

# THE LAPLACE TRANSFORM

## LEARNING GOALS

### Definition

The transform maps a function of time into a function of a complex variable

### Two important singularity functions

The unit step and the unit impulse

### Transform pairs

Basic table with commonly used transforms

### Properties of the transform

Theorem describing properties. Many of them are useful as computational tools

### Performing the inverse transformation

By restricting attention to rational functions one can simplify the inversion process

### Convolution integral

Basic results in system analysis

### Initial and Final value theorems

Useful result relating time and s-domain behavior

## ONE-SIDED LAPLACE TRANSFORM

$$\mathcal{L}[f(t)] = \mathbf{F}(s) = \int_0^{\infty} f(t)e^{-st} dt$$

$$s = \sigma + j\omega$$

$\forall s \ni$  the integral is well defined  
( $s \in \mathbf{RoC}$ )

It will be necessary to consider  $t = 0^-$  as the lower limit

To insure uniqueness of the transform one assumes  $f(t) = 0$  for  $t < 0$

## A SUFFICIENT CONDITION FOR EXISTENCE OF LAPLACE TRANSFORM

$$\int_0^{\infty} e^{-\sigma t} |f(t)| dt < \infty$$

Transform exists for  
 $\text{Re}\{s\} + \sigma > 0$

## THE INVERSE TRANSFORM

$$\mathcal{L}^{-1}[\mathbf{F}(s)] = f(t) = \frac{1}{2\pi j} \int_{\sigma_1 - j\infty}^{\sigma_1 + j\infty} \mathbf{F}(s)e^{st} ds$$

Contour integral  
in the complex plane

Evaluating the integrals can be quite time-consuming. For this reason we develop better procedures that apply only to certain useful classes of function

## TWO SINGULARITY FUNCTIONS

### Unit step

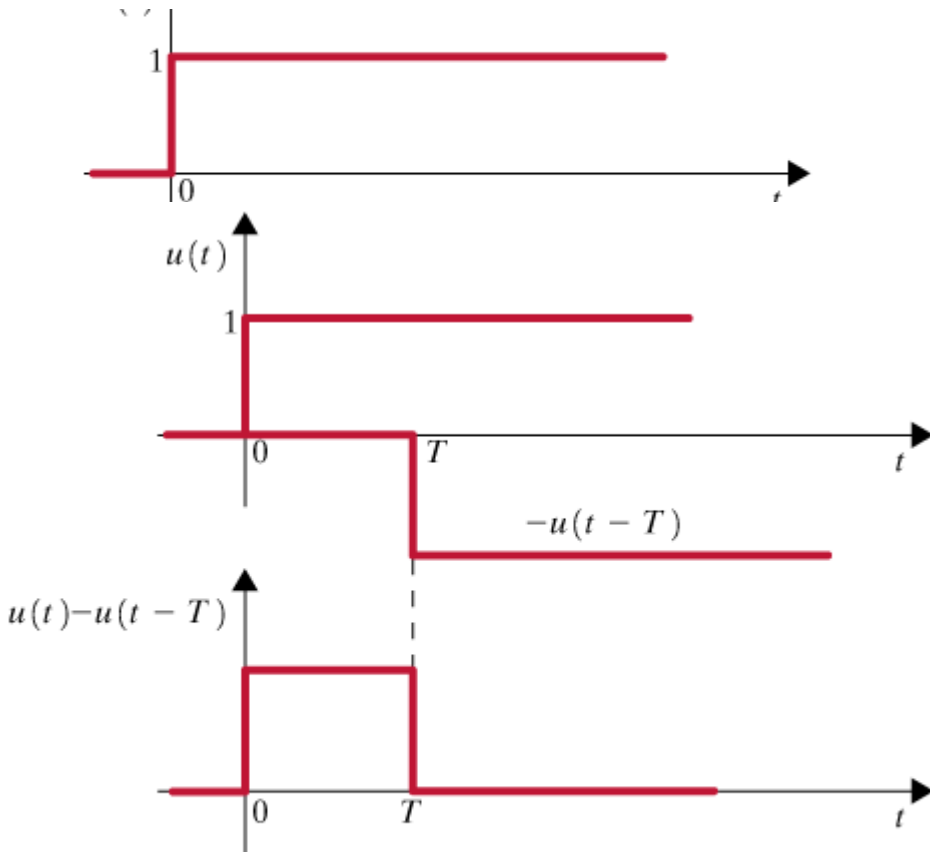
(Important “test” function in system analysis)

This function has derivative that is zero everywhere except at the origin. We will “define” a derivative for it

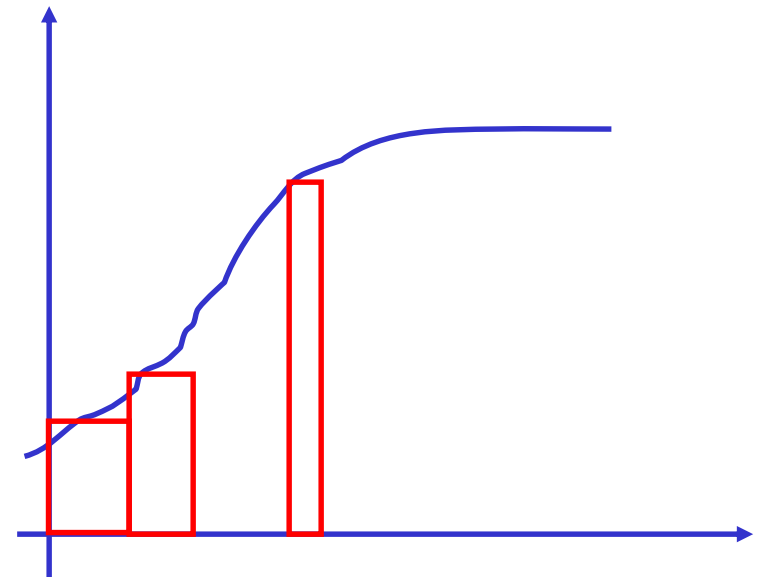
$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$

For positive time functions

$$f(t) = f(t)u(t)$$



Using the unit step to build functions



Using square pulses to approximate an arbitrary function

The narrower the pulse the better the approximation

## Computing the transform of the unit step

$$U(s) = \int_0^{\infty} 1 \times e^{-sx} dx = \lim_{T \rightarrow \infty} \int_0^T e^{-sx} dx$$

$$U(s) = \lim_{T \rightarrow \infty} \left( -\frac{1}{s} e^{-sx} \right)_0^T$$

$$U(s) = \frac{1}{s} - \lim_{T \rightarrow \infty} \frac{e^{-sT}}{s} \quad (s = \sigma + j\omega)$$

$$U(s) = \frac{1}{s} - \lim_{T \rightarrow \infty} \frac{e^{-\sigma T} e^{j\omega T}}{\sigma + j\omega}$$

$$U(s) = \frac{1}{s}; \forall s \ni \text{Re}\{s\} > 0$$

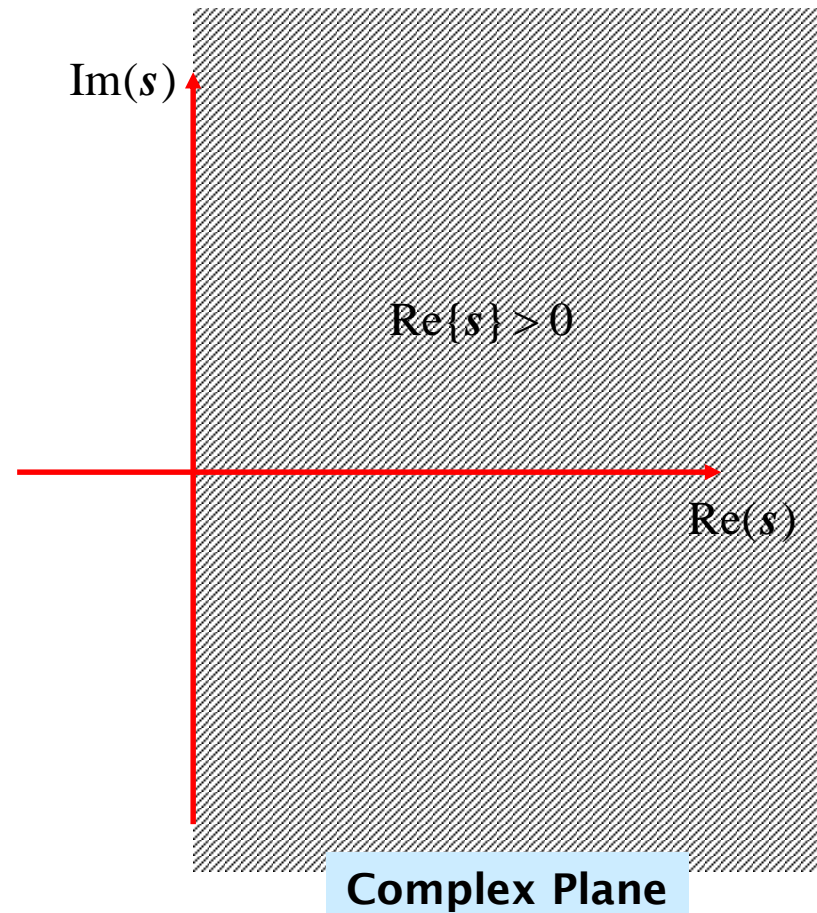
RoC

To simplify question of RoC:  
A special class of functions

$$\int_0^{\infty} e^{-\sigma t} |f(t)| dt < \infty \Rightarrow \text{RoC} \supset \{s : \text{Re}\{s\} > -\sigma\}$$

In this case the RoC is at least half a plane. And any linear combination of such signals will also have a RoC that is a half plane

## An example of Region of Convergence (RoC)



# THE IMPULSE FUNCTION

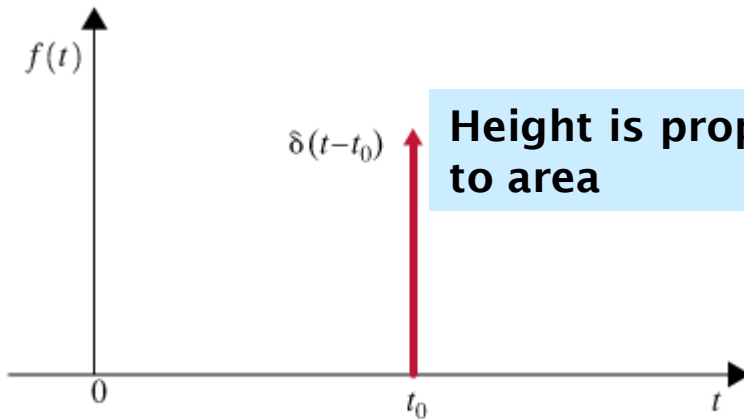
(Good model for impact, lightning, and other well known phenomena)

$$\delta(t - t_0) = 0 \quad t \neq t_0$$

$$\int_{t_0 - \varepsilon}^{t_0 + \varepsilon} \delta(t - t_0) dt = 1 \quad \varepsilon > 0$$

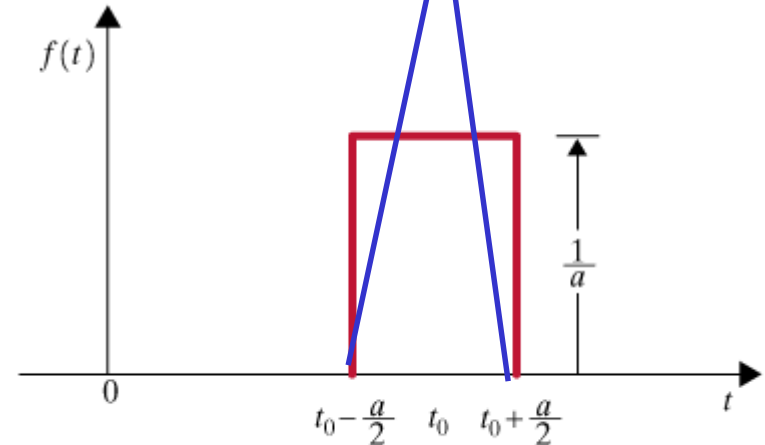
These two conditions are not feasible for “normal” functions

Approximations to the impulse



Height is proportional to area

Representation of the impulse



(a)

$$\int_{t_1}^{t_2} f(t) \delta(t - t_0) dt = \begin{cases} f(t_0) & t_1 < t_0 < t_2 \\ 0 & t_0 < t_1, t_0 > t_2 \end{cases}$$

Sifting or sampling property of the impulse

For  $t_0 = t_2$  or  $t_0 = t_1$  the integral is NOT defined

$$\mathbf{F}(s) = \int_0^{\infty} \delta(t - t_0) e^{-st} dt = e^{-st_0}$$

In order to have a valid transform for  $\delta(t)$  the lower limit is assumed  $t = 0^-$

Laplace transform

**EXAMPLE 1**

$$\int_{t_1}^{t_2} f(t) \delta(t - t_0) dt = \begin{cases} f(t_0) & t_1 < t_0 < t_2 \\ 0 & t_0 < t_1, t_0 > t_2 \end{cases}$$

Evaluate the  
integral

$$0 < t_0 = \pi < 2\pi$$

$$\int_0^{2\pi} \cos t \delta(t - \pi) dt = \cos \pi$$

Evaluate the

integral

$$t_0 = 10\pi > 2\pi$$

$$\int_0^{2\pi} e^{-t} \cos t \delta(t - 10\pi) dt = 0$$

**EXAMPLE 2**

find the Laplace transform of  $f(t) = t$ .

$$\mathbf{F}(s) = \int_0^{\infty} t e^{-st} dt = -t \frac{1}{s} e^{-st} \Big|_0^{\infty} - \int_0^{\infty} \left( -\frac{1}{s} e^{-st} dt \right) = -\frac{1}{s^2} e^{-st} \Big|_0^{\infty} = \frac{1}{s^2}$$

Integration by parts with  
 $u = t, dv = e^{-st} dt$

$$du = dt, v = -\frac{1}{s} e^{-st}$$

We will develop properties that will permit the determination of a large number of transforms from a small table of transform pairs

# Useful Properties of the Laplace Transform

$f(t)$	$F(s)$	
$Af(t)$ $f_1(t) \pm f_2(t)$	$AF(s)$ $F_1(s) \pm F_2(s)$	Linearity
$f(at)$	$\frac{1}{a} F\left(\frac{s}{a}\right), a > 0$	
$f(t - t_0)u(t - t_0), t_0 \geq 0$	$e^{-t_0 s} F(s)$	Time shifting
$f(t)u(t - t_0)$	$e^{-t_0 s} \mathcal{L}[f(t + t_0)]$	Time truncation
$e^{-at} f(t)$	$F(s + a)$	Multiplication by exponential
$\frac{d^n f(t)}{dt^n}$	$s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) \cdots s^0 f^{n-1}(0)$	
$tf(t)$	$-\frac{dF(s)}{ds}$	Multiplication by time
$\frac{f(t)}{t}$	$\int_s^\infty F(\lambda) d\lambda$	Some properties will be proved and used as efficient tools in the computation of Laplace transforms
$\int_0^t f(\lambda) d\lambda$	$\frac{1}{s} F(s)$	
$\int_0^t f_1(\lambda) f_2(t - \lambda) d\lambda$	$F_1(s) F_2(s)$	

### EXAMPLE 3

Find the transform for  $f(t) = e^{-at}$

$$F(s) = \int_0^{\infty} e^{-at} e^{-st} dt = \int_0^{\infty} e^{-(s+a)t} dt$$

$$F(s) = -\frac{1}{s+a} \bigg|_0^{\infty} = \frac{1}{s+a}$$

### Basic Table of Laplace Transforms

$f(t)$	$F(s)$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
$e^{-at}$	$\frac{1}{s+a}$

We develop properties that expand the table and allow computation of transforms without using the definition

### LINEARITY PROPERTY

$$\mathcal{L}[Af(t)] = A\mathbf{F}(s) \quad \text{Homogeneity}$$

$$\mathcal{L}[f_1(t) \pm f_2(t)] = \mathbf{F}_1(s) \pm \mathbf{F}_2(s) \quad \text{Additivity}$$

Follow immediately from the linearity properties of the integral

### APPLICATION

$$\begin{aligned} \mathbf{F}(s) &= \int_0^{\infty} \cos \omega t e^{-st} dt \\ &= \int_0^{\infty} \frac{e^{-j\omega t} + e^{j\omega t}}{2} e^{-st} dt \\ &= \frac{1}{2} \mathcal{L}[e^{-j\omega t}] + \frac{1}{2} \mathcal{L}[e^{j\omega t}] \end{aligned}$$

$$a = j\omega$$

$$a = -j\omega$$

$$\begin{aligned} F(s) &= \frac{1}{2} \frac{1}{s + j\omega} + \frac{1}{2} \frac{1}{s - j\omega} \\ &= \frac{1}{2} \frac{(s - j\omega) + (s + j\omega)}{(s + j\omega)(s - j\omega)} \\ &= \frac{s}{s^2 + \omega^2} \end{aligned}$$



With a similar use of linearity one shows

$$\mathcal{L}[\sin \omega t] = \frac{\omega}{s^2 + \omega^2}$$

Additional entries for the table

$\sin bt$

$\cos bt$

$$\frac{b}{s^2 + b^2}$$

$$\frac{s}{s^2 + b^2}$$

#### EXAMPLE 4

Find the Laplace transform for

$$x(t) = \cos(3t + \pi/3)$$

$$x(t) = \cos \pi/3 \cos 3t - \sin \pi/3 \sin 3t$$

$$X(s) = \cos \pi/3 \frac{s}{s^2 + 9} - \sin \pi/3 \frac{3}{s^2 + 9}$$

#### EXAMPLE 5

$$x(t) = 3 - \delta(t) + 3e^{-4t}$$

Application of Linearity

$$X(s) = 3\frac{1}{s} - 1 + 4\frac{1}{s+4}$$

Notice that the unit step is not shown explicitly. Hence

3 and  $3u(t)$   
are equivalent

## MULTIPLICATION BY EXPONENTIAL

$$\mathcal{L}[e^{-at}f(t)] = F(s + a)$$

$$L[e^{-at}f(t)] = \int_0^{\infty} e^{-at} f(t) e^{-st} dt = \int_0^{\infty} f(t) e^{-(s+a)t} dt$$

$F(s + a)$

### EXAMPLE 6

$$y(t) = e^{-3t} \cos(10t) \quad \alpha = -3$$

$$f(t) = \cos 10t \Rightarrow F(s) = \frac{s}{s^2 + 100} \quad (\text{From table})$$

$$y(t) = e^{-3t} f(t) \Rightarrow Y(s) = F(s + 3) = \frac{s + 3}{(s + 3)^2 + 100}$$

New entries for the table of transform pairs

$$e^{-at} \sin bt \quad \frac{b}{(s + a)^2 + b^2}$$

$$e^{-at} \cos bt \quad \frac{s + a}{(s + a)^2 + b^2}$$

## EXAMPLE 7

$$x(t) = e^{-2t} \cos(4t + \pi/3)$$

$$x(t) = e^{-2t} (\cos \pi/3 \cos 4t - \sin \pi/3 \sin 4t)$$

$$a = -2, b = 4$$

$$X(s) = \cos \pi/3 \frac{s + 2}{(s + 2)^2 + 16} - \sin \pi/3 \frac{4}{(s + 2)^2 + 16}$$

$f(t)$	$F(s)$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
$e^{-at}$	$\frac{1}{s + a}$
$\sin bt$	$\frac{b}{s^2 + b^2}$
$\cos bt$	$\frac{s}{s^2 + b^2}$
$e^{-at} \sin bt$	$\frac{b}{(s + a)^2 + b^2}$
$e^{-at} \cos bt$	$\frac{s + a}{(s + a)^2 + b^2}$

## MULTIPLICATION BY TIME

$$\mathcal{L}[y(t)] = Y(s) \implies \mathcal{L}[ty(t)] = -\frac{dY(s)}{ds}$$

### Differentiation under an integral

$$\frac{d}{ds} \left[ \int_{t_0}^{t_1} g(s, t) dt \right] = \int_{t_0}^{t_1} \frac{\partial g(s, t)}{\partial s} dt$$

$$\begin{aligned} \mathcal{L}[y(t)] = Y(s) &= \int_{0^-}^{\infty} y(t) e^{-st} dt \\ \frac{dY}{ds}(s) &= \int_{0^-}^{\infty} \frac{\partial y(t) e^{-st}}{\partial s} dt \\ &= \int_{0^-}^{\infty} (-t) y(t) e^{-st} dt \\ &= -\mathcal{L}[ty(t)] \end{aligned}$$

Remember that we consider the functions to be zero for  $t < 0$ . Hence

$$x(t) = x(t)u(t)$$

## EXAMPLE 8

Let  $u(t)$  be the unit step

Find the transform of the ramp function

$$r(t) = tu(t)$$

$$u(t) \leftrightarrow U(s) = \frac{1}{s}$$

$$tu(t) \leftrightarrow -\frac{d}{ds} \left( \frac{1}{s} \right) = \frac{1}{s^2}$$

$$t^2 u(t) \leftrightarrow -\frac{d}{ds} \left( \frac{1}{s^2} \right) = \frac{2}{s^3}$$

By successive application of the property one shows

$$t^n (u(t)) \leftrightarrow \frac{n!}{s^{n+1}}$$

This result, plus linearity, allows computation of the transform of any polynomial

## EXAMPLE 9

$$x(t) = 1 + 2t + 6t^3$$

$$X(s) = \frac{1}{s} + 2\frac{1}{s^2} + 3\frac{3!}{s^4}$$

## TIME SHIFTING PROPERTY

$$f(t)u(t) \leftrightarrow F(s) \Rightarrow f(t-t_0)u(t-t_0) \leftrightarrow e^{-st_0}F(s)$$

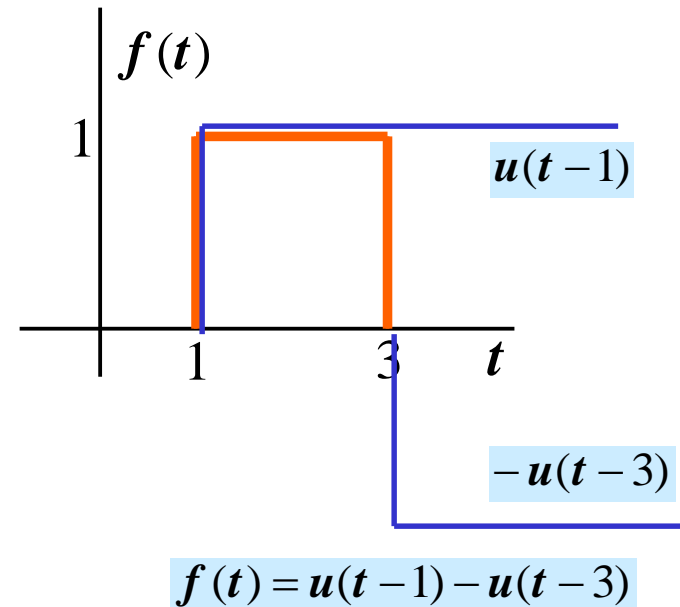
$$\begin{aligned}\mathcal{L}[f(t-t_0)u(t-t_0)] &= \int_0^{\infty} f(t-t_0)u(t-t_0)e^{-st} dt \\ &= \int_{t_0}^{\infty} f(t-t_0)e^{-st} dt\end{aligned}$$

let  $\lambda = t - t_0$  and  $d\lambda = dt$ , then

$$\begin{aligned}\mathcal{L}[f(t-t_0)u(t-t_0)] &= \int_0^{\infty} f(\lambda)e^{-s(\lambda+t_0)} d\lambda \\ &= e^{-t_0 s} \int_0^{\infty} f(\lambda)e^{-s\lambda} d\lambda \\ &= e^{-t_0 s} \mathbf{F}(s) \quad t_0 \geq 0\end{aligned}$$

## EXAMPLE 10

$$f(t) = \begin{cases} 1 & 1 \leq t \leq 3 \\ 0 & \text{elsewhere} \end{cases}$$



$$F(s) = e^{-s} \frac{1}{s} - e^{-3s} \frac{1}{s} = \frac{1}{s} (e^{-s} - e^{-3s})$$

**EXAMPLE 11**

FIND THE TRANSFORM FOR

$$f(t) = te^{-(t-1)}u(t-1) - e^{-(t-1)}u(t-1)$$

One can apply the time shifting property if the time variable always appears as it appears in the argument of the step. In this case as  $t-1$

$$f(t) = (t-1+1)e^{-(t-1)}u(t-1) - e^{-(t-1)}u(t-1)$$

$$\begin{aligned} f(t) &= (t-1)e^{-(t-1)}u(t-1) + e^{-(t-1)}u(t-1) \\ &\quad - e^{-(t-1)}u(t-1) \\ &= (t-1)e^{-(t-1)}u(t-1) \end{aligned}$$

$$tu(t) \leftrightarrow \frac{1}{s^2}$$

$$te^{-t}u(t) \leftrightarrow \frac{1}{(s+1)^2}$$

$$\therefore (t-1)e^{-(t-1)}u(t-1) \leftrightarrow \frac{e^{-s}}{(s+1)}$$

One could also write

$$f(t) = e^{-(t-1)}u(t-1)g(t)$$

And apply the time truncation property

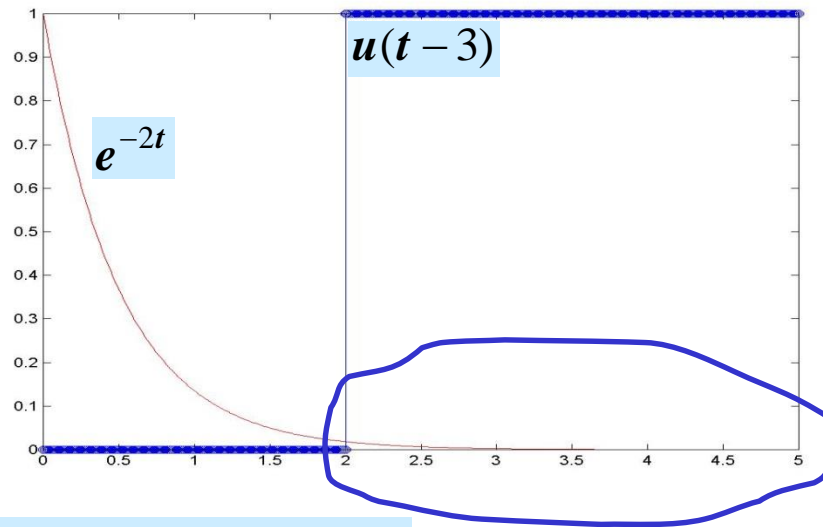
$$f(t) = g(t)u(t-1) \Rightarrow F(s) = e^{-s}\mathcal{L}[g(t+1)]$$

$$\begin{aligned} g(t+1) &= ete^{-(t+1)} = te^{-t} \\ \mathcal{L}[g(t+1)] &= \frac{1}{(s+1)^2} \end{aligned}$$

The two properties are only different representations of the same result

**EXAMPLE 12**

$$f(t) = e^{-2t}u(t-3)$$



$$\begin{aligned} f(t) &= e^{-2(t-3+3)}u(t-3) \\ &= e^{-6}e^{-2(t-3)}u(t-3) \end{aligned}$$

$$e^{-2t}u(t) \leftrightarrow \frac{1}{s+2} \Rightarrow e^{-2(t-3)}u(t-3) \leftrightarrow e^{-3s} \frac{1}{s+2}$$

$$F(s) = e^{-6} \frac{e^{-3s}}{s+2}$$

**Using time truncation**

$$f(t) = g(t)u(t-3) \Rightarrow F(s) = e^{-3s} \mathcal{L}[g(t+3)]$$

$$g(t) = e^{-2t} \Rightarrow g(t+3) = e^{-2(t+3)} = e^{-6}e^{-2t}$$

$$\mathcal{L}[g(t+3)] = e^{-6} \frac{1}{s+2}$$

**EXAMPLE 13**

$$g(t)$$

$$x(t) = \sin(2t - \pi/6)u(t-2)$$

$$x(t) = \sin(2(t-2+2) - \pi/6)u(t-2)$$

$$\theta = 4 - \pi/6$$

$$x(t) = \sin(2(t-2) + \theta)u(t-2)$$

$$\begin{aligned} x(t) &= \cos \theta \sin(2(t-2))u(t-2) \\ &\quad + \sin \theta \cos(2(t-2))u(t-2) \end{aligned}$$

$$\sin 2tu(t) \leftrightarrow \frac{2}{s^2 + 4}$$

$$\Rightarrow \sin(2(t-2))u(t-2) \leftrightarrow e^{-2s} \frac{2}{s^2 + 4}$$

$$\cos 2tu(t) \leftrightarrow \frac{s}{s^2 + 4}$$

$$\Rightarrow \cos(2(t-2))u(t-2) \leftrightarrow e^{-2s} \frac{s}{s^2 + 4}$$

$$X(s) = e^{-2s} \left( \cos \theta \frac{2}{s^2 + 4} + \sin \theta \frac{s}{s^2 + 4} \right)$$

$$X(s) = e^{-2s} \mathcal{L}[g(t+2)]$$

**EXAMPLE 14**

Compute the Laplace transform of the following functions

$$\text{A) } f(t) = e^{-4t}(t - e^{-t}) = te^{-4t} - e^{-5t} = \frac{1}{(s+4)^2} - \frac{1}{s+5}$$

$$\text{B) } g(t) = \frac{te^{-4x}}{a^2 + 4} \quad G(s) = \frac{e^{-4x}}{a^2 + 4} \mathcal{L}[t] \quad G(s) = \frac{e^{-4x}}{s^2(a^2 + 4)}$$

$$\text{C) } x(t) = \cos(bt)u(t-1) \quad X(s) = e^{-s} \mathcal{L}[\cos(b(t+1))]$$

$$\cos(b(t+1)) = \cos b \cos bt - \sin b \sin bt$$

$$\mathcal{L}[\cos(b(t+1))] = \cos b \frac{s}{s^2 + b^2} - \sin b \frac{b}{s^2 + b^2}$$

$$X(s) = e^{-s} \left( \cos b \frac{s}{s^2 + b^2} - \sin b \frac{b}{s^2 + b^2} \right)$$

**EXAMPLE 15****Compute the Laplace transform**

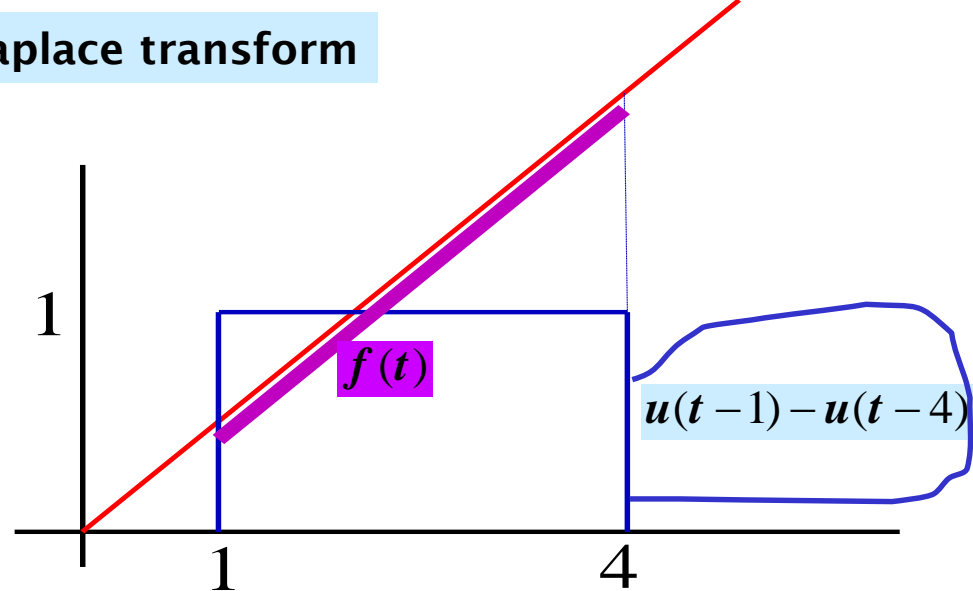
$$f(t) = \begin{cases} t & 1 \leq t \leq 4 \\ 0 & \text{elsewhere} \end{cases}$$

$$f(t) = t(u(t-1) - u(t-4))$$

$$f(t) = (t-1+1)u(t-1) - (t-4+4)u(t-4)$$

$$f(t) = (t-1)u(t-1) + u(t-1) - (t-4)u(t-4) - u(t-4)$$

$$F(s) = e^{-s} \frac{1}{s^2} + e^{-s} \frac{1}{s} - e^{-4s} \frac{1}{s^2} - e^{-4s} \frac{1}{s}$$



Using the definition

$$F(s) = \int_1^4 te^{-st} dt$$



## PERFORMING THE INVERSE TRANSFORM

**FACT:** Most of the Laplace transforms that we encounter are proper rational functions of the form

$$F(s) = \frac{P(s)}{Q(s)} = \frac{a_m s^m + a_{m-1} s^{m-1} + \dots + a_1 s + a_0}{b_n s^n + b_{n-1} s^{n-1} + \dots + b_1 s + b_0}$$

**Zeros** = roots of numerator  
**Poles** = roots of denominator

$$m \leq n$$

## KNOWN: PARTIAL FRACTION EXPANSION

If  $Q(s) = Q_1(s)Q_2(s)$  is a COPRIME factorization of the denominator with

$\deg(Q_i) = n_i$  ( $\because \sum n_i = n$ ), then

$$F(s) = K_0 + \frac{P_1(s)}{Q_1(s)} + \frac{P_2(s)}{Q_2(s)}; \deg(P_i) < n_i$$

If  $m < n$  and the poles are simple

$$\frac{P_1(s)}{Q(s)} = \frac{K_1}{s + p_1} + \frac{K_2}{s + p_2} + \dots + \frac{K_n}{s + p_n}$$

## Simple, complex conjugate poles

$$\frac{P_1(s)}{Q_1(s)(s + \alpha - j\beta)(s + \alpha + j\beta)}$$

$$= \frac{K_1}{s + \alpha - j\beta} + \frac{K_1^*}{s + \alpha + j\beta} + \dots$$

$$= \frac{C_1(s + \alpha)}{(s + \alpha)^2 + \beta^2} + \frac{C_2\beta}{(s + \alpha)^2 + \beta^2} + \dots$$

## Pole with multiplicity $r$

$$\frac{P_1(s)}{Q_1(s)(s + p_1)^r}$$

$$= \frac{K_{11}}{(s + p_1)} + \frac{K_{12}}{(s + p_1)^2} + \dots + \frac{K_{1r}}{(s + p_1)^r} + \dots$$

**THE INVERSE TRANSFORM OF EACH PARTIAL FRACTION IS IMMEDIATE. WE ONLY NEED TO COMPUTE THE VARIOUS CONSTANTS**

## SIMPLE POLES

$$\mathbf{F}(s) = \frac{\mathbf{P}(s)}{\mathbf{Q}(s)} = \frac{K_1}{s + p_1} + \frac{K_2}{s + p_2} + \cdots + \frac{K_n}{s + p_n} \quad \times / (s + p_i)$$

$$\left. \frac{(s + p_i)\mathbf{P}(s)}{\mathbf{Q}(s)} \right|_{s=-p_i} = 0 + \cdots + 0 + K_i + 0 + \cdots + 0 \quad i = 1, 2, \dots, n$$

### EXAMPLE 1

$$F(s) = \frac{12(s+1)(s+3)}{s(s+2)(s+4)(s+5)}$$

**Write the partial fraction expansion**

$$F(s) = \frac{K_1}{s} + \frac{K_2}{s+2} + \frac{K_3}{s+4} + \frac{K_4}{s+5}$$

**Determine the coefficients (residues)**

$$K_1 = sF(s)|_{s=0} = \frac{12 \times 1 \times 3}{2 \times 4 \times 5} = \frac{9}{10}$$

$$K_2 = (s+2)F(s)|_{s=-2} = \frac{12(-1)(1)}{(-2)(2)(3)} = 1$$

$$K_3 = (s+4)F(s)|_{s=-4} = \frac{12(-3)(-1)}{(-4)(-2)(1)} = \frac{36}{8}$$

$$K_4 = (s+5)F(s)|_{s=-5} = \frac{12(-4)(-2)}{(-5)(-3)(-1)} = -\frac{32}{5}$$

**Get the inverse of each term and write the final answer**

$$f(t) = \left( \frac{9}{10} + e^{-2t} + \frac{36}{8}e^{-4t} - \frac{32}{5}e^{-5t} \right) u(t)$$

**The step function is necessary to make the function zero for  $t < 0$**

**“FORM” of the inverse transform**

$$f(t) = (K_1 + K_2e^{-2t} + K_3e^{-4t} + K_4e^{-5t})u(t)$$

**EXAMPLE 2****Find the inverse transform**

1. Partial fraction
2. Residues
3. Inverse of each term

$$\text{A) } F(s) = \frac{10(s+6)}{(s+1)(s+3)} = \frac{K_1}{s+1} + \frac{K_2}{s+3}$$

**Partial fraction**

$$K_1 = (s+1)F(s)|_{s=-1} = \frac{10(-1+6)}{(-1+3)}$$

**residues****Inverse of each term**

$$f(t) = (25e^{-t} - 15e^{-3t})u(t)$$

$$K_2 = (s+3)F(s)|_{s=-3} = \frac{10(-3+6)}{(-3+1)}$$

**Makes the function zero for  $t < 0$** **Form of solution:  $f(t) = (K_1e^{-t} + K_2e^{-3t})u(t)$** 

$$\text{B) } F(s) = \frac{12(s+2)}{s(s+1)} = \frac{K_1}{s} + \frac{K_2}{s+1} \quad f(t) = (K_1 + K_2e^{-t})u(t)$$

$$K_1 = sF(s)|_{s=0} = \frac{12(2)}{1}$$

$$f(t) = (24 - 12e^{-t})u(t)$$

$$K_2 = (s+1)F(s)|_{s=-1} = \frac{12(-1+2)}{-1}$$

## COMPLEX CONJUGATE POLES

$$\mathbf{F}(s) = \frac{\mathbf{P}_1(s)}{\mathbf{Q}_1(s)(s + \alpha - j\beta)(s + \alpha + j\beta)} = \frac{K_1}{s + \alpha - j\beta} + \frac{K_1^*}{s + \alpha + j\beta} + \dots$$

$$(s + \alpha - j\beta)\mathbf{F}(s) \Big|_{s=-\alpha+j\beta} = K_1 = |K_1| \angle \theta \quad \Rightarrow \quad \mathbf{F}(s) = \frac{|K_1| \angle \theta}{s + \alpha - j\beta} + \frac{|K_1| \angle -\theta}{s + \alpha + j\beta} + \dots$$

$$= \frac{|K_1|e^{j\theta}}{s + \alpha - j\beta} + \frac{|K_1|e^{-j\theta}}{s + \alpha + j\beta} + \dots$$

$$f(t) = \mathcal{L}^{-1}[\mathbf{F}(s)] = |K_1|e^{j\theta}e^{-(\alpha-j\beta)t} + |K_1|e^{-j\theta}e^{-(\alpha+j\beta)t}$$

$$= |K_1|e^{-\alpha t}[e^{j(\beta t + \theta)} + e^{-j(\beta t + \theta)}] + \dots$$

Euler's Identity

$$\cos \phi = \frac{e^{j\phi} + e^{-j\phi}}{2}$$

$$f(t) = 2|K_1|e^{-\alpha t} \cos(\beta t + \theta) + \dots$$

## USING QUADRATIC FACTORS

$$\mathbf{F}(s) = \frac{\mathbf{P}_1(s)}{\mathbf{Q}_1(s)(s + \alpha - j\beta)(s + \alpha + j\beta)} = \frac{P_1(s)}{Q_1(s)[(s + \alpha)^2 + \beta^2]} = \frac{C_1(s + \alpha)}{(s + \alpha)^2 + \beta^2} + \frac{C_2\beta}{(s + \alpha)^2 + \beta^2} + \dots$$

$$f(t) = C_1e^{-\alpha t} \cos \beta t + C_2e^{-\alpha t} \sin \beta t + \dots$$

**Avoids using complex algebra.  
Must determine the coefficients in different way**

**The two forms are equivalent !**

**EXAMPLE 3**

$$Y(s) = \frac{10(s+2)}{s(s^2+4s+5)}$$

$$\begin{aligned} s^2 + 4s + 5 &= (s+2-j1)(s+2+j1) \\ &= (s+2)^2 + 1 \end{aligned}$$

$$Y(s) = \frac{10(s+2)}{s(s+2-j1)(s+2+j1)} = \frac{K_0}{s} + \frac{K_1}{s+2-j1} + \frac{K_1^*}{s+2+j1}$$

$$K_0 = sY(s)|_{s=0} = \frac{10(2)}{(2-j1)(2+j1)} = \frac{20}{5} = 4$$

$$K_1 = (s+2-j1)Y(s)|_{s=-2+j1} = \frac{10(j1)}{(-2+j1)(j2)} = \frac{5}{\sqrt{5} \angle 153.43^\circ}$$

$$y(t) = (4 + 2 \times 2.236 \cos(t - 2.678))u(t)$$

$$f(t) = 2|K_1|e^{-\alpha t} \cos(\beta t + \theta) + \dots$$

**MUST use radians in exponent**

$$= 2.236 \angle -153.43^\circ = 2.236e^{-j2.678}$$

**Using quadratic factors**

$$Y(s) = \frac{10(s+2)}{s(s^2+4s+5)} = \frac{C_0}{s} + \frac{C_1(s+2)}{(s+2)^2+1} + \frac{C_2}{(s+2)^2+1} = \frac{C_0((s+2)^2+1) + C_1(s+2)s + C_2s}{s(s^2+4s+5)}$$

$$\therefore 10(s+2) = C_0((s+2)^2+1) + C_1(s+2)s + C_2s$$

$$s^2: 0 = C_0 + C_1 \Rightarrow C_1 = -C_0 = -4$$

$$s: 10 = 4C_0 + 2C_1 + C_2 \Rightarrow C_2 = 2$$

$$s^0: 20 = 5C_0 \Rightarrow C_0 = 4$$

$$y(t) = (C_0 + C_1 e^{-2t} \cos t + C_2 e^{-t} \sin t)u(t)$$

**Alternative way to determine coefficients**

$$\text{For } s=0: 20 = 5C_0$$

$$\text{For } s=-2: 0 = C_0 - 2C_2$$

$$\text{For } s=-1: 10 = 2C_0 - C_1 - C_2$$

## MULTIPLE POLES

$$\mathcal{L}^{-1}\left[\frac{1}{(s+p)^n}\right] = \frac{1}{(n-1)!} t^{n-1} e^{-pt}$$

$$F(s) = \frac{\mathbf{P}_1(s)}{\mathbf{Q}_1(s)(s+p_1)^r} = \frac{K_{11}}{s+p_1} + \frac{K_{12}}{(s+p_1)^2} + \dots + \frac{K_{1r}}{(s+p_1)^r} + \dots \times / (s+p_1)^r$$

$$(s+p_1)^r F(s) \Big|_{s=-p_1} = K_{1r}$$

$$\frac{d}{ds} [(s+p_1)^r F(s)] \Big|_{s=-p_1} = K_{1r-1}$$

$$\frac{d^2}{ds^2} [(s+p_1)^r F(s)] \Big|_{s=-p_1} = (2!) K_{1r-2}$$

$$K_{1j} = \frac{1}{(r-j)!} \frac{d^{r-j}}{ds^{r-j}} [(s+p_1)^r F(s)] \Big|_{s=-p_1}$$

The method of identification of coefficients, or even the method of selecting values of  $s$ , may provide a convenient alternative for the determination of the residues

### EXAMPLE 4

$$F(s) = \frac{(s+2)^2}{s^3(s+5)} = \frac{K_{11}}{s} + \frac{K_{12}}{s^2} + \frac{K_{13}}{s^3} + \frac{K_2}{s+5} \quad K_2 = (s+5)F(s) \Big|_{s=-5} = \frac{(-3)^2}{(-5)^3}$$

$$s^3 F(s) = \frac{(s+2)^2}{s+5} = K_{11}s^2 + K_{12}s + K_{13} + K_2 \frac{s^3}{s+5} = \frac{K_{11}s^2(s+5) + K_{12}s(s+5) + K_{13}(s+5) + K_2s^3}{s+5}$$

$$s^3 : 0 = K_{11} + K_2$$

$$s^2 : 1 = 5K_{11} + K_{12}$$

$$s^1 : 2 = 5K_{12} + K_{13}$$

$$s^0 : 4 = 5K_{13}$$

For  $K_{11}$

$r=3, j=1$

For  $K_{11}$  must differentiate

one more time

$$K_{13} = s^3 F(s) \Big|_{s=0} = \frac{4}{5} \quad K_{12} = \frac{d}{ds} \left( \frac{(s+2)^2}{s+5} \right) \Big|_{s=0}$$

$$= \frac{2(s+2)(s+5) - (s+2)^2}{(s+5)^2} \Big|_{s=0}$$

**EXAMPLE 5**

$$\mathbf{F}(s) = \frac{10(s+3)}{(s+1)^3(s+2)}$$

$$\mathbf{F}(s) = \frac{10(s+3)}{(s+1)^3(s+2)} = \frac{K_{11}}{s+1} + \frac{K_{12}}{(s+1)^2} + \frac{K_{13}}{(s+1)^3} + \frac{K_2}{s+2}$$

$$f(t) = \left( K_{11}e^{-t} + K_{12}te^{-t} + K_{13}\left(\frac{1}{2}t^2e^{-t}\right) + K_2e^{-2t} \right) u(t)$$

Using identification of coefficients

$$K_2 = (s+2)\mathbf{F}(s)\Big|_{s=-2} = \frac{10(1)}{(-1)^3} = -10$$

$$K_{13} = (s+1)^3\mathbf{F}(s)\Big|_{s=-1} = \frac{10(2)}{(1)} = 20$$

$$K_{12} = \frac{d}{ds} \left( (s+1)^3 \mathbf{F}(s) \right) \Big|_{s=-1} = \frac{d}{ds} \left( \frac{10(s+3)}{s+2} \right) \Big|_{s=-1}$$

$$= \frac{10(s+2) - 10(s+3)}{(s+2)^2} \Big|_{s=-1} = \frac{-10}{(s+2)^2} \Big|_{s=-1} = -10$$

$$K_{11} = \frac{1}{2!} \frac{d^2}{ds^2} \left( (s+1)^3 \mathbf{F}(s) \right) \Big|_{s=-1} = \frac{1}{2!} \frac{d}{ds} \frac{-10}{(s+2)^2}$$

$$K_{11} = \frac{1}{2} \frac{10(2(s+2))}{(s+2)^4} \Big|_{s=-1} = \frac{10}{(s+2)^3} \Big|_{s=-1} = 10$$

$$(s+1)^3 \mathbf{F}(s) = \frac{10(s+3)}{(s+2)}$$

$$= \frac{K_{11}(s+1)^2(s+2) + K_{12}(s+1)(s+2) + K_{13}(s+2) + K_2(s+1)^3}{(s+2)}$$

$$s^3 : 0 = K_{11} + K_2$$

$$s^2 : 0 = 4K_{11} + K_{12} + 3K_2$$

$$s^1 : 10 = 5K_{11} + 3K_{12} + K_{13} + 3K_2$$

$$s^0 : 30 = 2K_{11} + 2K_{12} + 2K_{13} + K_2$$

$$K_{1j} = \frac{1}{(r-j)!} \frac{d^{r-j}}{ds^{r-j}} \left[ (s+p_1)^r \mathbf{F}(s) \right] \Big|_{s=-p_1}$$

## EXAMPLE 6

Find the inverse transform

$$F(s) = \frac{s}{(s+1)^2}$$

Partial fraction

$$F(s) = \frac{s}{(s+1)^2} = \frac{K_{11}}{s+1} + \frac{K_{12}}{(s+1)^2}$$

Form of the inverse

$$f(t) = (K_{11}e^{-t} + K_{12}te^{-t})u(t)$$

Residues

$$K_{12} = (s+1)^2 F(s) \big|_{s=-1} = -1$$

$$K_{1j} = \frac{1}{(r-j)!} \frac{d^{r-j}}{ds^{r-j}} [(s+p_1)^r F(s)] \bigg|_{s=-p_1}$$

alternatively

$$(s+1)^2 F(s) = s = K_{11}(s+1) + K_{12}$$

$$\therefore \frac{d}{ds} (s+1)^2 F(s) = 1 = K_{11}$$

$$f(t) = (e^{-t} - te^{-t})u(t)$$



**EXAMPLE 7**

Find the inverse transform

$$F(s) = \frac{(s+2)}{s^2(s+1)}$$

**Partial fraction expansion**

$$F(s) = \frac{(s+2)}{s^2(s+1)} = \frac{K_{11}}{s} + \frac{K_{12}}{s^2} + \frac{K_2}{s+1}$$

Form of the inverse

$$f(t) = (K_{11} + K_{12}t + K_2e^{-t})u(t)$$

**Residues**

$$K_2 = (s+1)F(s)|_{s=-1} = \frac{(-1+2)}{(-1)^2}$$

$$K_{12} = s^2F(s)|_{s=0} = \frac{2}{1}$$

$$s^2F(s) = \frac{s+2}{s+1} = sK_{11} + K_{12} + K_2 \frac{s^2}{(s+1)}$$

$$\frac{d}{ds}(s^2F(s))|_{s=0} = K_{11}$$

$$\frac{d}{ds}\left(\frac{s+2}{s+1}\right)\bigg|_{s=0} = \frac{(s+1)-(s+2)}{(s+1)^2}\bigg|_{s=0} = -1$$

**inverse**

$$f(t) = (-1 + 2t + e^{-t})u(t)$$

# CONVOLUTION INTEGRAL

CLAIM: Given an ODE

$$\frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \dots a_0 y = b_m \frac{d^m u}{dt^m} + \dots + b_0 u$$

there exists a function,  $h(t), t \geq 0$ , such that

$$y(t) = \int_0^t h(t-x)u(x)dx = h(t) \otimes u(t)$$

is a particular solution of the equation for  $t \geq 0$

(Actually, the zero state response)

RESULT: If  $f_1, f_2$ , are positive time functions

$$f(t) = f_1(t) \otimes f_2(t) = \int_0^t f_1(t-\lambda)f_2(\lambda) d\lambda = \int_0^t f_1(\lambda)f_2(t-\lambda) d\lambda$$

$$F(s) = F_1(s)F_2(s)$$

PROOF

$$\mathcal{L}[f(t)] = \int_0^\infty \left[ \int_0^t f_1(t-\lambda)f_2(\lambda) d\lambda \right] e^{-st} dt$$

$$\mathcal{L}[f(t)] = \int_0^\infty \left[ \int_0^\infty f_1(t-\lambda)u(t-\lambda)f_2(\lambda) d\lambda \right] e^{-st} dt$$

$$\mathcal{L}[f(t)] = \int_0^\infty f_2(\lambda) \left[ \int_0^\infty f_1(t-\lambda)u(t-\lambda)e^{-st} dt \right] d\lambda$$

**Shifting**

$$\mathcal{L}[f(t)] = \int_0^\infty f_2(\lambda) \mathbf{F}_1(s) e^{-s\lambda} d\lambda$$

$$= \mathbf{F}_1(s) \int_0^\infty f_2(\lambda) e^{-s\lambda} d\lambda$$

$$= \mathbf{F}_1(s) \mathbf{F}_2(s)$$

EXAMPLE 1 FIND  $Y(s)$

$$y(t) + \int_0^t e^{-(t-x)} y(x) dx = t; \quad t > 0$$

$$y(t) + e^{-t} \otimes y(t) = t \Rightarrow Y(s) + \frac{1}{s+2} Y(s) = \frac{1}{s^2}$$

$$\left( 1 + \frac{1}{s+2} \right) Y(s) = \frac{1}{s^2}$$

$$Y(s) = \frac{s+2}{s^2(s+2)}$$

## EXAMPLE 2

## Using convolution to determine a network response

Network function

$$H(s) = \frac{V_0(s)}{V_S(s)} = \frac{10}{s+5}$$

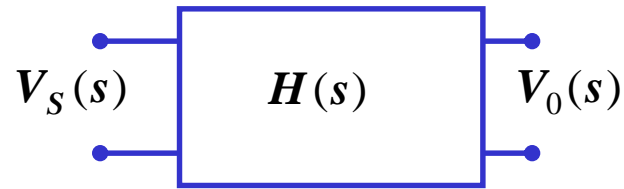
$$V_0(s) = \frac{10}{s+5} \times \frac{1}{s}$$

$$\frac{10}{s+5} \leftrightarrow 10e^{-5t}u(t)$$

$$\frac{1}{s} \leftrightarrow u(t)$$

Input

$$V_S(s) = \frac{1}{s}$$



$$V_0(s) = H(s)V_S(s)$$

$$v_0(t) = e^{-5t}u(t) \otimes u(t)$$

RESULT : If  $f_1, f_2$ , are positive time functions

$$f(t) = \int_0^t f_1(t-\lambda)f_2(\lambda)d\lambda = \int_0^t f_1(\lambda)f_2(t-\lambda)d\lambda$$

$$F(s) = F_1(s)F_2(s)$$

For  $t \geq 0$

$$v_0(t) = 10 \int_0^t e^{-5(t-\lambda)} d\lambda = 10e^{-5t} \int_0^t e^{5\lambda} d\lambda = 10e^{-5t} \left[ \frac{1}{5} e^{\lambda} \right]_0^t$$

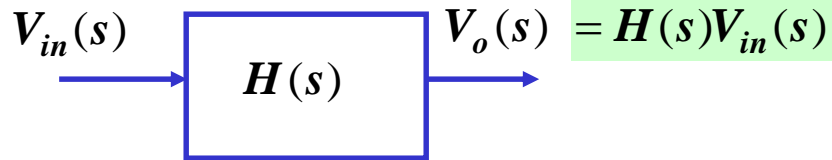
$$v_0(t) = 2e^{-5t} [e^t - 1] = 2(1 - e^{-5t}); t \geq 0$$

In general convolution is not an efficient approach to determine the output of a system. But it can be a very useful tool in special cases

### EXAMPLE 3

This example illustrates an idealized modeling approach and the use of convolution as a system simulation tool.

This slide shows how one can obtain a “black box” model for a system



Unknown linear system represented in the Laplace domain

The black box model is a description of the system based only on input/output data. There is no information on what is “inside the box”

Using the step response

$$v_{in}(t) = u(t) \Rightarrow V_{in}(s) = \frac{1}{s}, V_{os}(s) = \frac{H(s)}{s}$$

$$\therefore H(s) = sV_{os}(s) \Rightarrow h(t) = \frac{d}{dt}v_{os}(t)$$

The impulse response is the derivative of the step response of a system

Once the impulse response is obtained, the convolution can be evaluated numerically

Ideal approach to modeling

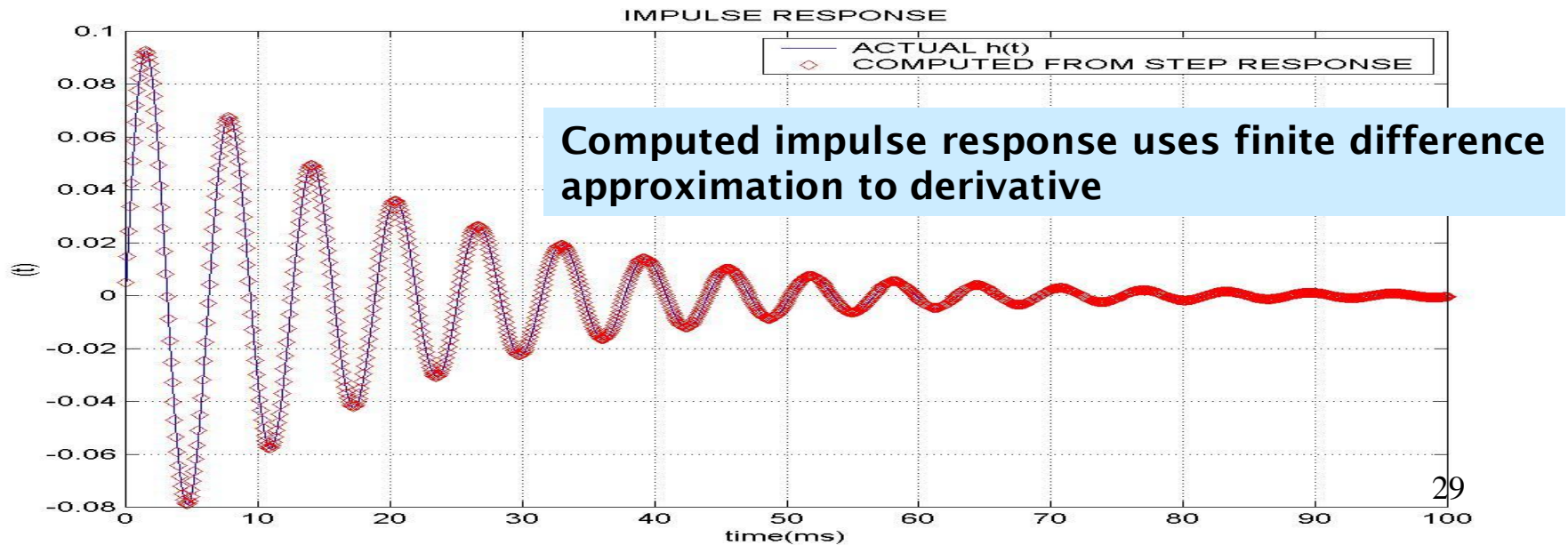
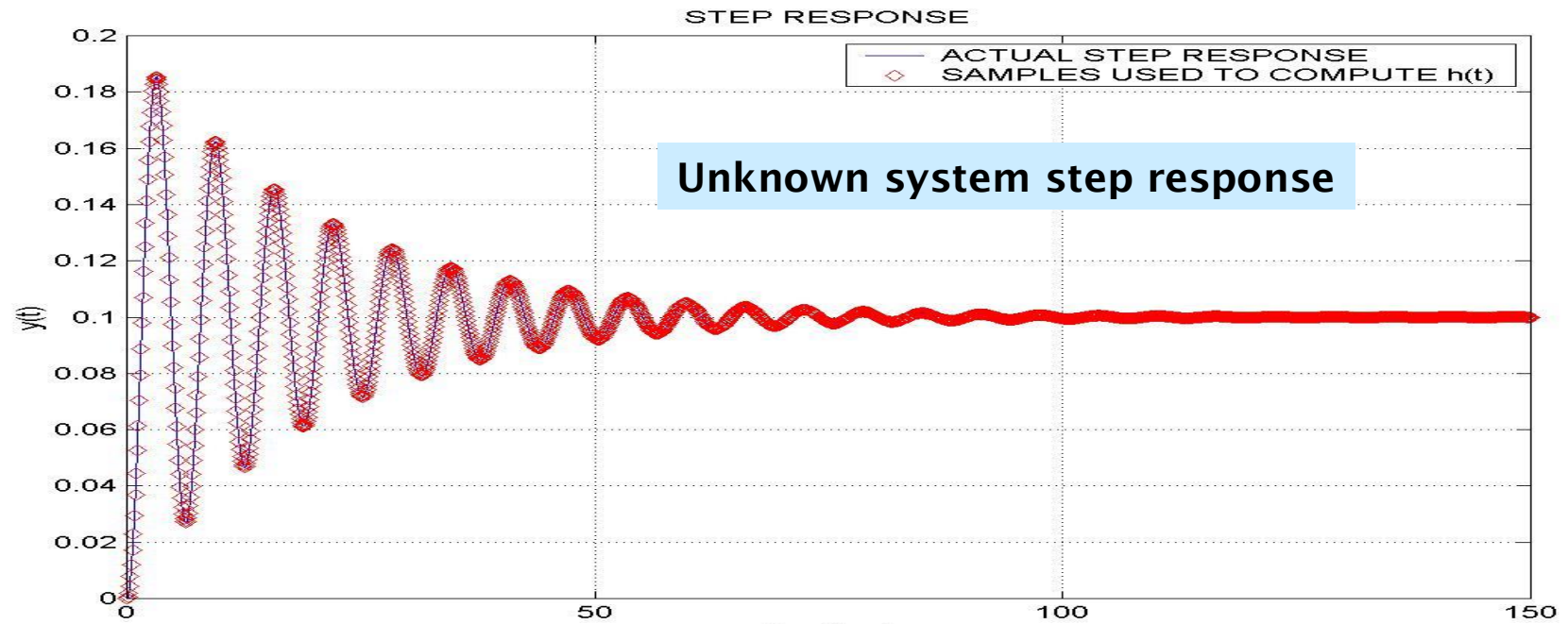
Measure the impulse response  $v_{in}(t) = \delta(t) \Rightarrow V_{in}(s) = 1,$   
 $\therefore V_o(s) = H(s), v_o(t) = h(t)$

For any other input one has

$$v_o(t) = \int_0^t h(t-x)v_{in}(x)dx$$

In practice, a good approximation to an impulse may be difficult, or impossible to apply. Hence we try to use “more sensible inputs.”

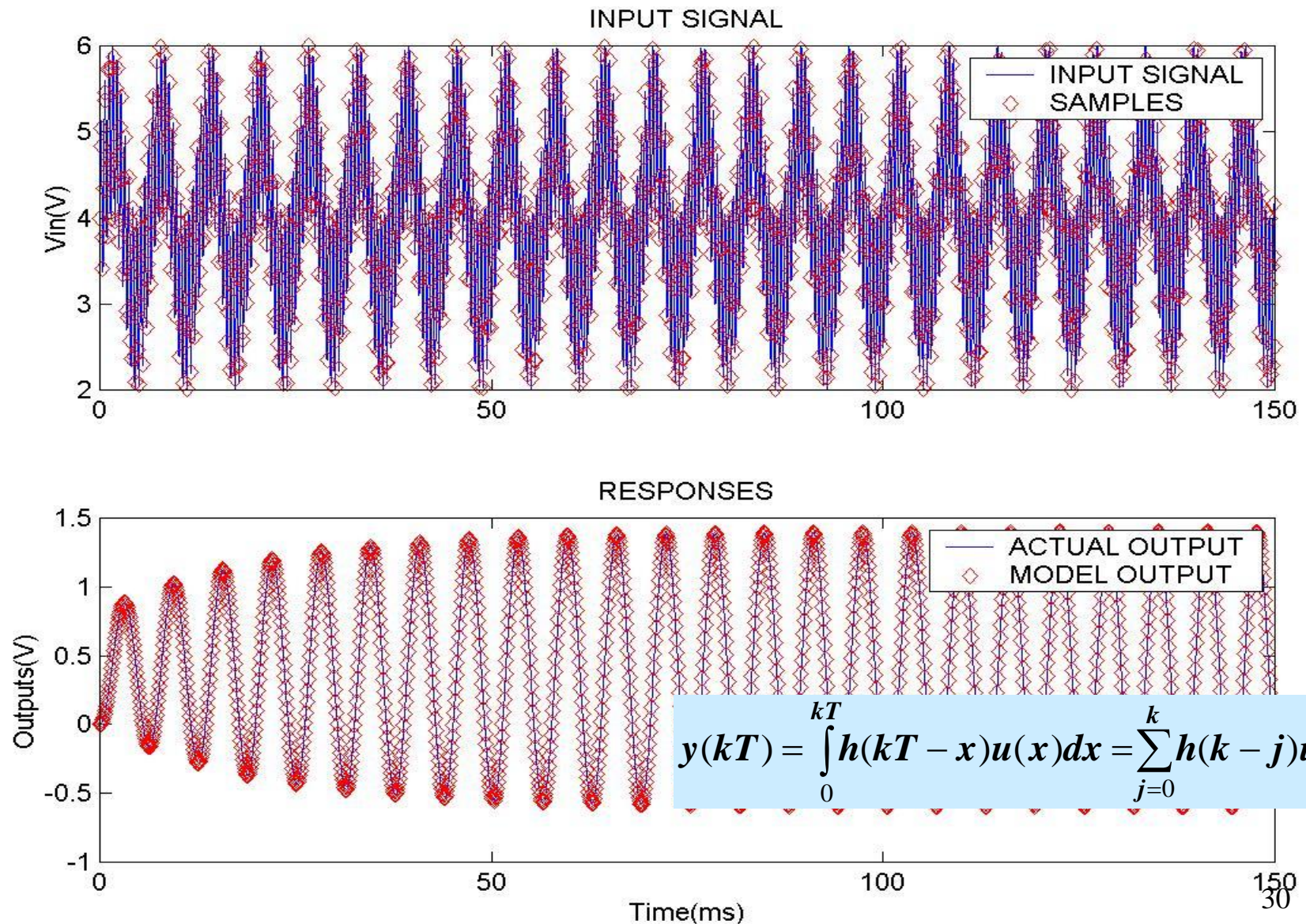
# A CASE STUDY IN MODELING



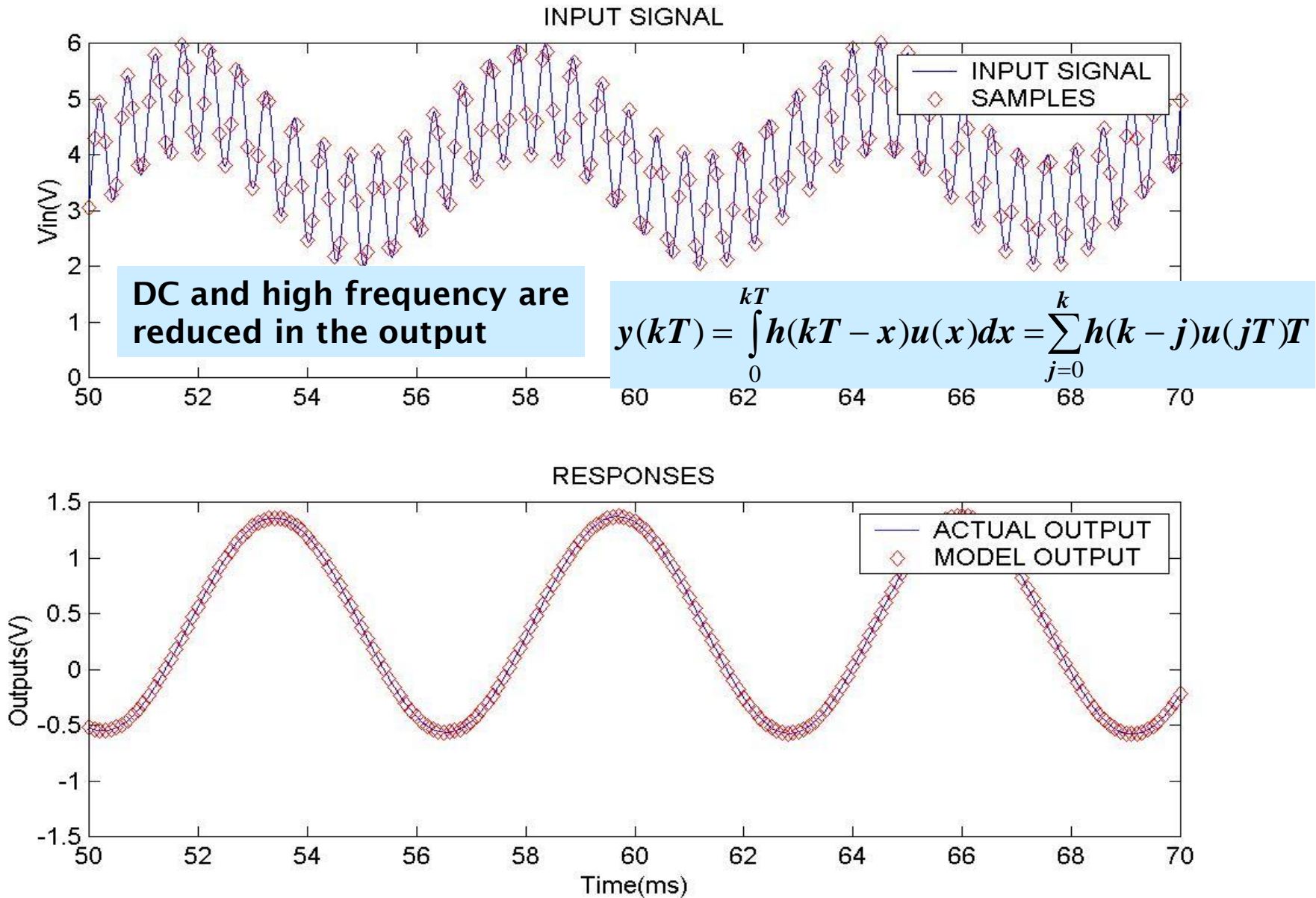


## Test of the model

The model output uses the computed impulse response and samples of the input signal. Convolution integral is evaluated numerically



## Detailed view of a segment of the signals showing bandpass action



## INITIAL AND FINAL VALUE THEOREMS

These results relate behavior of a function in the time domain with the behavior of the Laplace transform in the s-domain

### INITIAL VALUE THEOREM

Assume that both  $f(t)$ ,  $\frac{df}{dt}$ , have Laplace transform. Then

$$\lim_{t \rightarrow 0} f(t) = \lim_{s \rightarrow \infty} sF(s)$$

$$\mathcal{L}\left[\frac{df}{dt}\right] = sF(s) - f(0)$$

And if the derivative is transformable then

$$\lim_{s \rightarrow \infty} \mathcal{L}\left[\frac{df}{dt}\right] = 0$$

### FINAL VALUE THEOREM

Assume that both  $f(t)$ ,  $\frac{df}{dt}$ , have Laplace transform and that  $\lim_{t \rightarrow \infty} f(t)$  exists. Then

$$\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s)$$

$$\int_0^{\infty} \frac{df}{dt}(t) e^{-st} dt = sF(s) - f(0)$$

Taking limits as  $s \rightarrow 0$

$$\int_0^{\infty} \frac{df}{dt}(t) dt = \lim_{s \rightarrow 0} sF(s) - f(0)$$

NOTE:  $\lim_{t \rightarrow \infty} f(t)$  will exist if  $F(s)$  has poles with negative real part and at most a single pole at  $s = 0$



**EXAMPLE 1**

Given  $F(s) = \frac{10(s+1)}{s(s^2+2s+2)}$ .

Determine the initial and final values for  $f(t)$

Clearly,  $f(t)$  has Laplace transform. And  $sF(s) - f(0)$  is also defined.

$$f(0) = \lim_{s \rightarrow \infty} sF(s)$$

$$f(0) = \lim_{s \rightarrow \infty} \frac{10(s+1)}{s^2+2s+2} = 0$$

$F(s)$  has one pole at  $s=0$  and the others have negative real part. The final value theorem can be applied.

$$\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s)$$

$$\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} \frac{10(s+1)}{s^2+2s+2} = 5$$

NOTE: Computing the inverse one gets

$$f(t) = 5 + 5\sqrt{2}e^{-t} \cos\left(t - \frac{3\pi}{4}\right)$$

**EXAMPLE 2**

Given  $F(s) = \frac{(s+1)^2}{s(s+2)(s^2+2s+2)}$ .

Determine the initial and final values for  $f(t)$

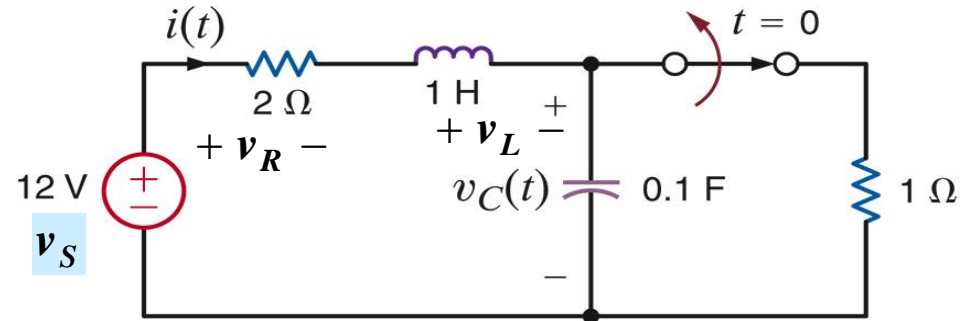
$$f(0) = \lim_{s \rightarrow \infty} \frac{(s+1)^2}{(s+2)(s^2+2s+2)} = 0$$

$$\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} \frac{(s+1)^2}{(s+2)(s^2+2s+2)} = \frac{1}{4}$$

## PROBLEM-SOLVING STRATEGY

### The Laplace Transform and Transient Circuits

- Step 1.** Assume that the circuit has reached steady state before a switch is moved. Draw the circuit valid for  $t = 0^-$  replacing capacitors with open circuits and inductors with short circuits. Solve for the initial conditions: voltages across capacitors and currents flowing through inductors. Remember  $v_C(0^-) = v_C(0^+) = v_C(0)$  and  $i_L(0^-) = i_L(0^+) = i_L(0)$ .
- Step 2.** Draw the circuit valid for  $t > 0$ . Use circuit analysis techniques to determine the differential or integrodifferential equation that describes the behavior of the circuit.
- Step 3.** Convert this differential/integrodifferential equation to an algebraic equation using the Laplace transform.
- Step 4.** Solve this algebraic equation for the variable of interest. Your result will be a ratio of polynomials in the complex variable  $s$ .
- Step 5.** Perform an inverse Laplace transform to solve for the circuit response in the time domain.

**EXAMPLE 1**FIND  $i(t), t > 0$ 

We will write the equation for  $i(t), t > 0$  and solve it using Laplace Transform

For  $t > 0$   $v_S = v_R + v_L + v_C$

$$v_S = 12u(t), t > 0$$

$$v_R = Ri(t)$$

$$v_L = L \frac{di}{dt}(t)$$

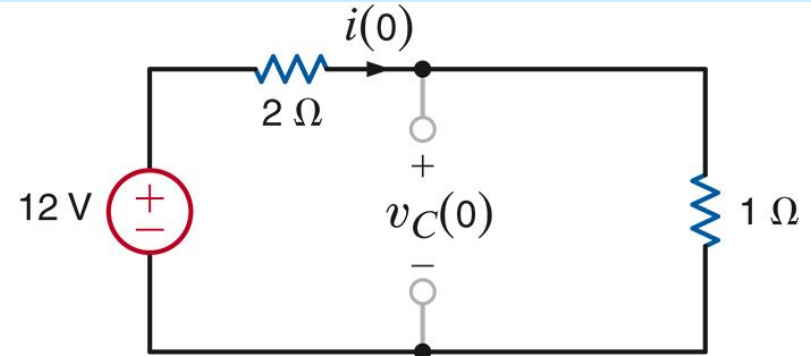
$$v_C = v_C(0) + \frac{1}{C} \int_0^t i(x) dx$$

One could write KVL in the Laplace domain and skip the time domain

$$12u(t) = 2i(t) + \frac{di}{dt}(t) + v_C(0) + \frac{1}{C} \int_0^t i(x) dx$$

$$\frac{12}{s} = 2I(s) + sI(s) - i(0) + \frac{v_C(0)}{s} + \frac{I(s)}{0.1s}$$

To find the initial conditions we use the steady state assumption for  $t < 0$



Circuit in steady state for  $t < 0$

$$i(0) = \frac{12(V)}{3\Omega} = 4(A); \quad v_C(0) = \frac{1}{1+2} \times 12(V) = 4(V)$$

$$\frac{12}{s} + 4 - \frac{4}{s} = \left( 2 + s + \frac{10}{s} \right) I(s)$$

Replace and rearrange

$$I(s) = \frac{4(s+2)}{s^2 + 2s + 10} = \frac{4(s+2)}{(s+1-j3)(s+1+j3)}$$

$$I(s) = \frac{K_1}{s+1-j3} + \frac{K^*}{s+1+j3} \quad K_1 = \left. \frac{4(s+2)}{s+1+j3} \right|_{s=-1+j3}$$

$$K_1 = \frac{4(1+j3)}{j6} = \frac{4 \times 3.16 \angle 71.57^\circ}{6 \angle 90^\circ} = 2.11 \angle -18.43^\circ$$

$$i(t) = 2 |K_1| e^{-\sigma t} \cos(\omega t + \theta)$$

$$\sigma = 1$$

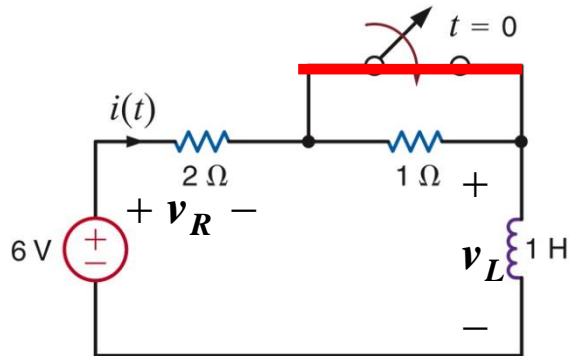
$$\omega = 3$$

$$\theta = -18.43^\circ$$

35

## EXAMPLE 2

Assuming the circuit in steady state for  $t < 0$ , determine  $i(t), t > 0$ .

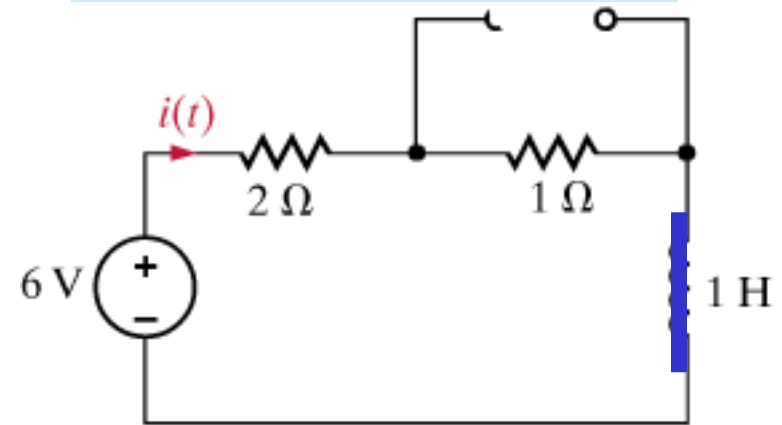


Equation for  $t > 0$   $6u(t) = 2i(t) + \frac{di}{dt}(t)$

Transforming to the Laplace domain

$$\frac{6}{s} = 2I(s) + sI(s) - i(0)$$

First we must determine  $i(0)$



Circuit in steady state for  $t < 0$

$$i(0) = \frac{6V}{3\Omega} = 2(A)$$

$$\frac{6}{s} + 2 = (s + 2)I(s)$$

$$I(s) = \frac{2(s + 3)}{s(s + 2)} = \frac{K_1}{s} + \frac{K_2}{s + 2}$$

$$K_1 = sI(s)|_{s=0} = 3 \quad K_2 = (s + 2)I(s)|_{s=-2} = -1$$

$$i(t) = 3 - e^{-2t} (A); t > 0$$