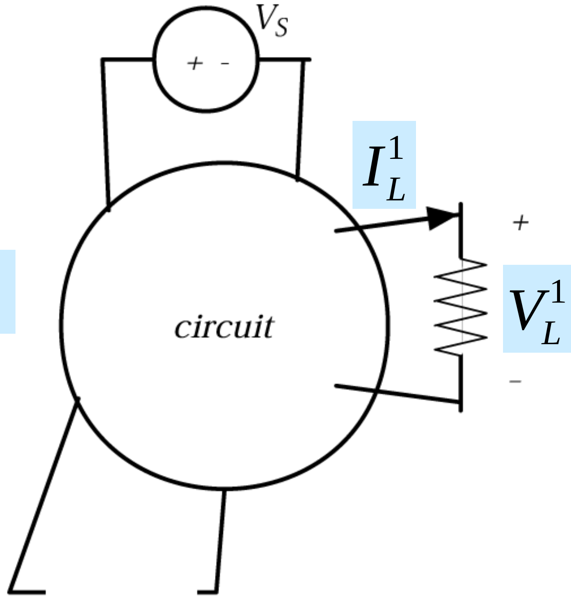

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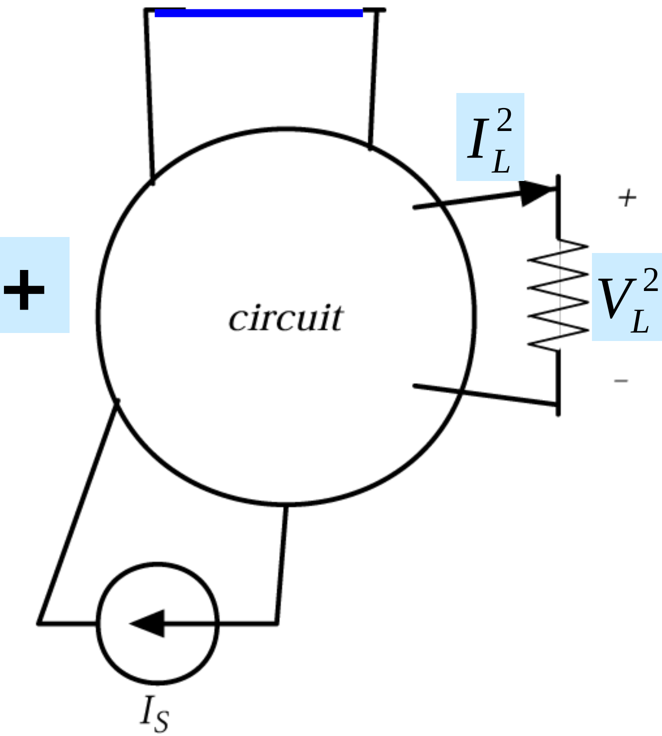


Circuit with current source set to zero(OPEN)

Due to the linearity of the models we must have

$$I_L = I_L^1 + I_L^2 \qquad V_L = V_L^1 + V_L^2$$

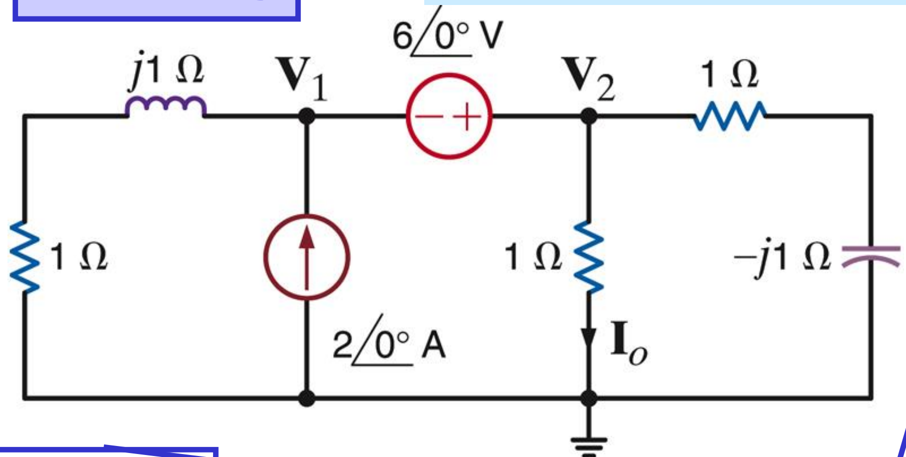
Principle of Source Superposition

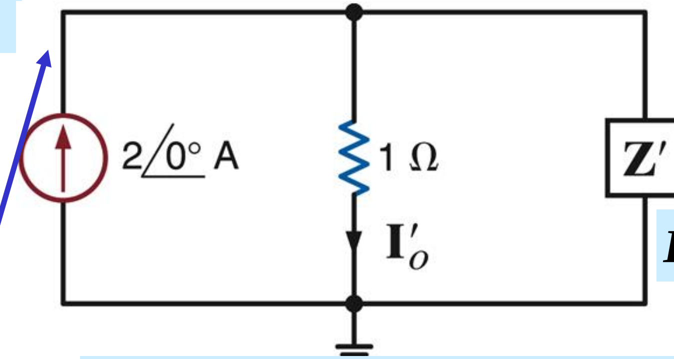
The approach will be useful if solving the two circuits is simpler, or more convenient, than solving a circuit with two sources

We can have any combination of sources. And we can partition any way we find convenient

EXAMPLE 3

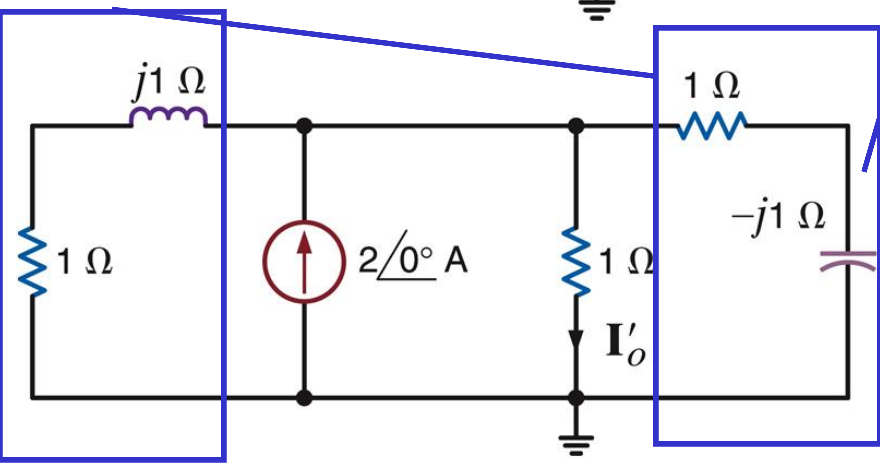
SOURCE SUPERPOSITION





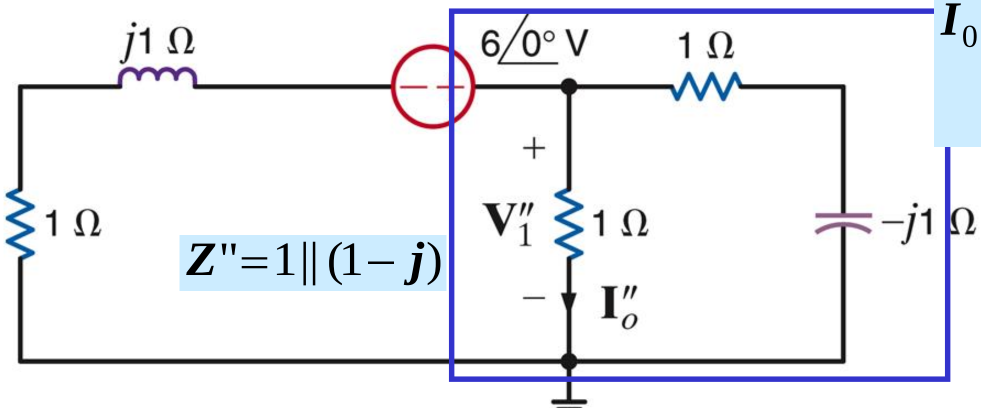
$I_0^{'} = 1\angle 0°(A)$

$$Z' = (1+j)\,\|\,(1-j) = \frac{(1+j)(1-j)}{(1+j)-(1-j)} = 1$$



COULD USE SOURCE TRANSFORMATION TO COMPUTE $I_0^{''}$

$$V_1^{''} = \frac{Z^{''}}{Z^{''}+1+j}6\angle 0°(V) \qquad I_0^{''} = \frac{Z^{''}}{Z^{''}+1+j}6\angle 0°(A)$$



$Z'' = 1\,\|\,(1-j)$

$$I_0^{''} = \frac{\dfrac{1-j}{2-j}}{\dfrac{1-j}{2-j}+1+j}6(A)$$

$$I_0^{''} = \frac{1-j}{(1-j)+3+j}6$$

$$I_0^{''} = \frac{6}{4} - \frac{6}{4}j(A)$$

$$I_0 = I_0^{'} + I_0^{''} = \left(\frac{5}{2} - \frac{3}{2}j\right)(A)$$

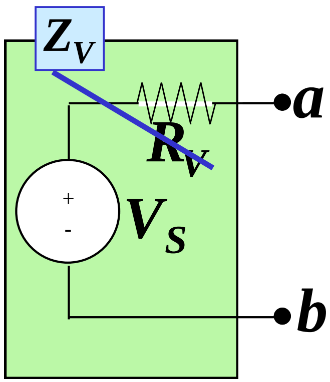
NEXT: SOURCE TRANSFORMATION

Source transformation is a good tool to reduce complexity in a circuit ...

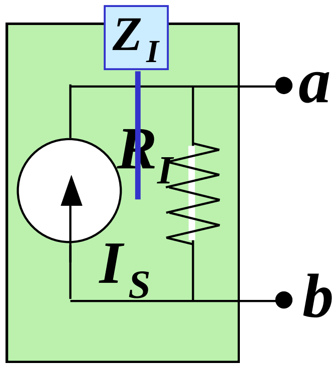
WHEN IT CAN BE APPLIED!!

"ideal sources" are not good models for real behavior of sources

A real battery does not produce infinite current when short-circuited



Improved model
for voltage source



Improved model
for current source

THE MODELS ARE EQUIVALENTS WHEN

$R_V = R_I = R$

$V_S = RI_S$

$Z_V = Z_I = Z$

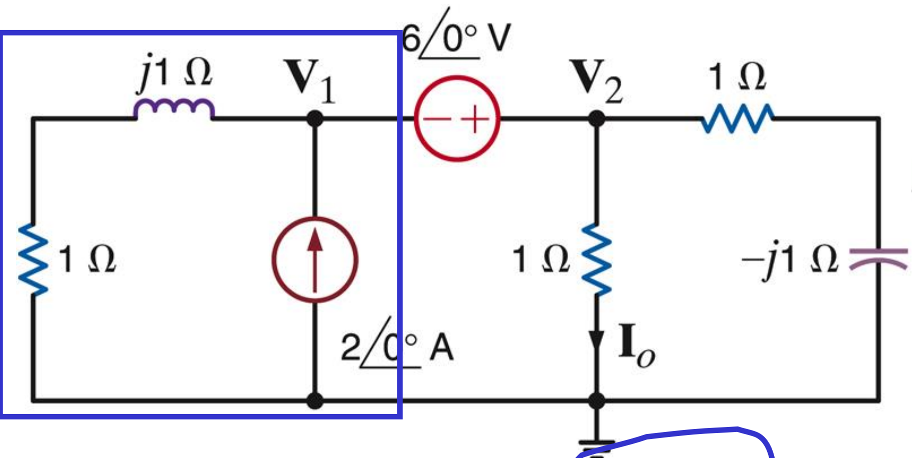
$V_S = ZI_S$

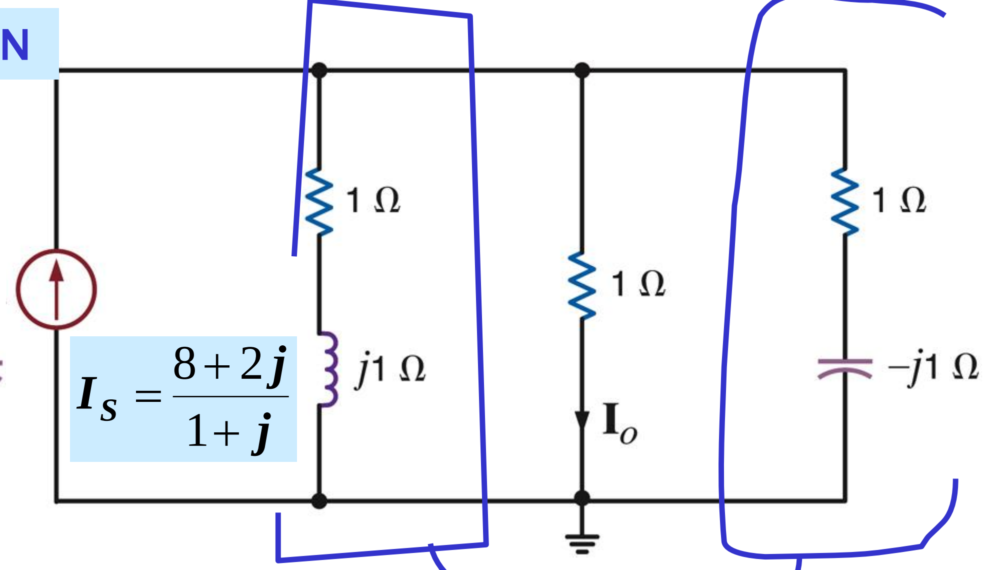
Source Transformationcan be used to determine the Thevenin or Norton Equivalent...

BUT THERE MAY BE MORE EFFICIENT TECHNIQUES

EXAMPLE 3

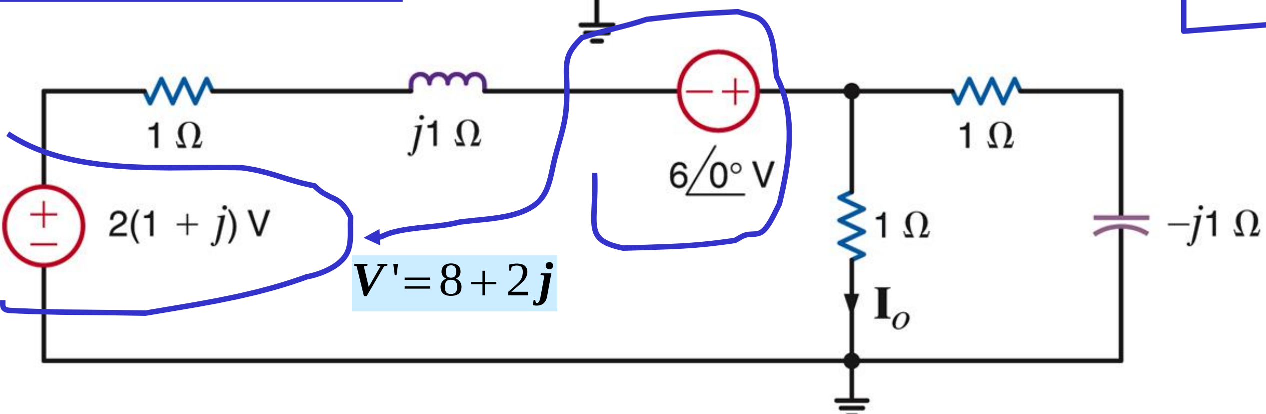
SOURCE TRANSFORMATION





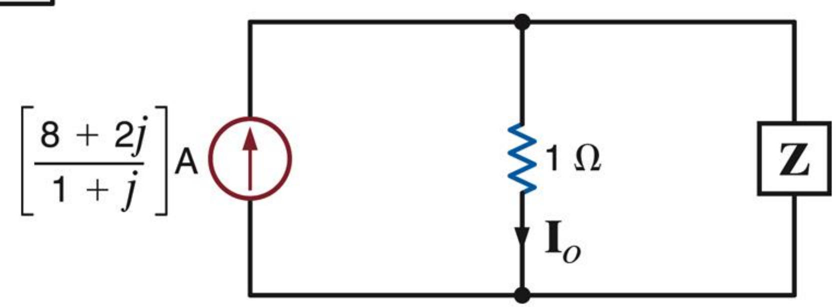
$$I_S = \frac{8+2j}{1+j}$$

$$Z = (1+j)\,\|\,(1-j) = 1\Omega$$



$$V' = 8+2j$$

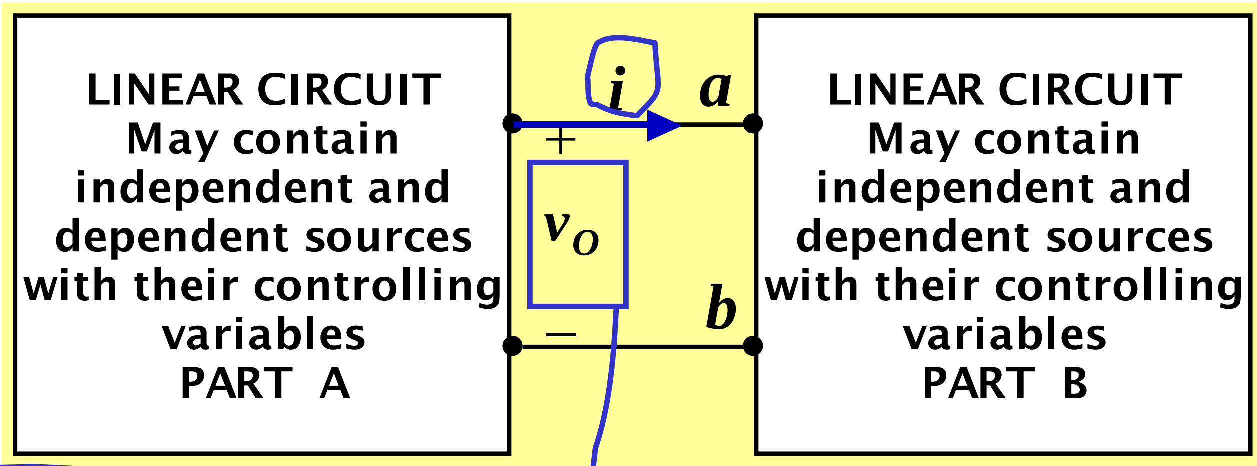
Now a voltage to current transformation

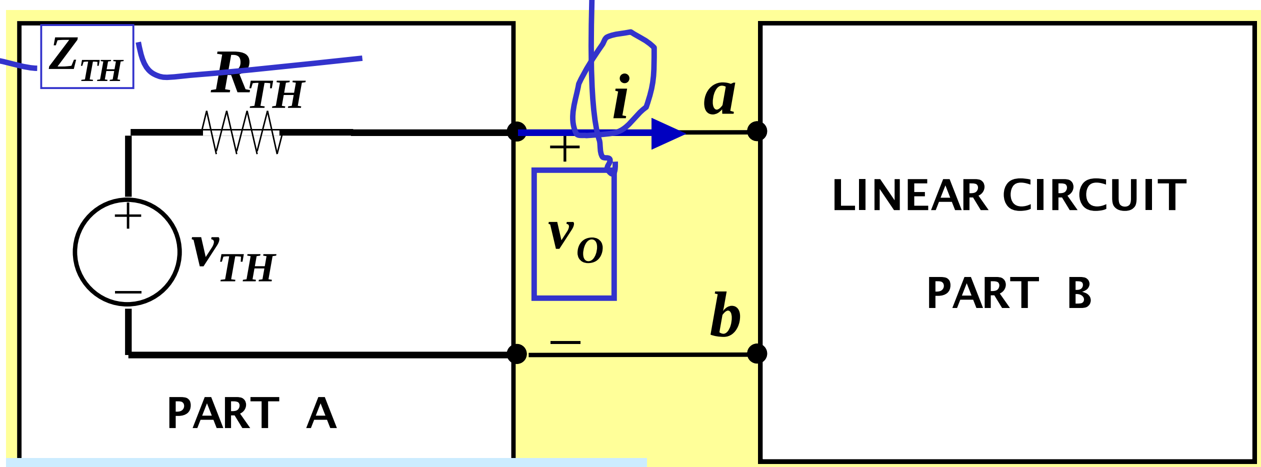


NEXT: THEVENIN

$$I_0 = \frac{I_S}{2} = \frac{4+j}{1+j} = \frac{(4+j)(1-j)}{(1+j)(1-j)} = \frac{5-3j}{2}$$

THEVENIN'S EQUIVALENCE THEOREM





Thevenin Equivalent Circuit

for PART A

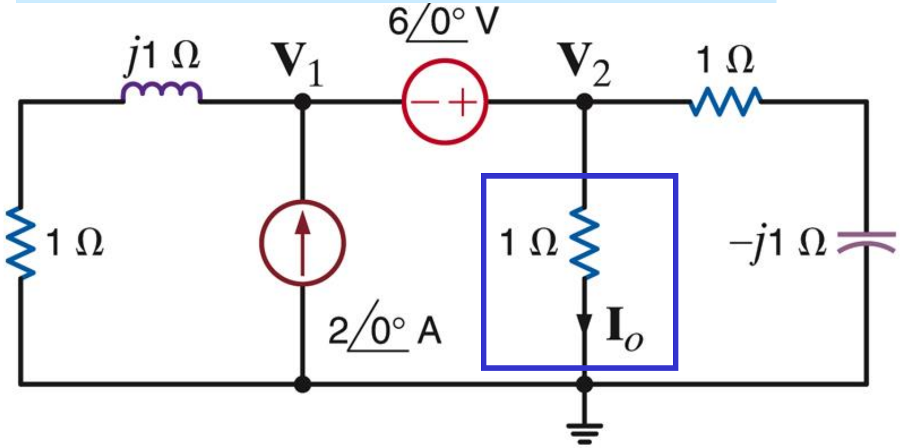
Phasor

v_{TH} Thevenin Equivalent Source

R_{TH} Thevenin Equivalent Resistance

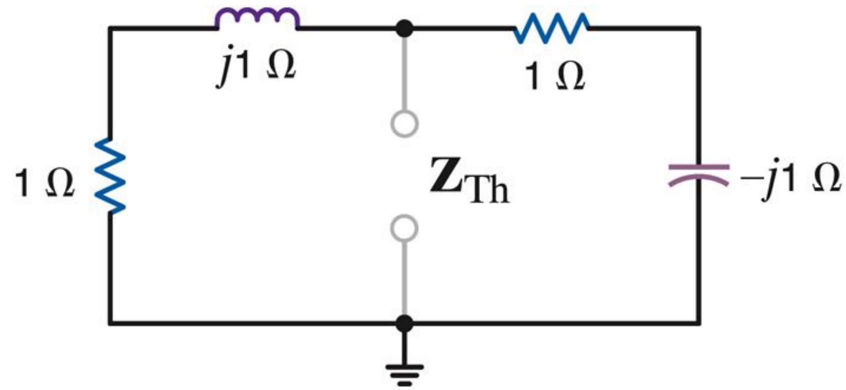
Impedance

EXAMPLE 3: THEVENIN ANALYSIS

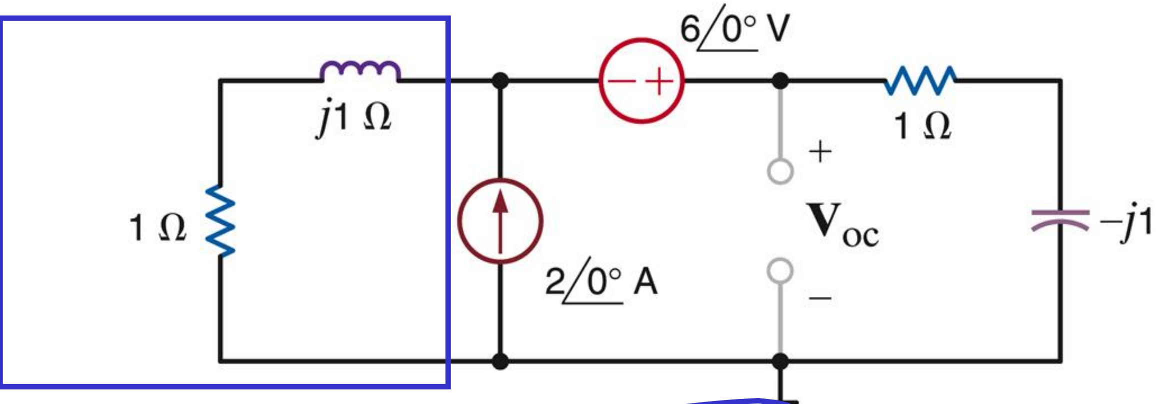


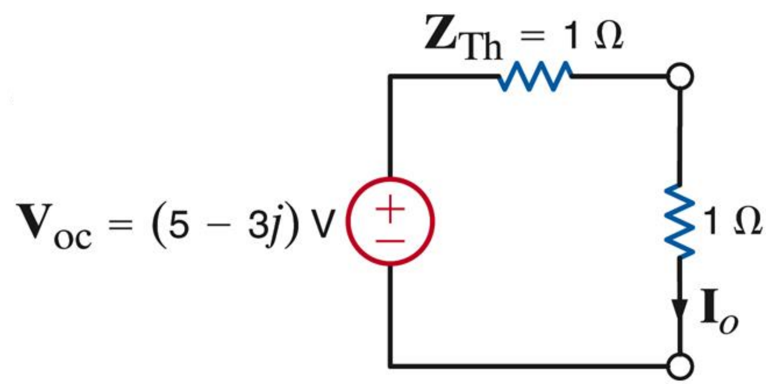
Voltage Divider

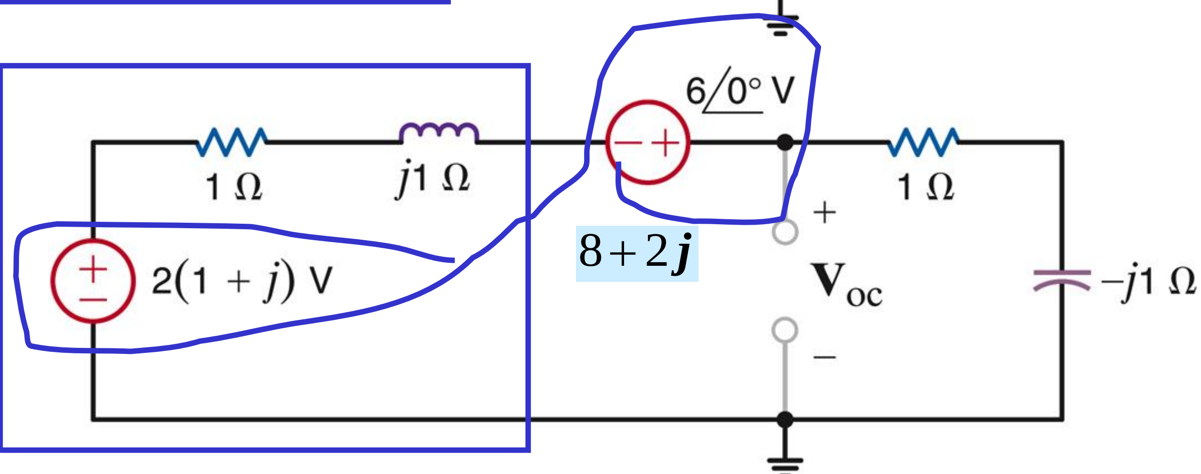
$$V_{OC} = \frac{1-j}{(1+j)+(1-j)}(8+2j) = \frac{10-6j}{2}$$



$$Z_{TH} = (1+j)\,\|\,(1-j) = 1\Omega$$





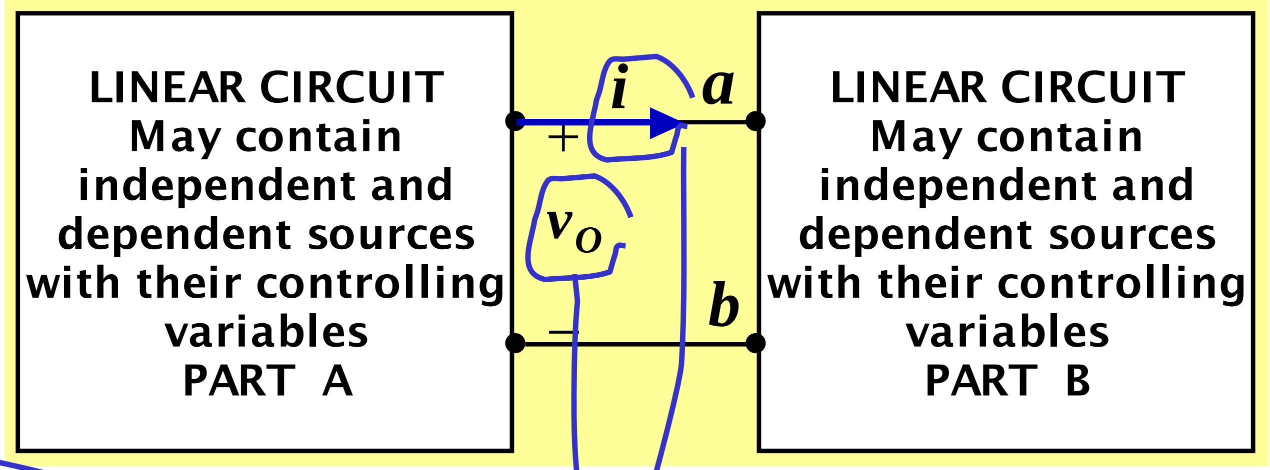


$8+2j$

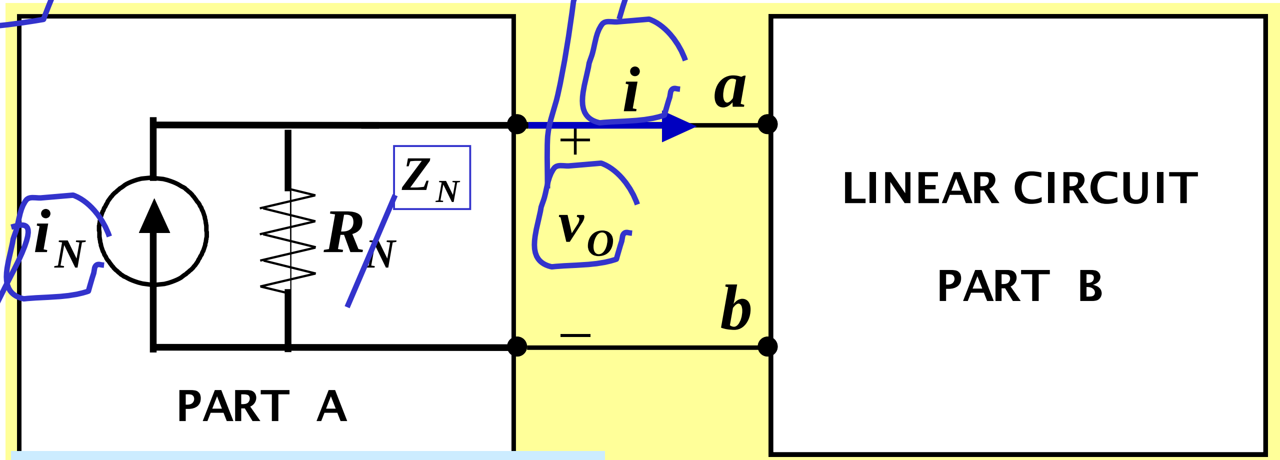
$$I_0 = \frac{5-3j}{2}(A)$$

NEXT: NORTON

NORTON'S EQUIVALENCE THEOREM



Phasors



Norton Equivalent Circuit

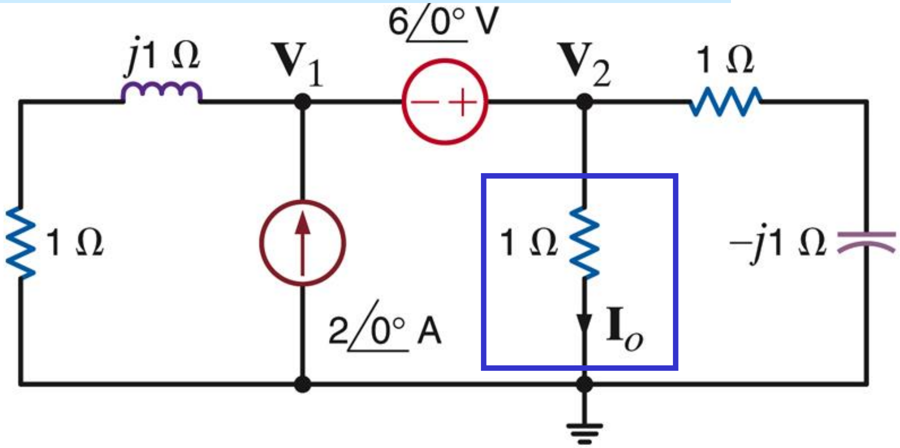
for PART A

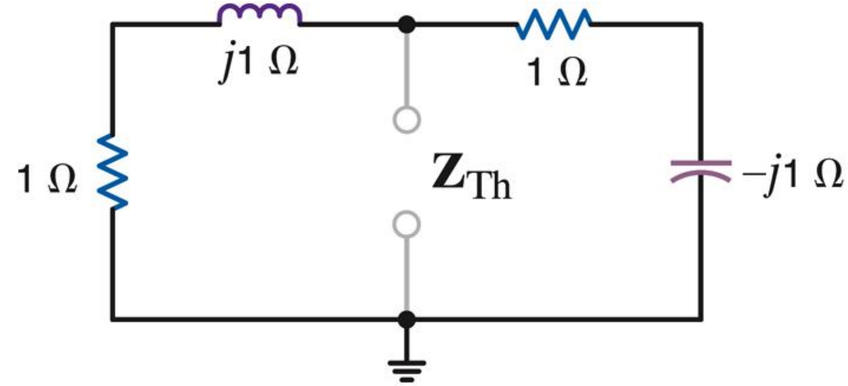
i_N Thevenin Equivalent Source

R_N $\boxed{Z_N}$ Thevenin Equivalent Resistance

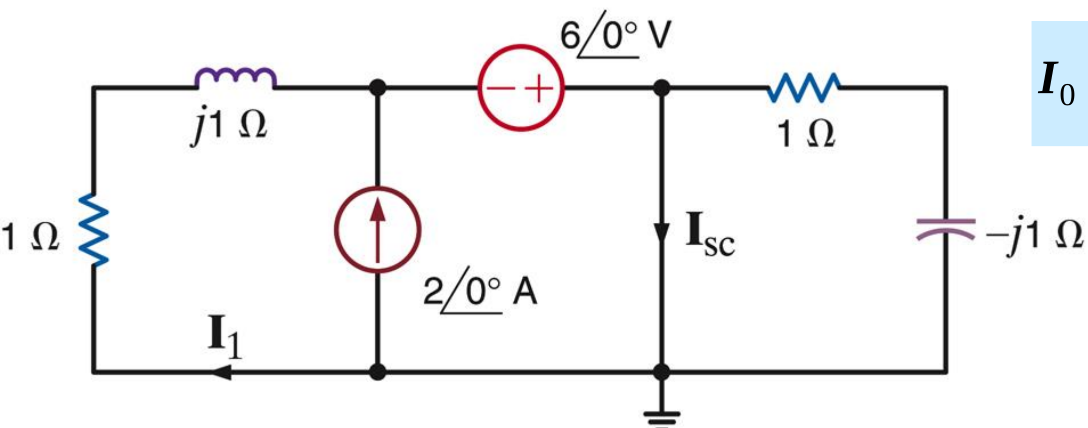
Impedance

EXAMPLE 3: NORTON ANALYSIS

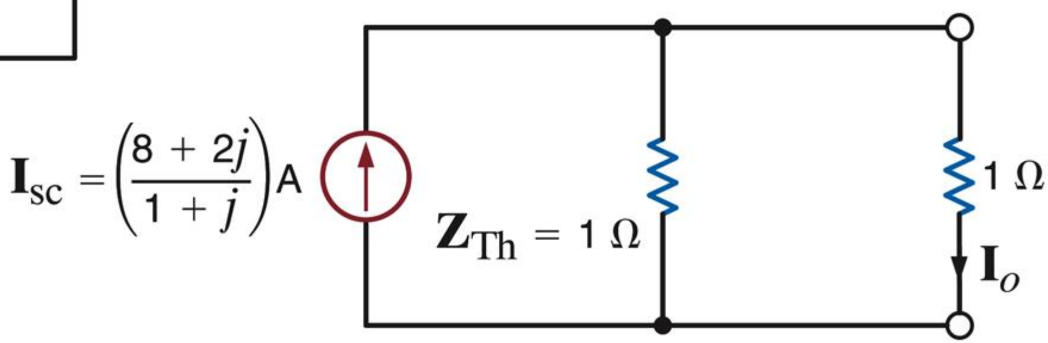




$$Z_{TH} = (1+j) \,||\, (1-j) = 1\Omega$$



$$I_0 = \frac{I_{SC}}{2} = \frac{4+j}{1-j} = \frac{(4+j)(1-j)}{(1+j)(1-j)} = \frac{5-3j}{2}$$



Possible techniques: loops, source transformation, superposition

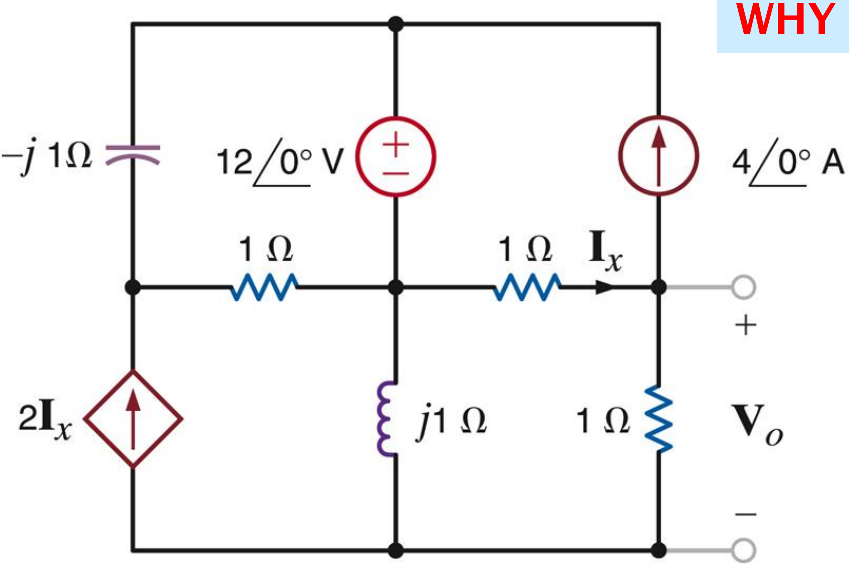
BY SUPERPOSTION

$$I_{SC} = 2\angle 0^\circ + \frac{6\angle 0^\circ}{1+j} = \frac{8+2j}{1+j}(A)$$

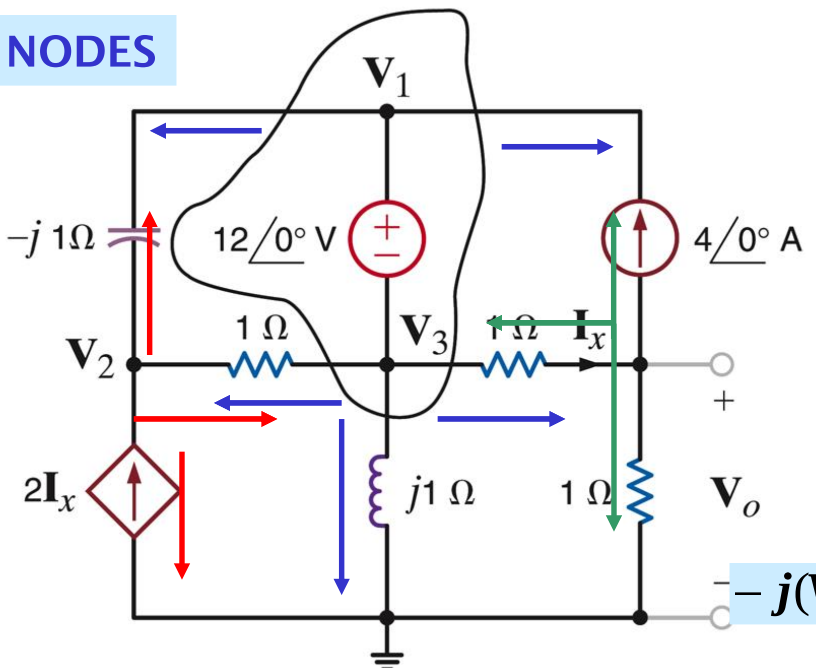
EXAMPLE 4

FIND V_0 USING NODES, LOOPS, THEVENIN, NORTON

WHY SKIP SUPERPOSITION AND TRANSFORMATION?



NODES



Notice choice of ground

Supernode constraint: $V_1 - V_3 = 12\angle 0°$

KCL @ Supernode

$$-4\angle 0° + \frac{V_3 - V_0}{1} + \frac{V_3 - V_2}{1} + \frac{V_1 - V_2}{-j} + \frac{V_3}{j} = 0$$

KCL@V_2

$$\frac{V_2 - V_1}{-j} - 2I_x + \frac{V_2 - V_3}{1} = 0$$

KCL@ V_0

$$\frac{V_0}{1} + \frac{V_0 - V_3}{1} + 4\angle 0° = 0$$

$$\Rightarrow V_3 = 2V_0 + 4$$

Controlling variable

$$I_x = \frac{V_3 - V_0}{1}$$

$$V_1 = V_3 + 12$$

$$V_1 = 2V_0 + 16$$

$$V_3 - V_0 = V_0 + 4$$

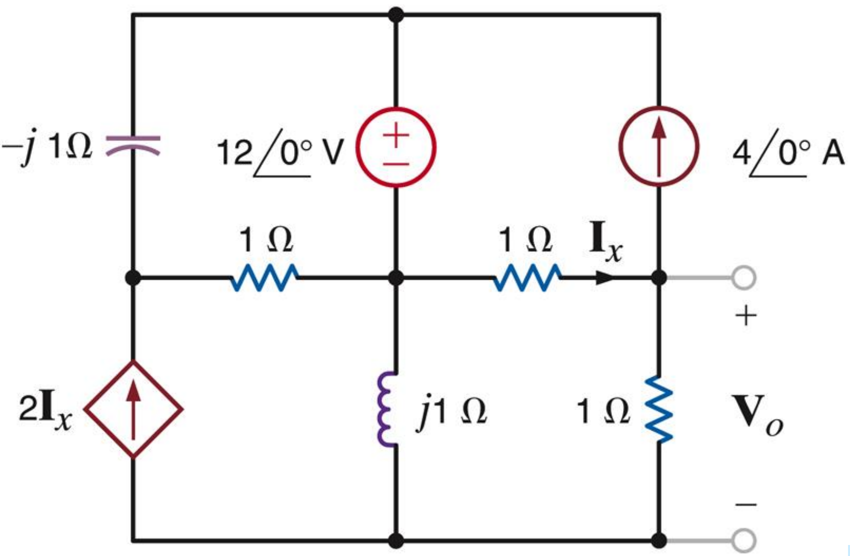
$$j(V_2 - 2V_0 - 16) - 2(V_0 + 4) + (V_2 - 2V_0 - 4) = 0$$

$$-j(V_2 - 2V_0 - 16) - (V_2 - 2V_0 - 4) + (V_0 + 4) - j(2V_0 + 4) = 4$$

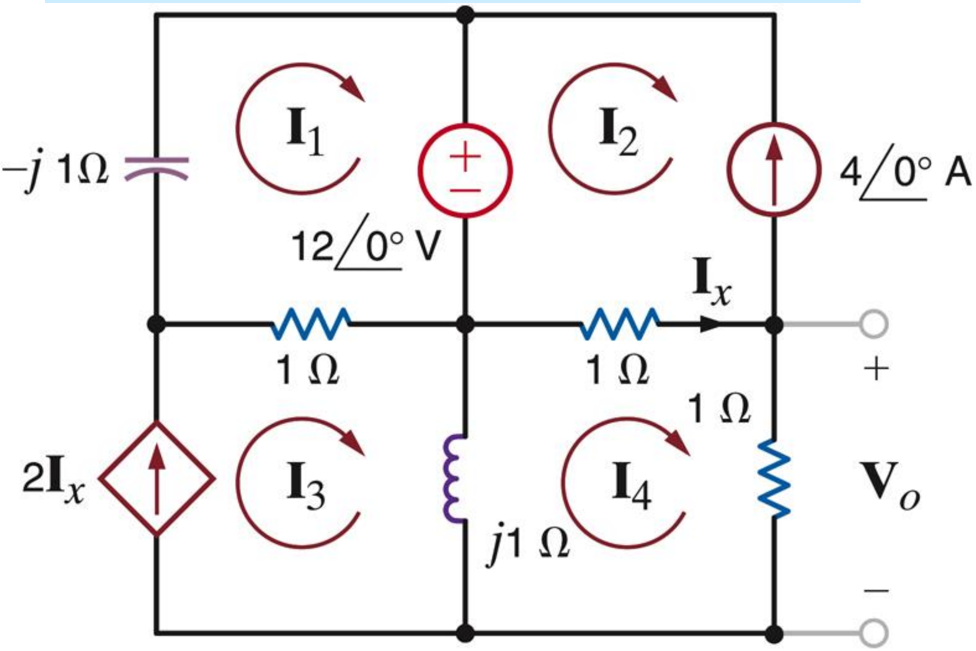
Adding: $V_0 = -\dfrac{8 + 4j}{1 + 2j}$

EXAMPLE 4

LOOP ANALYSIS



MESH CURRENTS ARE ACCEPTABLE



MESH CURRENTS DETERMINED BY SOURCES

$$\boldsymbol{I}_2 = -4\angle 0°$$
$$\boldsymbol{I}_3 = 2\boldsymbol{I}_x \qquad \Rightarrow \boldsymbol{I}_3 = 2(\boldsymbol{I}_4 + 4)$$

MESH 1:

$$-\boldsymbol{j}\boldsymbol{I}_1 + 12\angle 0° + 1(\boldsymbol{I}_1 - \boldsymbol{I}_3) = 0$$

MESH 4:

$$1(\boldsymbol{I}_4 - \boldsymbol{I}_2) + 1\times \boldsymbol{I}_4 + \boldsymbol{j}(\boldsymbol{I}_4 - \boldsymbol{I}_3) = 0$$

CONTROLLING VARIABLE: $\boldsymbol{I}_x = \boldsymbol{I}_4 - \boldsymbol{I}_2$

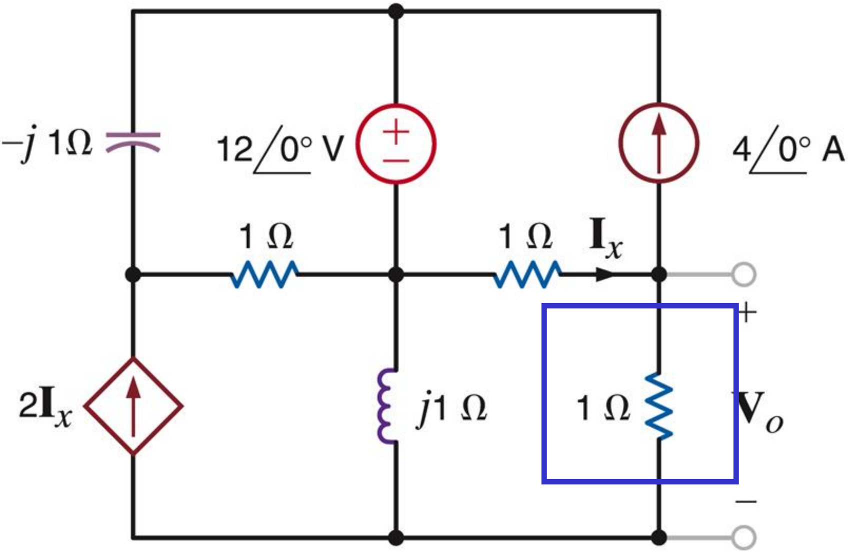
VARIABLE OF INTEREST: $\boldsymbol{V}_0 = 1\times \boldsymbol{I}_4(\boldsymbol{V})$

$$\boldsymbol{I}_4 + 4 + \boldsymbol{I}_4 + \boldsymbol{j}(\boldsymbol{I}_4 - 2(\boldsymbol{I}_4 + 4)) = 0$$

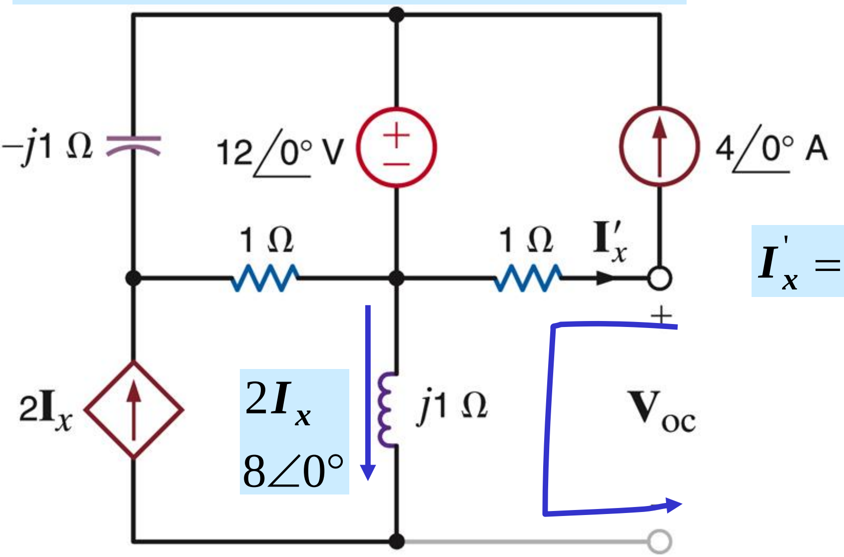
$$(2-\boldsymbol{j})\boldsymbol{I}_4 = -(4-8\boldsymbol{j}) \Rightarrow \boldsymbol{I}_4 = -\frac{4-8\boldsymbol{j}}{2-\boldsymbol{j}} \times \frac{\boldsymbol{j}}{\boldsymbol{j}}$$

$$\boldsymbol{V}_0 = -\frac{8+4\boldsymbol{j}}{1+2\boldsymbol{j}}$$

EXAMPLE 4 **THEVENIN**



FOR OPEN CIRCUIT VOLTAGE



$I'_x = 4\angle 0°$

$2I_x$

$8\angle 0°$

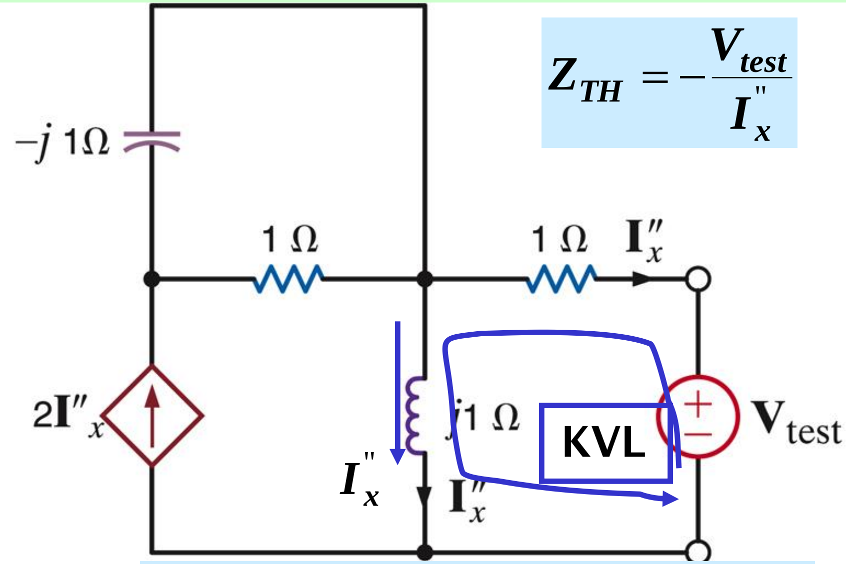
$V_{OC} = -4 + 8j(V)$

Alternative procedure to compute Thevenin impedance:

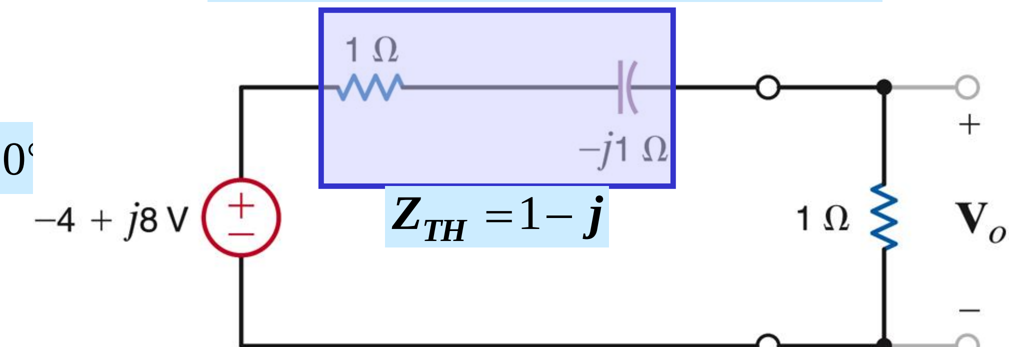
1. Set to zero all INDEPENDENT sources

2. Apply an external probe

$$Z_{TH} = -\frac{V_{test}}{I''_x}$$



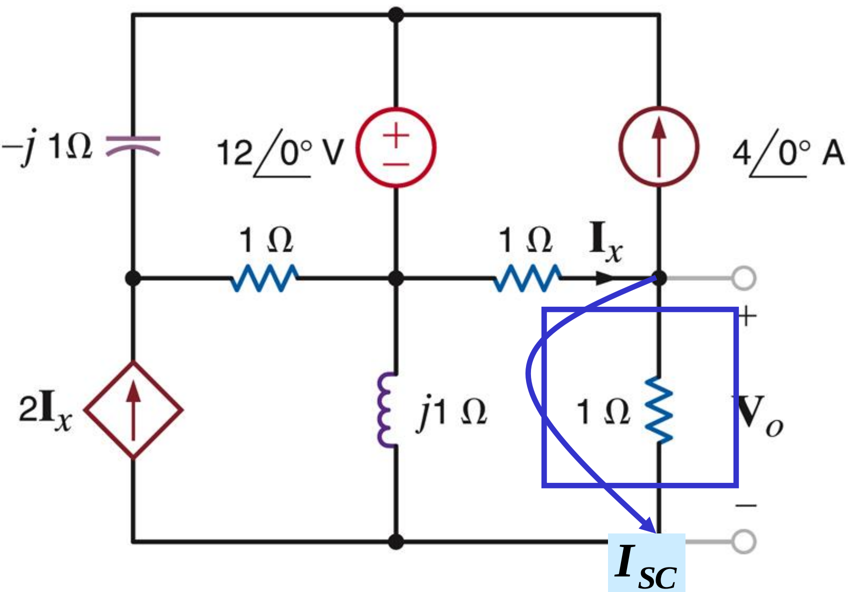
$$V_{test} = -I''_x + jI''_x \Rightarrow Z_{TH} = 1 - j(\Omega)$$



$Z_{TH} = 1 - j$

$$V_0 = \frac{1}{2-j}(-4+8j)(V)$$

EXAMPLE 4 **NORTON**



Supernode constraint

$$V_1 - V_3 = 12\angle 0° \quad \Rightarrow V_1 = V_3 + 12$$

KCL@ Supernode

$$\frac{V_3}{1} + \frac{V_3}{j} + \frac{V_3 - V_2}{1} + \frac{V_1 - V_2}{-j} - 4\angle 0° = 0 \quad /\times j$$

$$\text{KCL@ } V_2:\ -2I_x''' + \frac{V_2 - V_3}{1} + \frac{V_2 - V_1}{-j} = 0 \ /\times(-j)$$

$$\text{Controlling Variable: } I_x''' = \frac{V_3}{1}$$

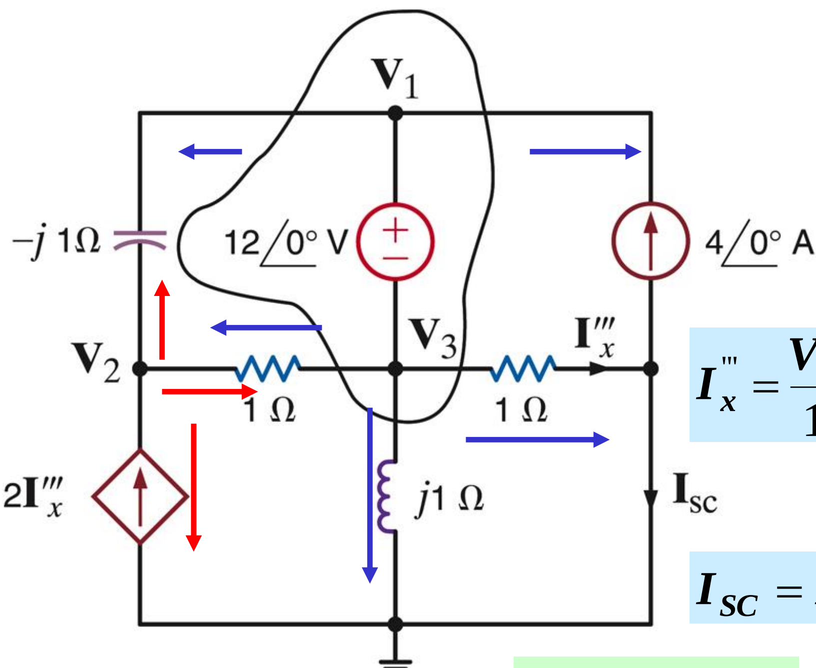
$$2jV_3 - j(V_2 - V_3) + (V_2 - V_3 - 12) = 0$$

$$(1-j)V_2 - (1-3j)V_3 = 12$$

$$(1+j)V_3 + jV_3 - jV_2 - (V_3 + 12) + V_2 = 4j$$

$$(1-j)V_2 + 2jV_3 = 12 + 4j$$

$$(1-j)V_3 = 4j \Rightarrow V_3 = \frac{4j}{1-j} \Rightarrow I_{SC} = \frac{-4+8j}{1-j}$$



$$I_x''' = \frac{V_3}{1}(A)$$

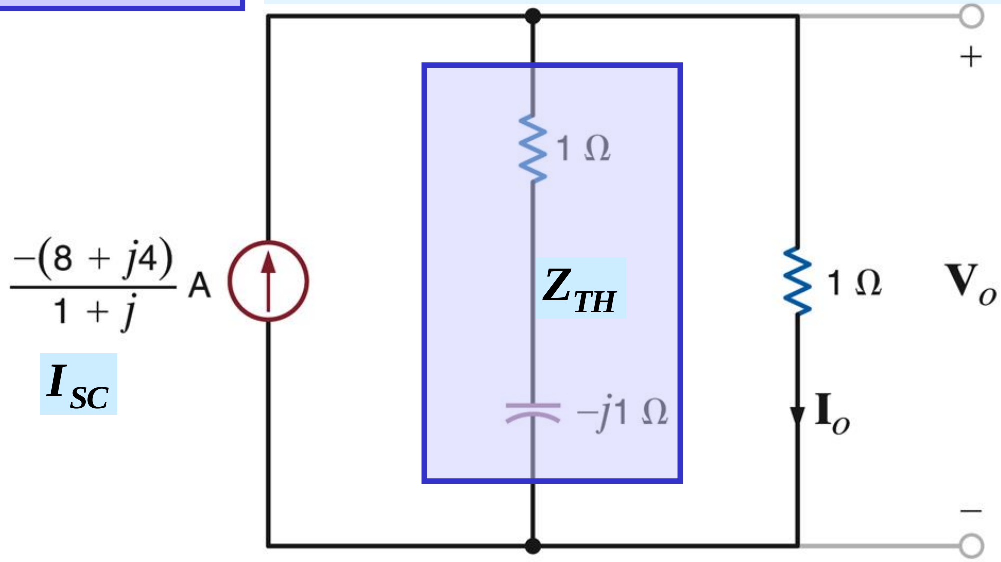
$$I_{SC} = I_x''' - 4$$

USE NODES

$$I_{SC} = \frac{(-4+8j)j}{(1-j)j} = -\frac{8+4j}{1+j}$$

Now we can draw the Norton Equivalent circuit ...

EXAMPLE 4 NORTON'S EQUIVALENT CIRCUIT



$$V_0 = (1) I_0 (V) = \frac{1-j}{2-j}\left(-\frac{8+4j}{1+j}\right)(V)$$

Current Divider

EXAMPLE 4 EQUIVALENCE OF SOLUTIONS

Using Norton's method

$$V_0 = -\frac{12-4j}{3+j} = -\frac{(8+4j)(1-j)}{(1+2j)(1-j)}$$

Using Thevenin's

$$V_0 = \frac{-4+8j}{2-j} \times \frac{j}{j}$$

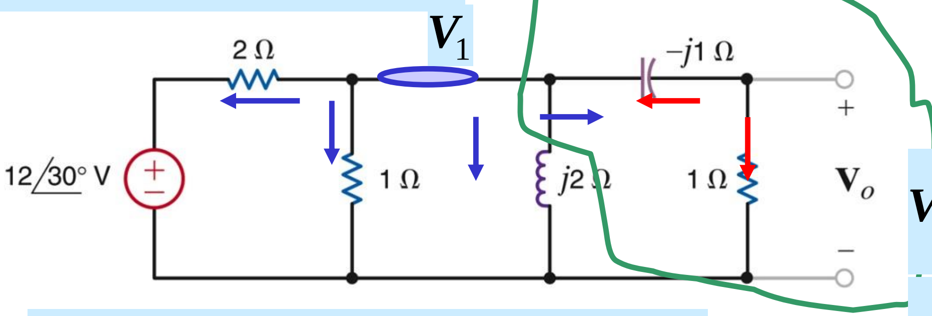
Using Node and Loop methods

$$V_0 = -\frac{8+4j}{1+2j}$$

EXAMPLE 5

COMPUTE V_0

USE NODAL ANALYSIS



$$\frac{V_1 - 12\angle 30^\circ}{2} + \frac{V_1}{1} + \frac{V_1}{j2} + \frac{V_1 - V_0}{-j} = 0 \quad /\times 2j$$

$$\frac{V_0 - V_1}{-j} + \frac{V_0}{1} = 0 \Rightarrow V_1 = (1-j)V_0$$

$$j(V_1 - 12\angle 30^\circ) + 2jV_1 + V_1 - 2(V_1 - V_0) = 0$$

$$2V_0 + (1 - 2 + 2j + j)(1-j)V_0 = j12\angle 30^\circ$$

$$(2 + (-1+3j)(1-j))V_0 = 1\angle 90^\circ \times 12\angle 30^\circ$$

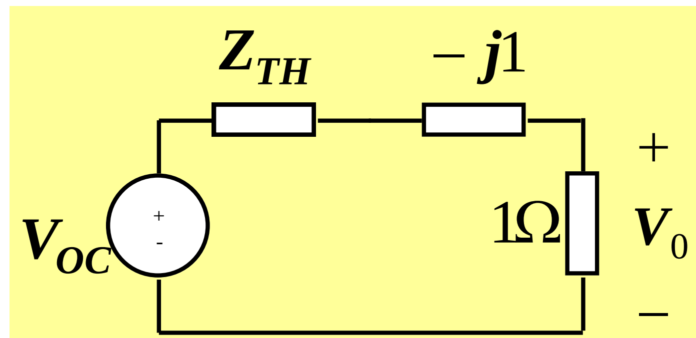
$$V_0 = \frac{12\angle 120^\circ}{4+4j} = \frac{12\angle 120^\circ}{5.66\angle 45^\circ} = 2.12\angle 75^\circ (V)$$

USE THEVENIN

$$Z_{TH} = 2 \| 1 \| j2 = \frac{4j/3}{2/3 + j2} = \frac{4j}{2+6j} = \frac{4j(2-6j)}{40}$$

$$V_{OC} = \frac{1\| j2}{2 + (1\| j2)} 12\angle 30^\circ = \frac{j2}{2(1+2j)+2j} 12\angle 30^\circ$$

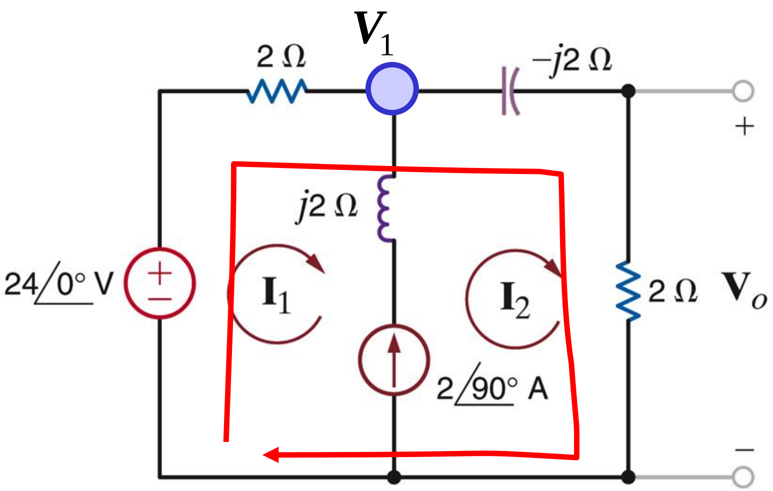
$$V_{OC} = \frac{24\angle 120^\circ}{2+6j} = \frac{12\angle 120^\circ}{1+3j}$$



$$V_0 = \frac{1}{Z_{TH} + 1 - j} V_{OC}$$

EXAMPLE 6

COMPUTE V_0 USING MESH ANALYSIS



CONSTRAINT

$$-I_1 + I_2 = 2\angle 90^\circ \Rightarrow I_1 = I_2 - 2j$$

SUPERMESH

$$-24\angle 0^\circ + 2I_1 - 2jI_2 + 2I_2 = 0$$

$$2(I_2 - 2j) + (2 - 2j)I_2 = 24 \Rightarrow (4 - 2j)I_2 = 24 + 4j$$

$$V_0 = 2I_2 = \frac{24 + 4j}{2 - j} = \frac{24.33\angle 9.46^\circ}{2.24\angle -26.57^\circ} = 10.86\angle 36.03^\circ$$

USING NODES

$$\frac{V_1 - 24\angle 0^\circ}{2} - 2\angle 90^\circ + \frac{V_1}{2 - 2j} = 0$$

$$V_0 = \frac{2}{2 - 2j} V_1$$

USING SOURCE SUPERPOSITION

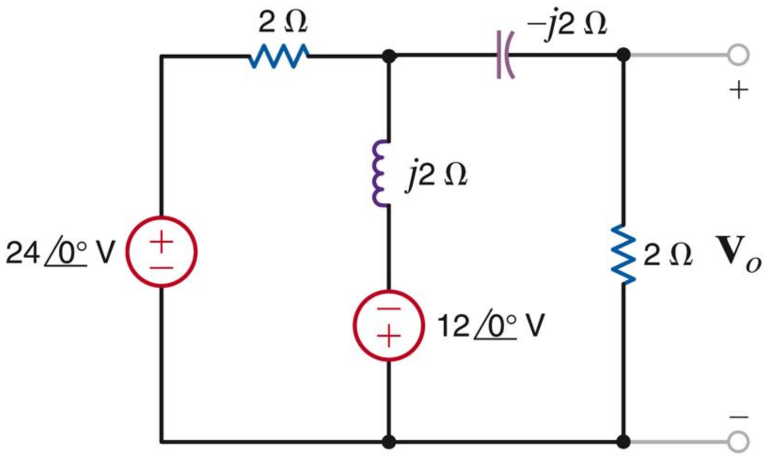
$$V_0^V = \frac{2}{2 + 2 - 2j} 24\angle 0^\circ$$

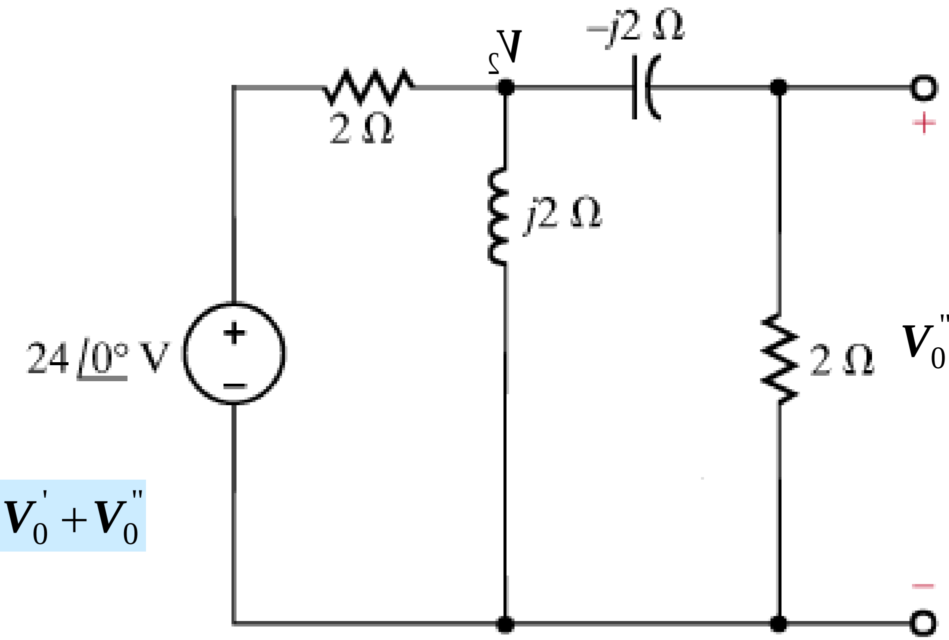
$$V_0^I = 2 \times \frac{2}{4 - 2j} 2\angle 90^\circ$$

$$V_0 = V_0^V + V_0^I$$

EXAMPLE 7

COMPUTE V_0

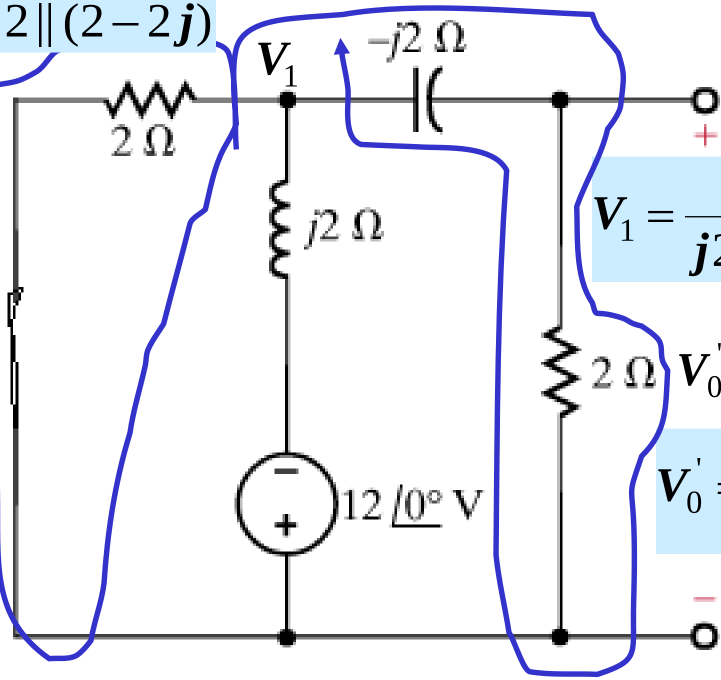




$$V_0 = V_0' + V_0''$$

USING SUPERPOSITION

$2 \| (2-2j)$



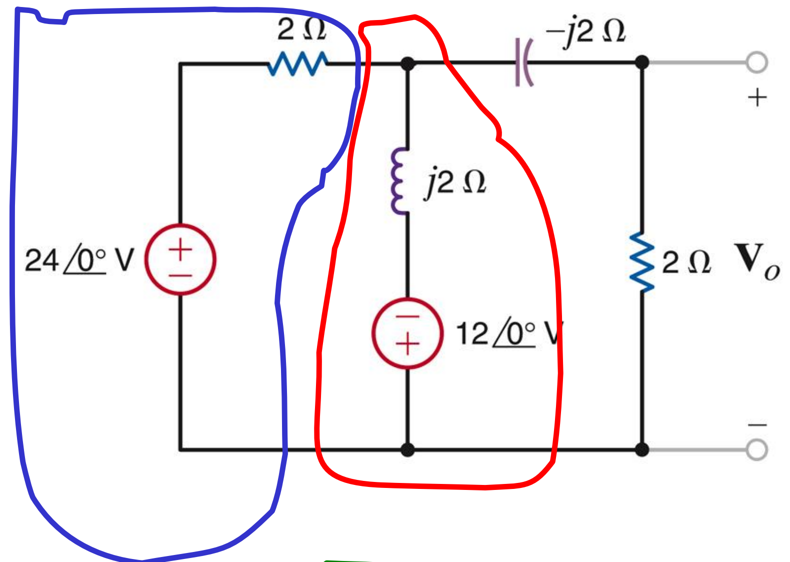
$$V_1 = \frac{2 \| (2-2j)}{j2 + (2 \| 2-2j)}(-12\angle 0°)$$

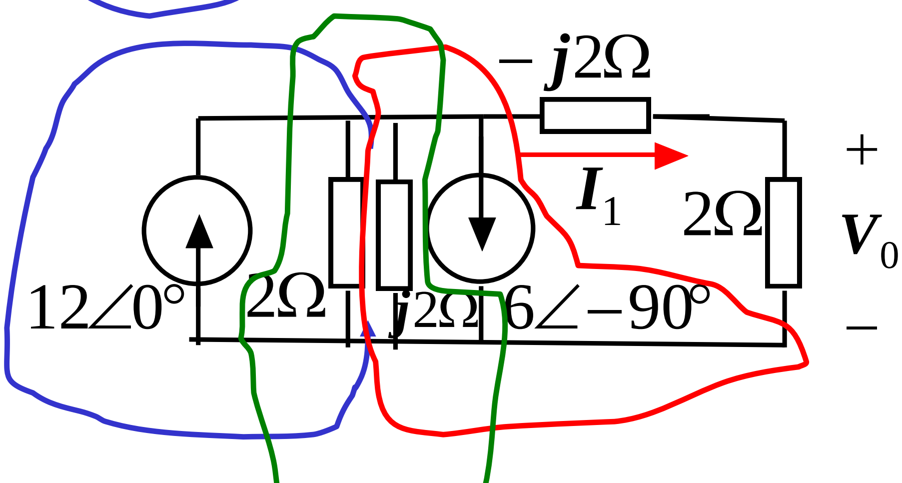
$$V_0' = \frac{2}{2-2j} V_1$$

$$V_2 = \frac{(2j) \| (2-2j)}{2+(2j \| (2-2j)} 24\angle 0°$$

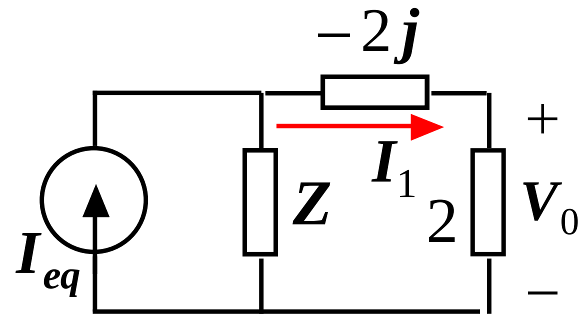
$$V_0'' = \frac{2}{2-2j} V_2$$

EXAMPLE 7 USE SOURCE TRANSFORMATION





$Z = 2 \| j2$



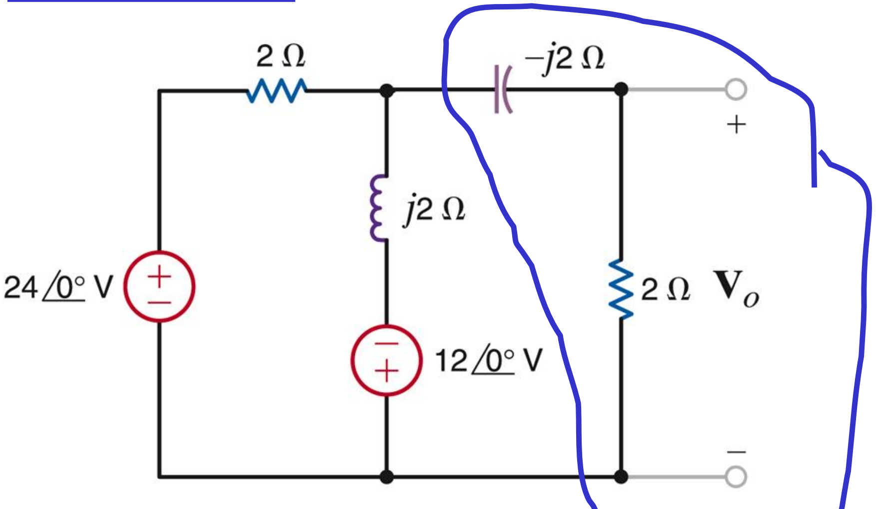
$$I_{eq} = 12\angle 0° - 6\angle -90° = 12 + 6j$$

$$I_1 = \frac{Z}{Z + 2 - 2j} I_{eq}$$

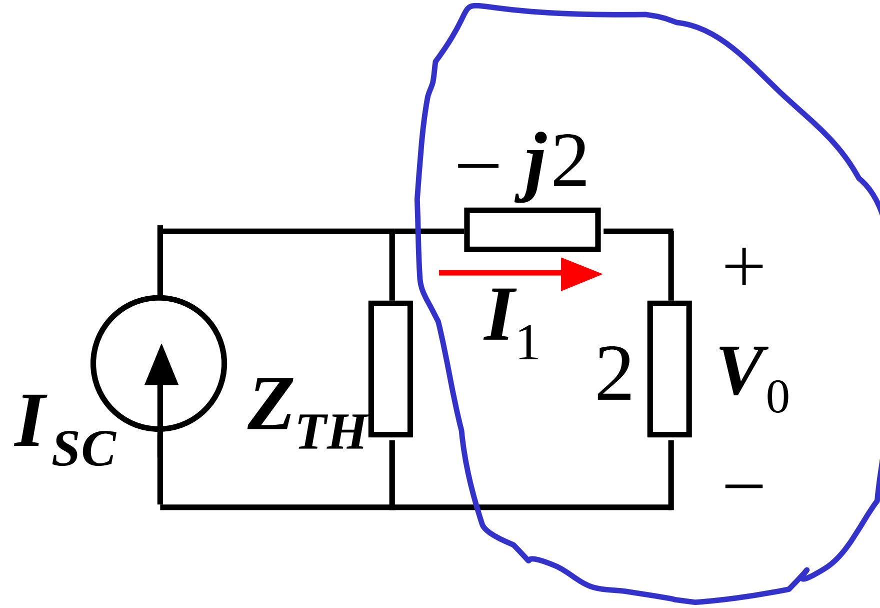
$$V_0 = 2I_1$$

EXAMPLE 7

USE NORTON'S THEOREM

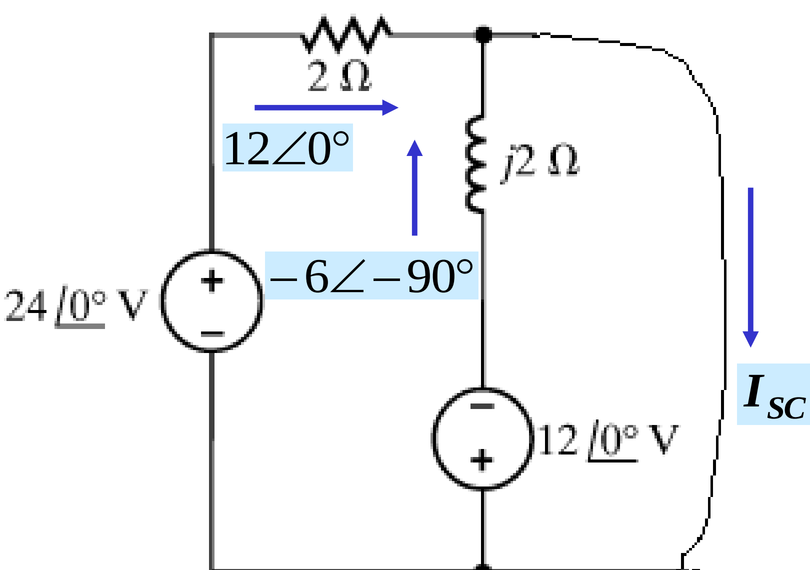


$$Z_{TH} = 2 \| j2$$



$$I_1 = \frac{Z_{TH}}{Z_{TH} + 2 - 2j} I_{SC}$$

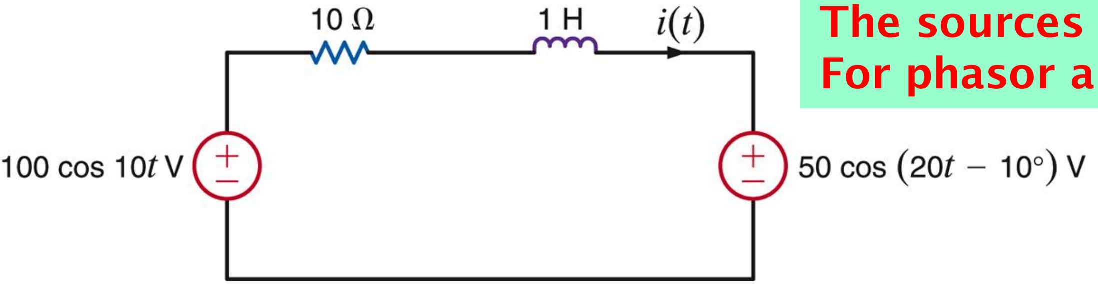
$$V_0 = 2I_1$$

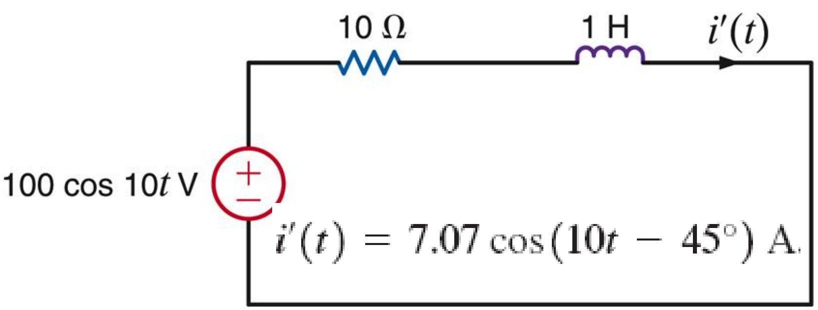


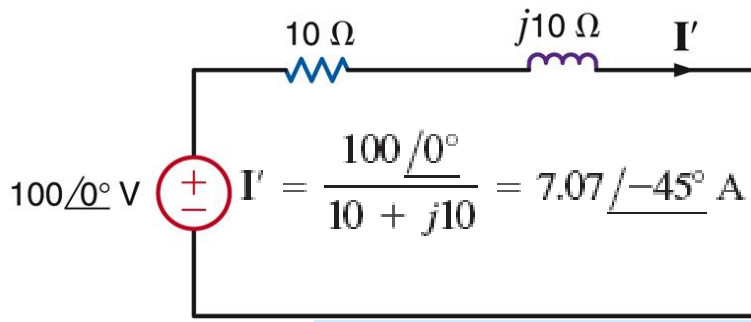
EXAMPLE 8

Find the current i(t) in steady state

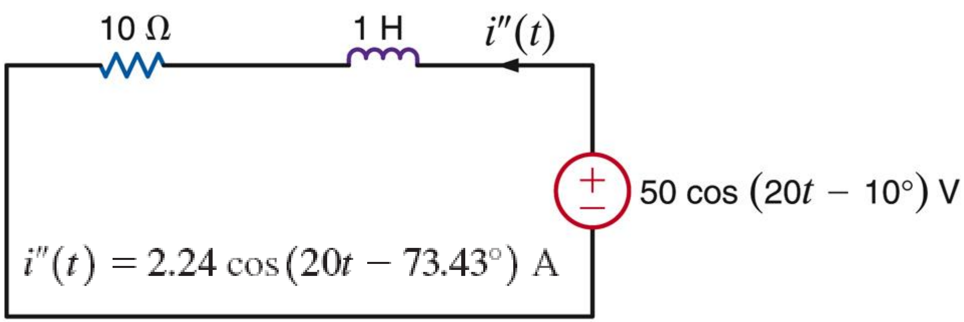
The sources have different frequencies!
For phasor analysis MUST use source superpositio

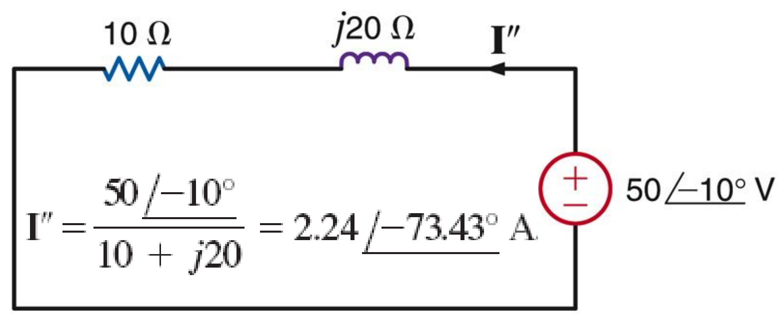






Frequency domain





SOURCE 2: FREQUENCY 20r/s

$$i'(t) - i''(t) = 7.07\cos(10t - 45^\circ) - 2.24\cos(20t - 73.43^\circ)\ \mathrm{A}$$

Principle of superposition

USING MATLAB

MATLAB recognizes complex numbers in rectangular representation. It does NOT recognize Phasors

Unless previously re-defined, MATLAB recognizes "i" or "j" as imaginary units

```
» z2=3+4j

z2 =

   3.0000 + 4.0000i

» z1=4+6i

z1 =

   4.0000 + 6.0000i
```

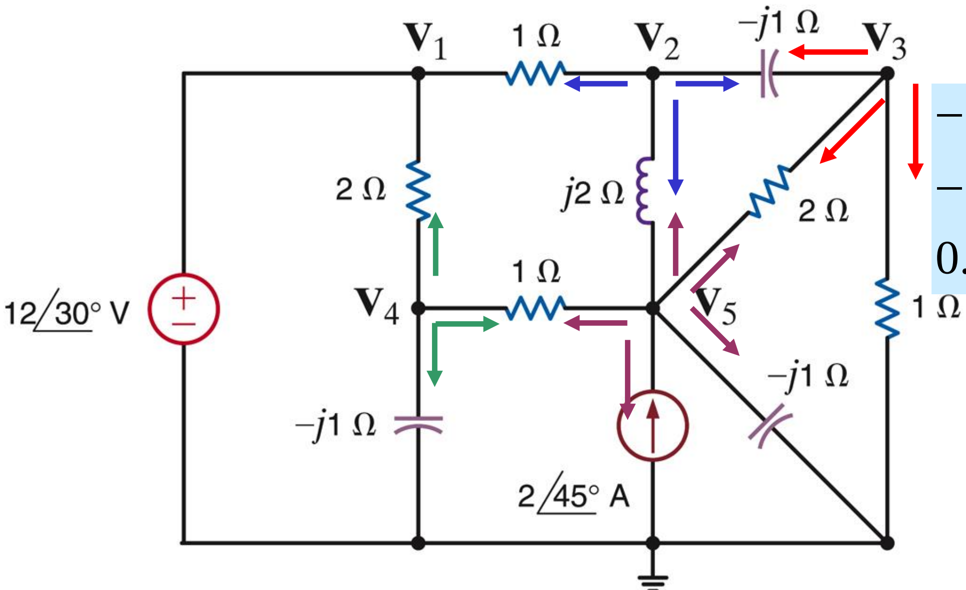
In its output MATLAB always uses "i" for the imaginary unit

Phasors $\leftrightarrow$ Rectangular $z = 10\angle 45°$

```
» a=45; % angle in degrees
» ar=a*pi/180, %convert degrees to radians
ar =
    0.7854
» m=10; %define magnitude
» x=m*cos(ar); %real part
x =
    7.0711
» y=m*sin(ar); %imaginary part
y =
    7.0711
» z=x+i*y
z =
   7.0711 + 7.0711i
z =  7.0711 + 7.0711i;
» mp=abs(z); %compute magnitude
mp =
    10
» arr=angle(z); %compute angle in RADIANS
arr =
0.7854
» adeg=arr*180/pi; %convert to degres
adeg =
    45
x=real(z)
x=
   7.0711
y=imag(z)
y=
   7.70711
```

EXAMPLE 1

COMPUTE ALL NODE VOLTAGES



$$V_1 = 12\angle 30^\circ$$

$$-V_1 + (1 + j - 0.5j)V_2 - jV_3 - 0.5jV_5 = 0$$

$$-jV_2 + (j + 0.5 + 1)V_3 - 0.5V_5 = 0$$

$$-0.5V_1 + (0.5 + 1 + j)V_4 - V_5 = 0$$

$$0.5jV_2 - 0.5V_3 - V_4 + (1 - 0.5j + 0.5 + j)V_5 = 2\angle 45^\circ$$

$$V_1 = 12\angle 30^\circ$$

$$\frac{V_2 - V_1}{1} + \frac{V_2 - V_3}{-j1} + \frac{V_2 - V_5}{j2} = 0$$

$$\frac{V_3 - V_2}{-j1} + \frac{V_3 - V_5}{2} + \frac{V_3}{1} = 0$$

$$\frac{V_4 - V_1}{2} + \frac{V_4 - V_5}{1} + \frac{V_4}{-j1} = 0$$

$$-2\angle 45^\circ + \frac{V_5 - V_4}{1} + \frac{V_5 - V_2}{j2} + \frac{V_5 - V_3}{2} + \frac{V_5}{-j1} = 0$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 + j0.5 & -j1 & 0 & j0.5 \\ 0 & -j1 & 1.5 + j1 & 0 & -0.5 \\ -0.5 & 0 & 0 & 1.5 + j1 & -1 \\ 0 & j0.5 & -0.5 & -1 & 1.5 + j0.5 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{bmatrix} = \begin{bmatrix} 12\angle 30^\circ \\ 0 \\ 0 \\ 0 \\ 2\angle 45^\circ \end{bmatrix}$$

$$YV = I_R$$

$$V = Y^{-1} I_R$$

EXAMPLE 1

```
%example8p18
%define the RHS vector.
ir=zeros(5,1); %initialize and define non zero values
ir(1)=12*cos(30*pi/180)+j*12*sin(30*pi/180);
ir(5)=2*cos(pi/4)+j*2*sin(pi/4), %echo the vector
%now define the matrix
y=[1,0,0,0,0; %first row
   -1,1+0.5j,-j,0,0.5j; %second row
   0,-j,1.5+j,0,-0.5; %third row
   -0.5,0,0,1.5+j,-1; %fourth row
   0,0.5i,-0.5,-1,1.5+0.5i] %last row and do echo
v=y\ir %solve equations and echo the answer
```

Echo of Answer

```
v =

  10.3923 + 6.0000i
   7.0766 + 2.1580i
   1.4038 + 2.5561i
   3.7661 - 2.9621i
   3.4151 - 3.6771i
```

Echo of RHS

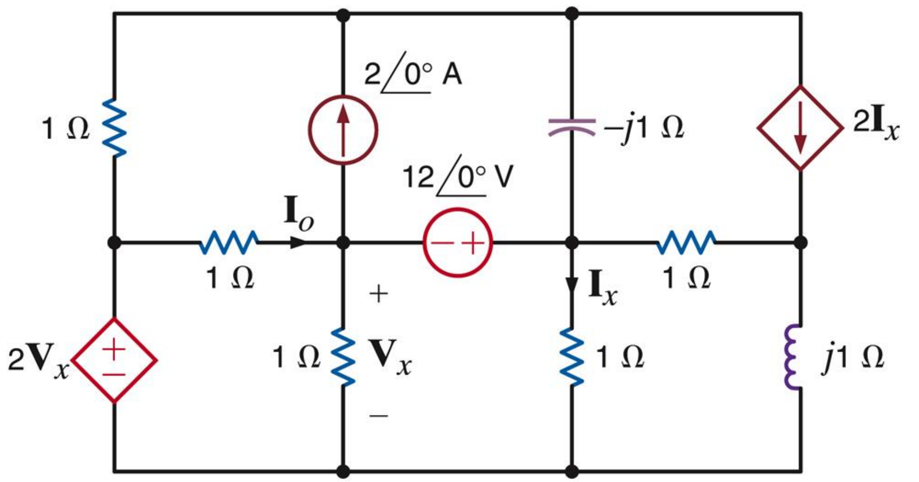
```
ir =
  10.3923 + 6.0000i
        0
        0
        0
   1.4142 + 1.4142i
```

Echo of Matrix

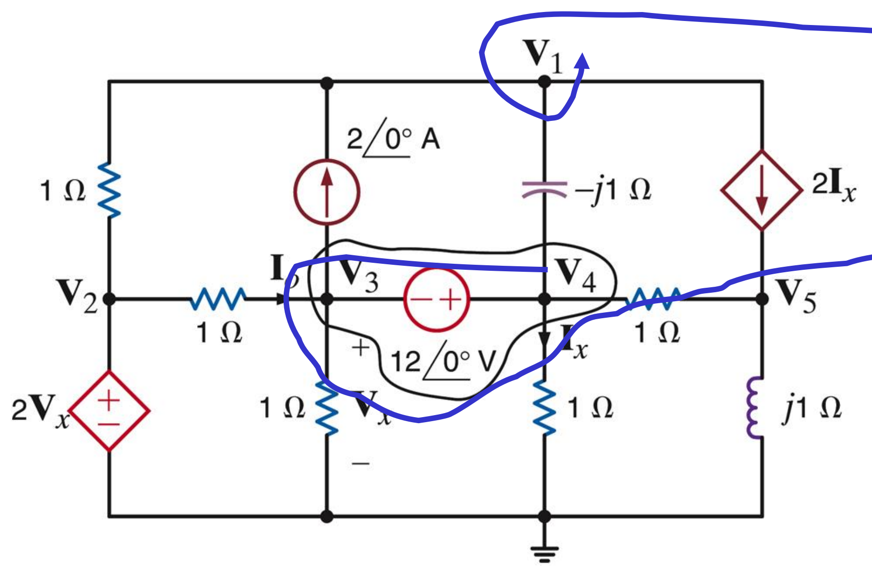
```
 y =
   Columns 1 through 4
    1.0000                  0                  0                  0
   -1.0000             1.0000 + 0.5000i        0 - 1.0000i        0
         0                  0 - 1.0000i   1.5000 + 1.0000i        0
   -0.5000                  0                  0             1.5000 + 1.0000i
         0                  0 + 0.5000i  -0.5000            -1.0000
   Column 5
         0
         0 + 0.5000i
   -0.5000
   -1.0000
    1.5000 + 0.5000i
```

EXAMPLE 2

FIND THE CURRENT I_o



6 meshes and two current sources
5 non-reference nodes and two voltage sources… slight advantage for nodes



$$\frac{\mathbf{V}_1 - \mathbf{V}_2}{1} - 2 + \frac{\mathbf{V}_1 - \mathbf{V}_4}{-j} + 2\mathbf{I}_x = 0$$

$$\mathbf{V}_2 = 2\mathbf{V}_x$$

$$\mathbf{V}_4 - \mathbf{V}_3 = 12$$

$$2 + \frac{\mathbf{V}_3 - \mathbf{V}_2}{1} + \frac{\mathbf{V}_3}{1} + \frac{\mathbf{V}_4}{1} + \frac{\mathbf{V}_4 - \mathbf{V}_5}{1} + \frac{\mathbf{V}_4 - \mathbf{V}_1}{-j} = 0$$

$$\frac{\mathbf{V}_5 - \mathbf{V}_4}{1} + \frac{\mathbf{V}_5}{j} = 2\mathbf{I}_x$$

$$\mathbf{V}_x = \mathbf{V}_3$$

$$\mathbf{I}_x = \frac{\mathbf{V}_4}{1}$$

EXAMPLE 2

NODE EQUATIONS

$$\frac{\mathbf{V}_1 - \mathbf{V}_2}{1} - 2 + \frac{\mathbf{V}_1 - \mathbf{V}_4}{-j} + 2\mathbf{I}_x = 0$$

$$\mathbf{V}_2 = 2\mathbf{V}_x$$

$$\mathbf{V}_4 - \mathbf{V}_3 = 12$$

$$2 + \frac{\mathbf{V}_3 - \mathbf{V}_2}{1} + \frac{\mathbf{V}_3}{1} + \frac{\mathbf{V}_4}{1} + \frac{\mathbf{V}_4 - \mathbf{V}_5}{1} + \frac{\mathbf{V}_4 - \mathbf{V}_1}{-j} = 0$$

$$\frac{\mathbf{V}_5 - \mathbf{V}_4}{1} + \frac{\mathbf{V}_5}{j} = 2\mathbf{I}_x$$

$$\mathbf{V}_x = \mathbf{V}_3$$

$$\mathbf{I}_x = \frac{\mathbf{V}_4}{1}$$

MATRIX FORM

$$\begin{bmatrix} 1+j & -1 & 0 & 2-j & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ -j & -1 & 2 & 2+j & -1 \\ 0 & 0 & 0 & -3 & 1-j \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 12 \\ -2 \\ 0 \end{bmatrix}$$

EXAMPLE 2

MATLAB COMMANDS

```
>> Y = [1+j*1 -1 0 2-j*1; 0 1 -2 0 0; 0 0 -1 1 0;
       -j*1 -1 2 2+j*1 -1; 0 0 0 -3 1-j*1]

Y =

Columns 1 through 4

 1.0000 +1.0000i    -1.0000            0       2.0000 -1.0000i
      0              1.0000      -2.0000            0
      0                   0      -1.0000       1.0000
      0 -1.0000i    -1.0000       2.0000       2.0000 +1.0000i
      0                   0            0     -3.0000

Column 5

      0
      0
      0
-1.0000
 1.0000  -1.0000i

>> I = [2; 0; 12; -2; 0]

I =
   -2
    0
   12
   -2
    0

>> V = inv(Y) *I

V =
    -5.5000    +4.5000i
   -26.0000   -24.0000i
   -13.0000   -12.0000i
    -1.0000   -12.0000i
    16.5000   -19.5000i

>>
```

LINEAR EQUATION

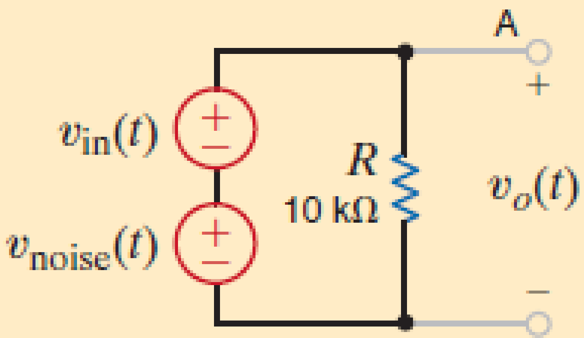
$$\begin{bmatrix} 1+j & -1 & 0 & 2-j & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ -j & -1 & 2 & 2+j & -1 \\ 0 & 0 & 0 & -3 & 1-j \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 12 \\ -2 \\ 0 \end{bmatrix}$$

ANSWER

$$\begin{aligned} \mathbf{I}_o &= (\mathbf{V}_2 - \mathbf{V}_3)/1 \\ &= -13 - j12 \text{ A} \end{aligned}$$

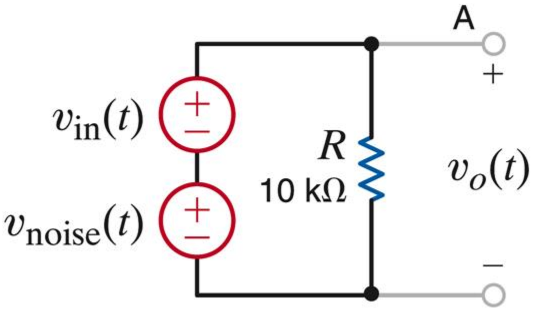
LEARNING APPLICATION 1 NOISE REJECTION

The network in Fig. 8.28 models an unfortunate situation that is all too common. Node A, which is the voltage $v_{in}(t)$ at the output of a temperature sensor, has "picked-up" a high-frequency voltage, $v_{noise}(t)$, caused by a nearby AM radio station. The noise frequency is 700 kHz. In this particular scenario, the sensor voltage, like temperature, tends to vary slowly. Our task then is to modify the circuit to reduce the noise at the output without disturbing the desired signal, $v_{in}(t)$.

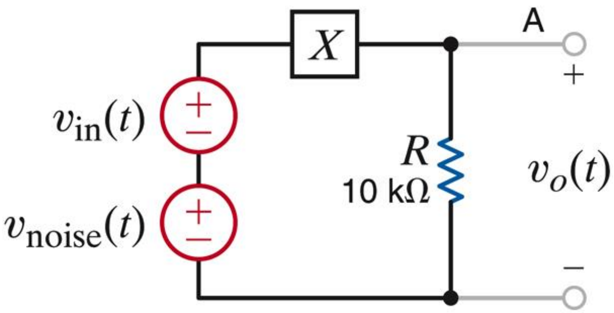


LEARNING APPLICATION 1

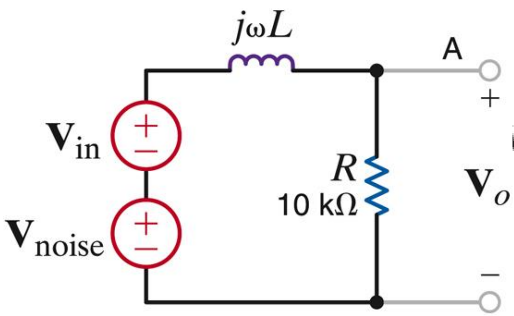
NOISE REJECTION



Noise has much higher frequency (700kHz) than signal.
Find a way to 'block' high frequencies



Impedance X should have low (zero) value at low frequencies and very high at noise frequency



$$V_o = \left[\frac{R}{R + j\omega L}\right] \mathbf{V}_1$$

Reduce amplitude of noise by 10

$$\left|\frac{R}{R + j\omega L}\right| = \frac{1}{10} \quad \text{at } f = 700 \text{ mHz}$$

$$L = 22.6 \text{ mH}$$

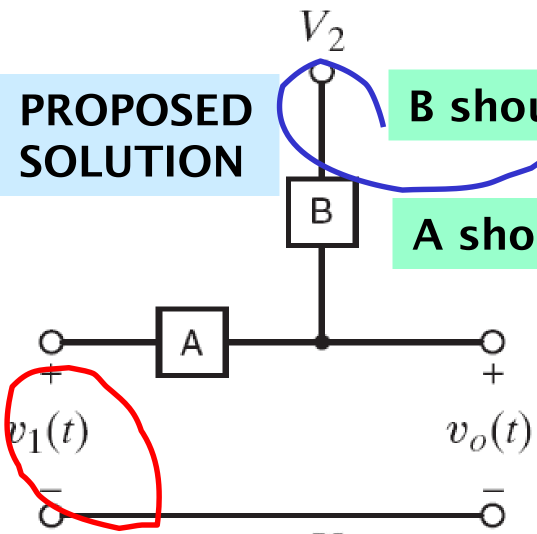
A sinusoidal signal, $v_1(t) = 2.5\cos(\omega t)$ when added to a dc level of $V_2 = 2.5$ V, provides a 0- to 5-V clock signal used to control a microprocessor. If the oscillation frequency of the signal is to be 1 GHz, let us design the appropriate circuit.

LEARNING BY DESIGN

PASSIVE SUMMING CIRCUIT - BIAS T NETWORK

$$v_o(t) = 2.5 + 2.5\cos\omega t, \quad \omega = 2\pi f; \; f = 1GHz$$

PROPOSED SOLUTION



B should have zero impedance for DC and block high frequencies

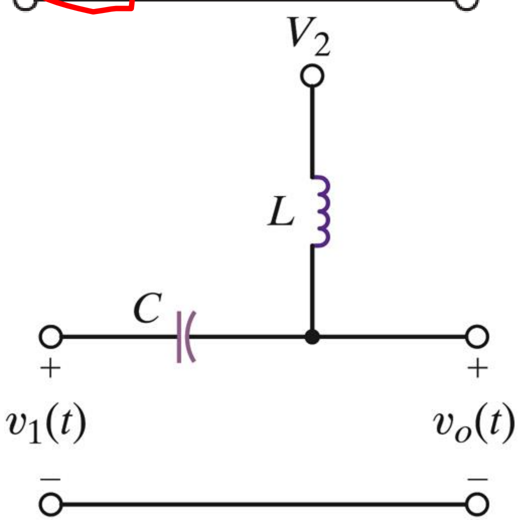
A should block DC and have very low impedance at 1GHz

AT DC THE CAPACITOR IS ALWAYS OPEN CIRCUIT
BUT AT 1GHz ONE WOULD NEED INFINITE INDUCTANCE.

JUST MAKE THE IMPEDANCES VERY DIFFERENT

PROPOSE

$$\omega C = 1; \; L\omega = 10k\Omega; \; \omega = 2\pi \times 10^9$$



$$C = \frac{1}{\omega X_C} = \frac{1}{2\pi \times 10^9} = 159 \text{ pF}$$

$$L = \frac{X_L}{\omega} = 1.59 \; \mu\text{H}$$

$$v_o(t) = 2.5 + 2.50025\cos\left[2\pi 10^9 t\right]$$

HOMEWORK I

8.4 Determine the phase angles by which $v_1(t)$ leads $i_1(t)$ and $v_1(t)$ leads $i_2(t)$, where

$$v_1(t) = 4 \sin (377t + 25°) \text{ V}$$
$$i_1(t) = 0.05 \cos (377t - 20°) \text{ A}$$
$$i_2(t) = -0.1 \sin (377t + 45°) \text{ A}$$

8.5 Calculate the current in the capacitor shown in Fig. P8.5 if the voltage input is

(a) $v_1(t) = 10 \cos (377t - 30°)$ V.

(b) $v_2(t) = 12 \sin (377t + 60°)$ V.

Give the answers in both the time and frequency domains.

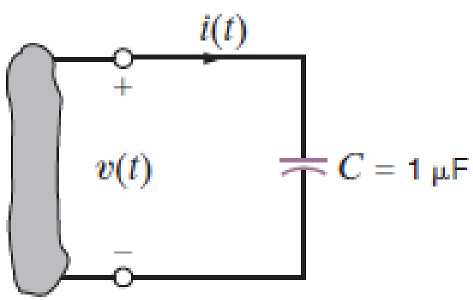


Figure P8.5

8.8 Find the equivalent admittance for the circuit in Fig. P8.8, if $\omega = 10$ radians/second.

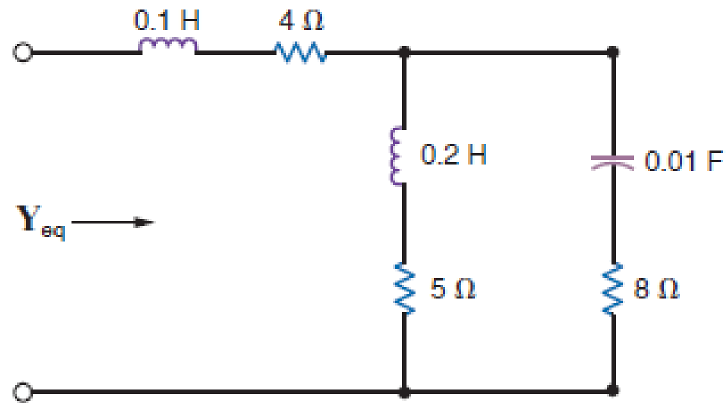


Figure P8.8

8.11 Find **Z** in the network in Fig. P8.11.

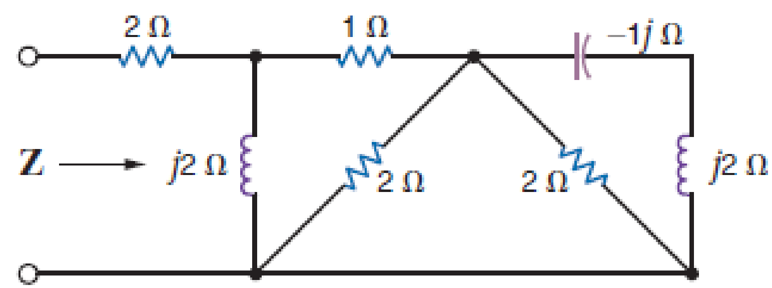


Figure P8.11

HOMEWORK I

8.18 The impedance of the network in Fig. P8.18 is found to be purely real at $f = 400$ Hz. What is the value of C?

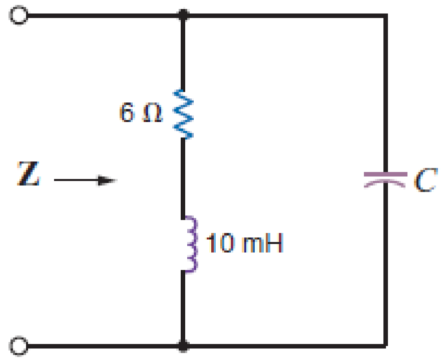


Figure P8.18

8.23 In the diagram in Fig. P8.23, $v(t) = 50\cos(10t + 10°)$ V and $i(t) = 25\cos(10t + 41°)$ A. Is the impedance of the BOX inductive or capacitive? Explain your answer.

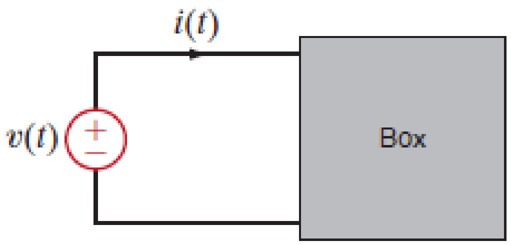


Figure P8.23

8.27 Find $v_x(t)$ and $v_R(t)$ in the circuit in Fig. P8.27 if $v(t) = 50\cos 10t$ V.

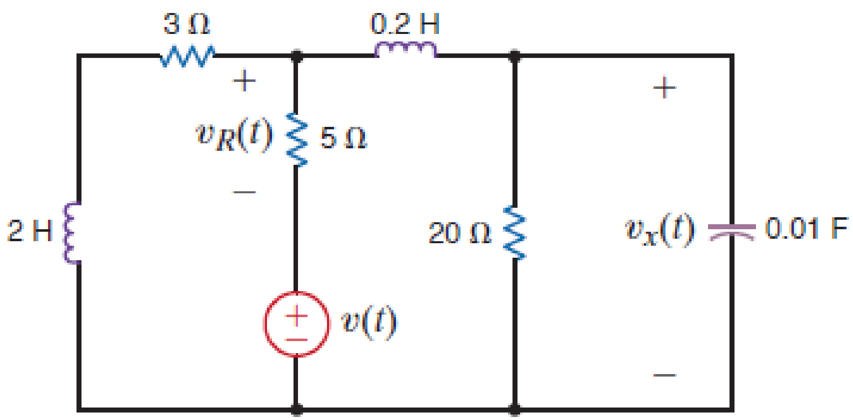


Figure P8.27

8.40 Find the voltage $\mathbf{V}_o$ shown in Fig. P8.40.

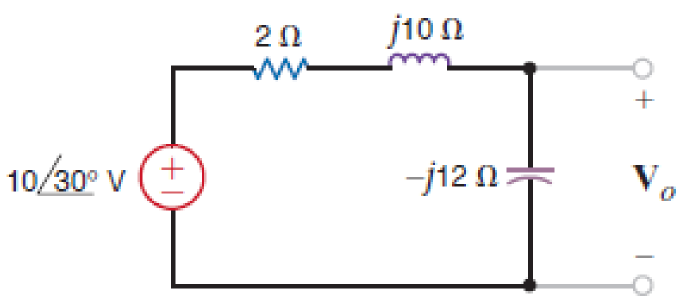


Figure P8.40

HOMEWORK I

8.51 Using nodal analysis, find $\mathbf{I}_o$ in the circuit in Fig. P8.51.

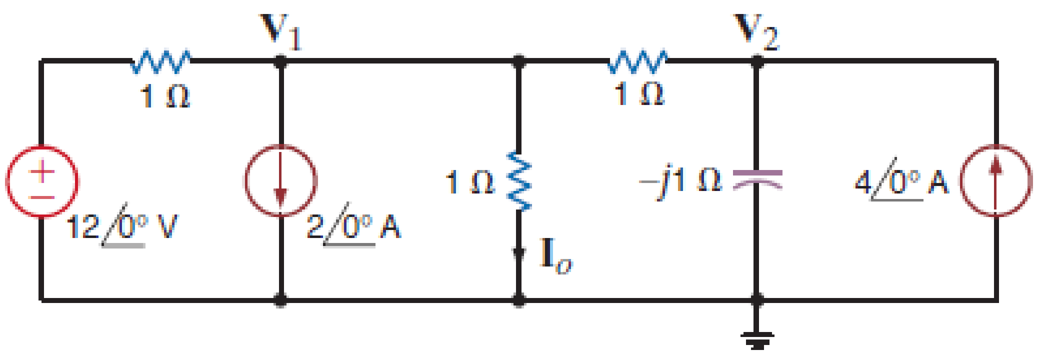


Figure P8.51

8.53 Find $\mathbf{V}_o$ in the network in Fig. P8.53 using nodal analysis.

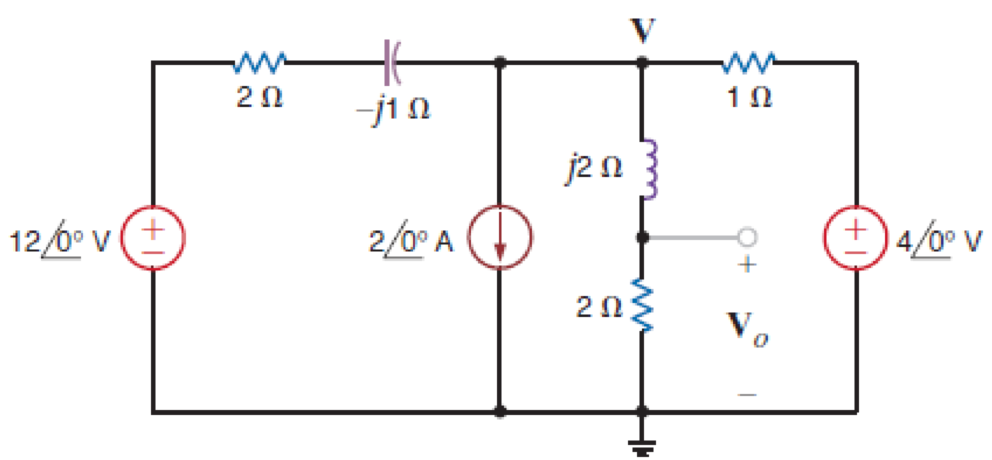


Figure P8.53

HOMEWORK I

8.71 Use loop analysis, to find $\mathbf{V}_o$ in the circuit in Fig. P8.71.

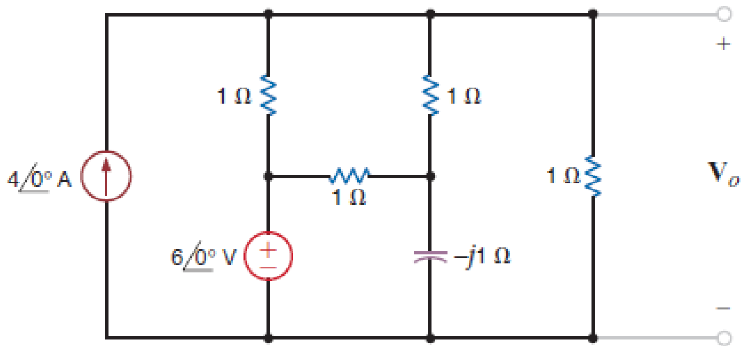


Figure P8.71

8.74 Use mesh analysis to find $\mathbf{V}_o$ in the circuit in Fig. P8.74.

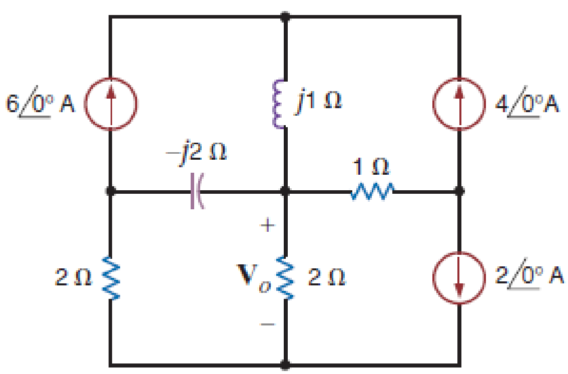


Figure P8.74

8.90 Use superposition to find $\mathbf{V}_o$ in the circuit in Fig. P8.90.

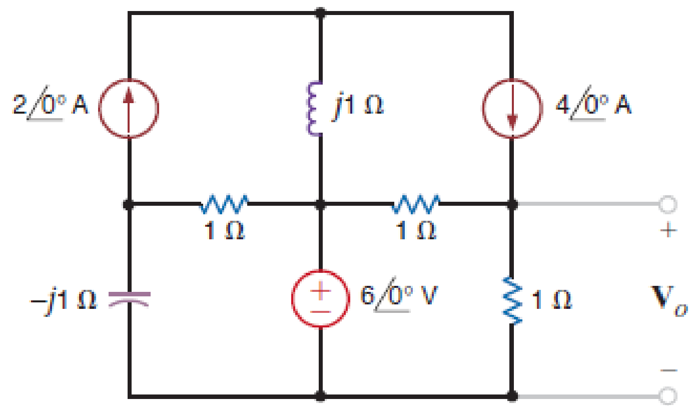


Figure P8.90

8.97 Use source transformation to find $\mathbf{V}_o$ in the circuit in Fig. P8.97.

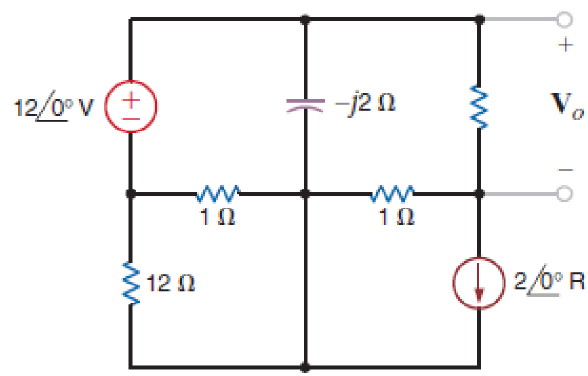


Figure P8.97

HOMEWORK I

8.105 Use Thévenin's theorem to find $\mathbf{V}_o$ in the circuit in Fig. P8.105.

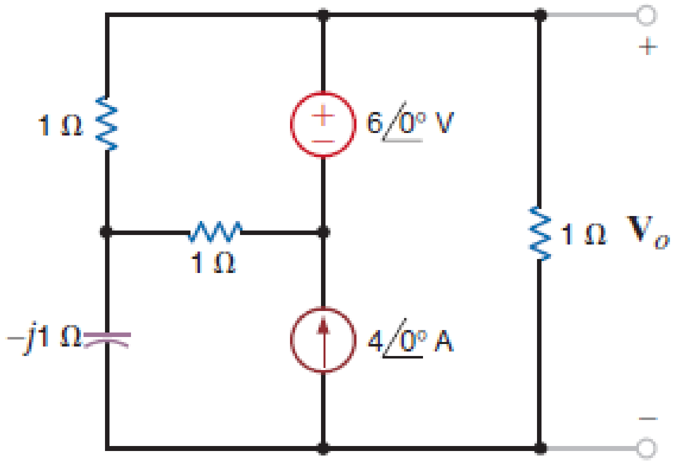


Figure P8.105

8.112 Find the Thévenin's equivalent for the network in Fig. P8.112 at terminals *A*–*B*.

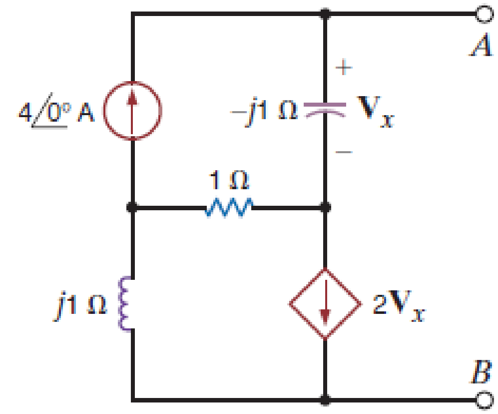


Figure P8.112

8.114 Find $\mathbf{I}_o$ in the network in Fig. P8.114 using Norton's theorem.

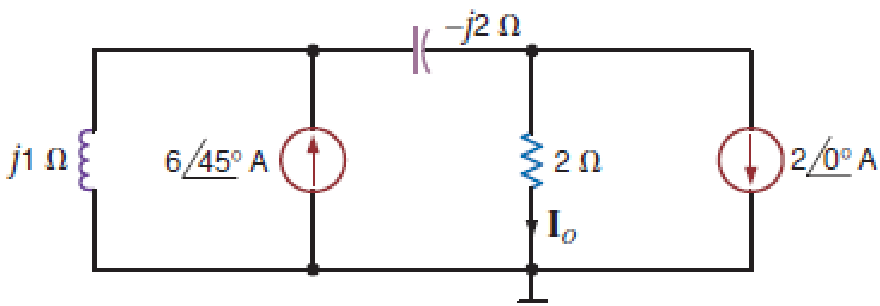


Figure P8.114

8.128 Apply Norton's theorem to find $\mathbf{V}_o$ in the network in Fig. P.8.128.

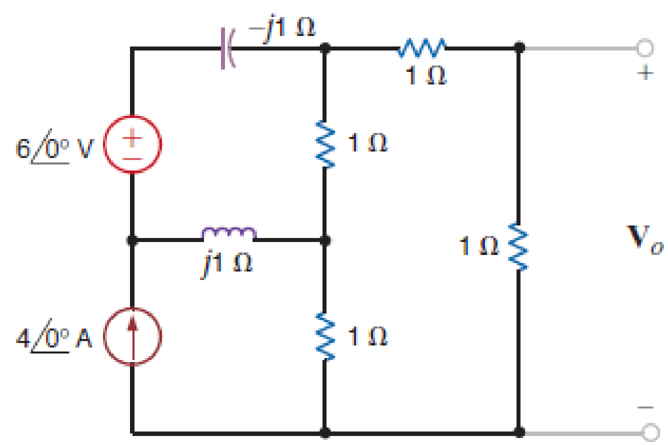


Figure P8.128