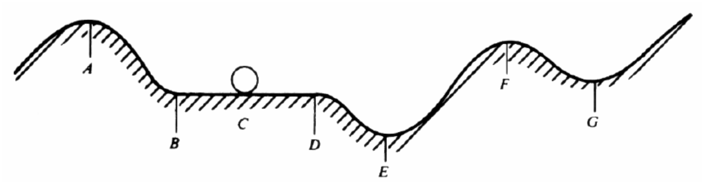
POWER SYSTEM DYNAMICS (STABILITY) AND CONTROL

Review of System Models, Dynamics and Stability Analysis Methods

Lecture Notes 2

Prof. Dr. Saffet AYASUN
Department of Electrical and Electronics Engineering
Gazi University

STABILITY CONCEPT



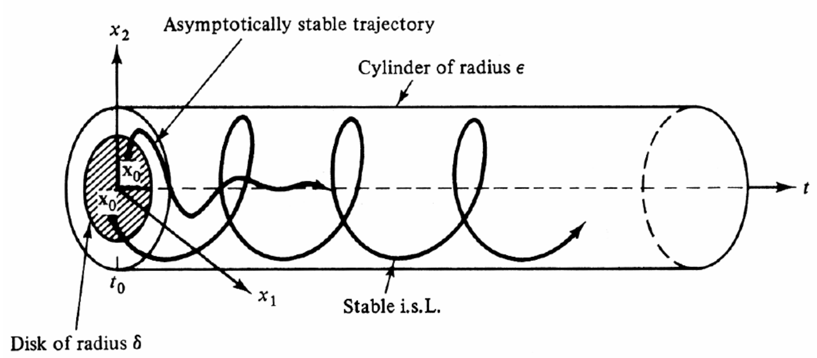
- Small perturbation
  - A and F are unstable equilibrium points
  - E and G are stable equilibrium points
  - C is a stable equilibrium point
  - B—D are neutrally stable equilibrium point

STABILITY DEFINITIONS

- Let an equilibrium point be transferred to the origin
- Definition 1: The origin is a stable equilibrium point if for any given value $\varepsilon > 0$ $\exists$ a number $\delta(\varepsilon, t_0) > 0$ s.t. if $\|\mathbf{x}(t_0)\| < \delta$, then the resultant motion $\mathbf{x}(t)$ satisfies

  $\|\mathbf{x}(t)\| < \varepsilon$ for $\forall\, t > t_0$
- (stability in the sense of Lyapunov: i.s.L.)
- Definition 2: The origin is an asymptotically stable equilibrium point if it is stable and $\exists$ a number $\delta(t_0) > 0$ s.t. whenever $\|\mathbf{x}(t_0)\| < \delta$ the resultant motion satisfies $\lim_{t\to\infty} \|\mathbf{x}(t)\| = 0$

STABILITY DEFINITIONS



NONLINEAR SYSTEMS

- State-space representation

$$\dot{x}_i = f_i(x_1, x_2, \dots, x_n; u_1, u_2, \dots, u_r; t) \qquad i = 1, 2, \dots, n$$

where n is the order of the system
r is the number of inputs

- Vector-matrix notation

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, t)$$

Nonautonomous

Where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_r \end{bmatrix} \qquad \mathbf{f} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}$$

AUTONOMOUS SYSTEMS

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}), \quad \mathbf{y} = \mathbf{g}(\mathbf{x}, \mathbf{u})$$

- where $\mathbf{y} = [y_1 \quad y_2 \quad \cdots \quad y_n]^T \qquad \mathbf{g} = [g_1 \quad g_2 \quad \cdots \quad g_r]^T$

- States:
  - minimum amount of information about the system at any instant in time that its future behavior can be determined
  - Any set of n linearly independent system variables may be used to describe the system behavior
  - The choice of states is not unique. If we define too many states, not all of them will be independent
  - When the system is not at equilibria, the system will change with time

EQUILIBRIUM POINTS

- The system is at rest (all variables are constant and unvarying with time)
- All derivatives are simultaneously zero

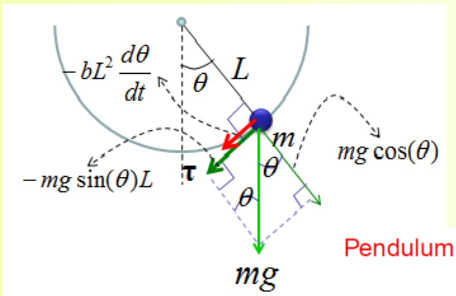
$$\mathbf{f}(\mathbf{x}_0) = 0$$

  - Where $\mathbf{x}_0$ is the state vector $\mathbf{x}$ at the equilibrium points
- Conclusions on the system stability can be drawn from equilibrium points
- Recall
  - Linear systems: the stability is entirely independent of the input and the stable state with zero input will always return to the origin
  - Nonlinear systems: the stability depends on type and magnitude of input and the initial state

STABILITY OF NONLINEAR SYSTEMS

- Local stability
  - The system is said to be *locally stable* about an equilibrium point if when subjected to small perturbation, it remains within a small region surrounding the equilibrium point
  - The system is said to be *asymptotically stable* if the system returns to the original state
- Finite stability
  - If the state of a system remains within a finite region R, it is said to be stable within R
- Global stability
  - The system is said to be globally stable if R includes the entire finite space

PENDULUM



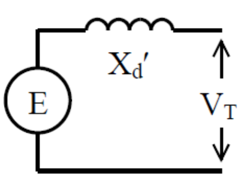
Pendulum without friction:

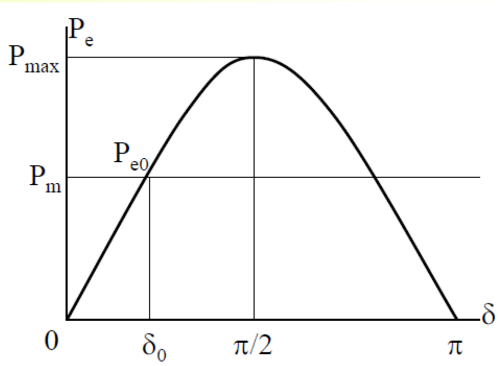
$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{g}{l}\sin x_1 \end{aligned}$$

Pendulum with torque input:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2 + \frac{1}{ml^2}T \end{aligned}$$

SYNCHRONNOUS MACHINE MODEL





power angle curve

Round Rotor Machine Model

$$E' = |E'|\angle\delta$$
$$V_G = |V_G|\angle 0°$$
$$B = 1/X_d'$$

$$P_e = |E'||V_G||B|\cos(\delta - 90°) = \frac{|E'||V_G|}{X_d'}\sin\delta = P_{max}\sin\delta$$

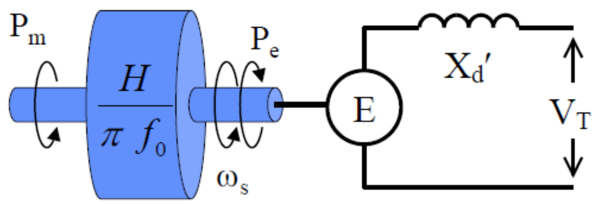
Power Systems I

SWING EQUATION

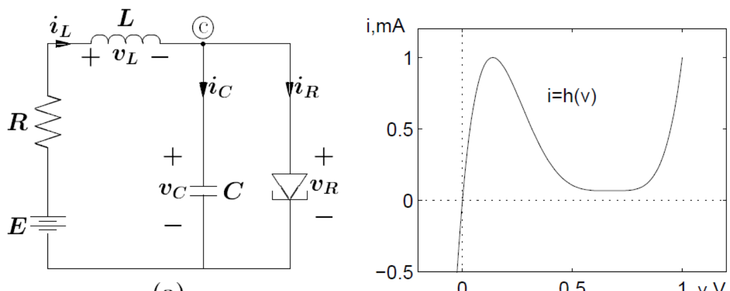
$$\frac{H}{\pi f_0}\frac{d^2\delta}{dt^2} = P_m - P_e$$ Dynamic Generator Model

$$P_e = P_{\max}\sin\delta$$ Synchronous Machine Model

$$\frac{H}{\pi f_0}\frac{d^2\delta}{dt^2} = P_m - P_{\max}\sin\delta$$ Forming the Swing Equation



TUNNEL-DIODE CIRCUIT



(a) (b)

$$i_C = C\frac{dv_C}{dt}, \qquad v_L = L\frac{di_L}{dt}$$

$$x_1 = v_C, \quad x_2 = i_L, \quad u = E$$

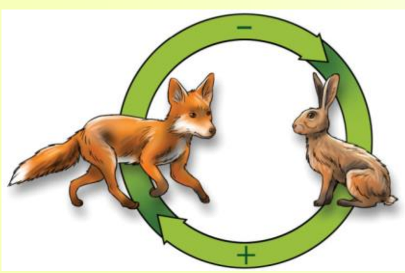
TUNNEL-DIODE CIRCUIT

$$i_C + i_R - i_L = 0 \quad \Rightarrow \quad i_C = -h(x_1) + x_2$$

$$v_C - E + Ri_L + v_L = 0 \quad \Rightarrow \quad v_L = -x_1 - Rx_2 + u$$

$$\begin{aligned} \dot{x}_1 &= \frac{1}{C}\left[-h(x_1) + x_2\right] \\ \dot{x}_2 &= \frac{1}{L}\left[-x_1 - Rx_2 + u\right] \end{aligned}$$

PREY-PREDATOR MODEL



$$\begin{aligned} \dot{x} &= ax - bxy \\ \dot{y} &= cxy - dy \end{aligned}$$

LINEARIZATION

- To investigate the small-signal performance

$$\dot{\mathbf{x}}_0 = \mathbf{f}(\mathbf{x}_0, \mathbf{u}_0) = 0$$

  - Let's perturb the system

$$\mathbf{x} = \mathbf{x}_0 + \Delta\mathbf{x} \qquad \mathbf{u} = \mathbf{u}_0 + \Delta\mathbf{u}$$

  - Hence

$$\dot{\mathbf{x}} = \dot{\mathbf{x}}_0 + \Delta\dot{\mathbf{x}} = \mathbf{f}\left[(\mathbf{x}_0 + \Delta\mathbf{x}), (\mathbf{u}_0 + \Delta\mathbf{u})\right]$$

  - Taylor's series expansion

$$\dot{x}_i = \dot{x}_{i0} + \Delta\dot{x}_i = f_i\left[(\mathbf{x}_0 + \Delta\mathbf{x}), (\mathbf{u}_0 + \Delta\mathbf{u})\right]$$

$$= f_i(\mathbf{x}_0, \mathbf{u}_0) + \frac{\partial f_i}{\partial x_1}\Delta x_1 + \ldots + \frac{\partial f_i}{\partial x_n}\Delta x_n$$

$$+ \frac{\partial f_i}{\partial u_1}\Delta u_1 + \ldots + \frac{\partial f_i}{\partial u_r}\Delta u_r$$

LINEARIZATION: LINEAR STATE-SPACE EQUATION (SSE) MODEL

- We obtain $\Delta\dot{x}_i = \frac{\partial f_i}{\partial x_1}\Delta x_1 + \ldots + \frac{\partial f_i}{\partial x_n}\Delta x_n + \frac{\partial f_i}{\partial u_1}\Delta u_1 + \ldots + \frac{\partial f_i}{\partial u_r}\Delta u_r$
- The linearized system is

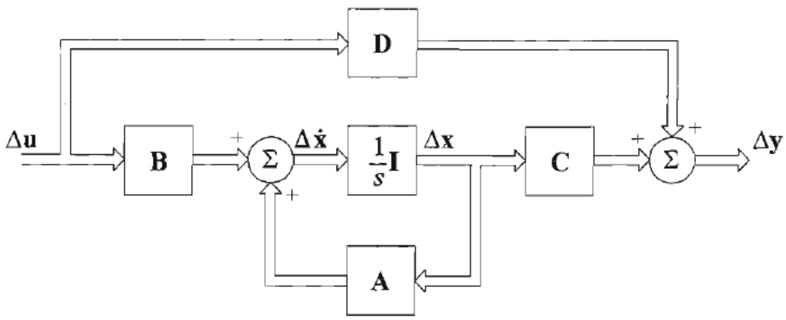
$$\Delta\dot{\mathbf{x}} = \mathbf{A}\Delta\mathbf{x} + \mathbf{B}\Delta\mathbf{u}$$

$$\Delta\mathbf{y} = \mathbf{C}\Delta\mathbf{x} + \mathbf{D}\Delta\mathbf{u}$$

- where

$$\mathbf{A} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \mathbf{B} = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_r} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial u_r} & \cdots & \frac{\partial f_n}{\partial u_r} \end{bmatrix} \mathbf{C} = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_m}{\partial x_1} & \cdots & \frac{\partial g_m}{\partial x_n} \end{bmatrix} \mathbf{D} = \begin{bmatrix} \frac{\partial g_1}{\partial u_1} & \cdots & \frac{\partial g_1}{\partial u_r} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_m}{\partial u_1} & \cdots & \frac{\partial g_m}{\partial u_r} \end{bmatrix}$$

BLOCK DIAGRAM OF SSE MODEL



LINEARIZATION: SECOND-ORDER SYSTEMS

Let $p = (p_1, p_2)$ be an equilibrium point of the system

$$\dot{x}_1 = f_1(x_1, x_2), \qquad \dot{x}_2 = f_2(x_1, x_2)$$

where f_1 and f_2 are continuously differentiable
Expand f_1 and f_2 in Taylor series about (p_1, p_2)

$$\begin{aligned}\dot{x}_1 &= f_1(p_1, p_2) + a_{11}(x_1 - p_1) + a_{12}(x_2 - p_2) + \text{H.O.T.} \\ \dot{x}_2 &= f_2(p_1, p_2) + a_{21}(x_1 - p_1) + a_{22}(x_2 - p_2) + \text{H.O.T.}\end{aligned}$$

$$a_{11} = \left.\frac{\partial f_1(x_1, x_2)}{\partial x_1}\right|_{x=p}, \qquad a_{12} = \left.\frac{\partial f_1(x_1, x_2)}{\partial x_2}\right|_{x=p}$$

$$a_{21} = \left.\frac{\partial f_2(x_1, x_2)}{\partial x_1}\right|_{x=p}, \qquad a_{22} = \left.\frac{\partial f_2(x_1, x_2)}{\partial x_2}\right|_{x=p}$$

PHASE PLANE ANALYSIS

Concept of Phase Plane Analysis:

- Phase plane method is applied to Autonomous Second Order System

$$\dot{x}_1 = f_1(x_1, x_2) \qquad \dot{x}_2 = f_2(x_1, x_2)$$

- System response $X(t) = (x_1(t), x_2(t))$ to initial condition $X_0 = \big(x_1(0), x_2(0)\big)$ is a mapping from $\mathbb{R}$(Time) to $\mathbb{R}^2(x_1, x_2)$
- The solution can be plotted in the $x_1 - x_2$ plane called ***State Plane*** or ***Phase Plane***
- The locus in the $x_1 - x_2$ plane is a curved named ***Trajectory*** that pass through point X_0
- The family of the phase plane trajectories corresponding to various initial conditions is called ***Phase portrait*** of the system.
- For a single DOF mechanical system, the phase plane is in fact is $(x, \dot{x})$ plane

SECOND-ORDER SYSTEMS

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) = f_1(x) \\ \dot{x}_2 &= f_2(x_1, x_2) = f_2(x) \end{aligned}$$

Let $x(t) = (x_1(t), x_2(t))$ be a solution that starts at initial state $x_0 = (x_{10}, x_{20})$. The locus in the x_1–x_2 plane of the solution $x(t)$ for all $t \geq 0$ is a curve that passes through the point x_0. This curve is called a *trajectory* or *orbit*

The x_1–x_2 plane is called the *state plane* or *phase plane*

The family of all trajectories is called the *phase portrait*

The *vector field* $f(x) = (f_1(x), f_2(x))$ is tangent to the trajectory at point x because

$$\frac{dx_2}{dx_1} = \frac{f_2(x)}{f_1(x)}$$

EXAMPLE 1: LINEARIZATION

1. In an ecological system, sheep and rabbit are in the same food chain and are competing each other. The population dynamics of these two species are defined by the following non-linear state-space equations:

$$\dot{x}_1 = x_1(3 - x_1 - 2x_2)$$
$$\dot{x}_2 = x_2(2 - x_1 - x_2)$$

$x_1(t)$: The population of rabbits at any time t

$x_2(t)$: The population of sheep at any time t

a) Find all equilibrium points and make some comment on these in terms of their population (10 p.)
b) For each equilibrium point, find a linear model valid around the corresponding equilibrium point and determine their types (10 p.)
c) Plot the phase portrait of the non-linear system using ***pplane*** program (5 p.)

EXAMPLE 1: LINEARIZATION

1 a) Denge noktaları:

$\dot{x}_1 = 0$ ve $\dot{x}_2 = 0$

$\dot{x}_1 = 0 \rightarrow x_1(3 - x_1 - 2x_2) = 0$

$\dot{x}_2 = 0 \rightarrow x_2(2 - x_1 - x_2) = 0$

$\rightarrow x_1 = 0$ veya $3 - x_1 - 2x_2 = 0$

$\rightarrow x_2 = 0$ veya $2 - x_1 - x_2 = 0$

Dolayısı ile (0,0) bir denge noktasıdır.

$\boxed{DN1: (x_1, x_2) = (0,0)}$

$x_1 = 0$ ve $2 - x_1 - x_2 = 0$ denklemlerinin çözümünden ikinci denge noktası bulunur.

$2 - 0 - x_2 = 0 \rightarrow x_2 = 2$

$\boxed{DN2: (0,2)}$

Benzer şekilde $x_2 = 0$ ve $3 - x_1 - 2x_2 = 0$ denkleminden üçüncü denge noktası bulunur.

$3 - x_1 - 0 = 0 \rightarrow x_1 = 3$

$\boxed{DN3: (3,0)}$

Son olarak

$3 - x_1 - 2x_2 = 0$

$2 - x_1 - x_2 = 0$

denklemlerinden dördüncü denge noktası bulunur.

$x_1 + 2x_2 = 3$

$\mp x_1 \mp x_2 = \mp 2$

$\boxed{x_2 = 1}$

$x_1 = 2 - x_2 = 2 - 1 = \boxed{1}$

$\boxed{DN4: (1,1)}$

DN1: (0,0): Her iki tür yok oluyor.

DN2 (0,2): Tavşan türü " "

DN3 (3,0): Koyun türü " "

DN4 (1,1): Her iki tür birlikte var oluyor.

EXAMPLE 1: LINEARIZATION

b) $\dot{x}_1 = 3x_1 - x_1^2 - 2x_1 x_2 = f_1$

$\dot{x}_2 = 2x_2 - x_1 x_2 - x_2^2 = f_2$

$$J(x) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}$$

$$J(x) = \begin{bmatrix} 3 - 2x_1 - 2x_2 & -2x_1 \\ -x_2 & 2 - x_1 - 2x_2 \end{bmatrix}$$

Her bir denge noktasında $J(x)$ matrisi hesaplanır.

DN1: (0,0)

$$J(0,0) = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

özdeğerler $\lambda_1 = 2, \lambda_2 = 3$

kararsız düğüm noktası

DN2 (0,2)

$$J(0,2) = \begin{bmatrix} -1 & 0 \\ -2 & -2 \end{bmatrix}$$

özdeğerler $\lambda_1 = -1, \lambda_2 = -2$

kararlı düğüm noktası

DN3 (3,0)

$$J(3,0) = \begin{bmatrix} -3 & -6 \\ 0 & -1 \end{bmatrix}$$

özdeğerler $\lambda_1 = -1, \lambda_2 = -3$

kararlı düğüm noktası

DN4 (1,1)

$$J(1,1) = \begin{bmatrix} -1 & -2 \\ -1 & -1 \end{bmatrix}$$

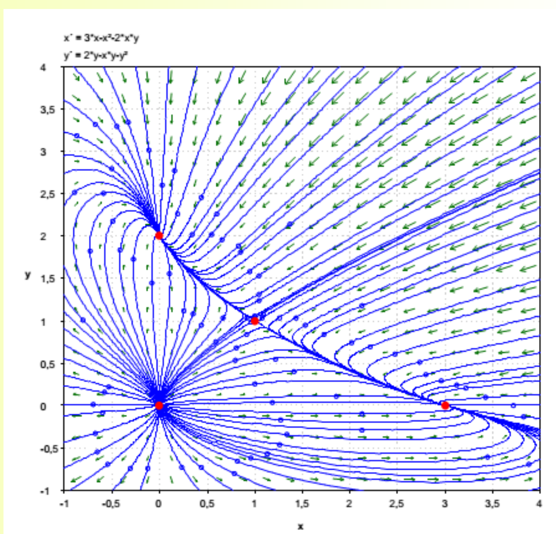
$\Delta(\lambda) = \det(\lambda I - A) = 0$

$\Delta(\lambda) = (\lambda + 1)^2 - 2 = 0$

$\Delta(\lambda) = \lambda^2 + 2\lambda - 1 = 0$

$$\lambda_{1,2} = \frac{-2 \pm \sqrt{4+4}}{2}$$

EXAMPLE 1: LINEARIZATION



EXAMPLE 2: LINEARIZATION

$$\dot{x}_1 = 2x_1x_2 - 4x_2 - 8$$
$$\dot{x}_2 = -x_1^2 + 4x_2^2$$

a) Denge noktaları (15)

$\dot{x}_1 = 0 \rightarrow 2x_1x_2 - 4x_2 - 8 = 0$

$\dot{x}_2 = 0 \rightarrow -x_1^2 + 4x_2^2 = 0 \rightarrow x_2^2 = \frac{x_1^2}{4}$ (5) $\rightarrow x_2 = \frac{x_1}{2}$, $x_2 = -\frac{x_1}{2}$

$\underline{x_2 = \frac{x_1}{2}} \rightarrow 2x_1\left(\frac{x_1}{2}\right) - 4\left(\frac{x_1}{2}\right) - 8 = 0 \rightarrow x_1^2 - 2x_1 - 8 = 0$

$x_1^2 - 2x_1 - 8 = (x_1 - 4)(x_1 + 2) = 0$

$x_1 = +4 \quad x_1 = -2$

$x_1 = +4 \rightarrow x_2 = +\frac{4}{2} = -2 \rightarrow DN1 : (+4, 2)$ (5)

$x_1 = -2 \rightarrow x_2 = \frac{-2}{2} = -1 \rightarrow DN2 : (-2, -1)$

$\underline{x_2 = -\frac{x_1}{2}} \rightarrow 2x_1\left(-\frac{x_1}{2}\right) - 4\left(-\frac{x_1}{2}\right) - 8 = 0$

$-x_1^2 + 2x_1 - 8 = 0 \rightarrow x_1^2 - 2x_1 + 8 = 0$
(Reel kökleri yoktur.) (5)

$\therefore$ Sadece (4, 2) ve (−2, −1) olmak üzere iki denge noktası

EXAMPLE 2: LINEARIZATION

b) Denge noktaları etrafında geçerli lineer modeller. (20)

$$J(x_1, x_2) = \begin{bmatrix} 2x_2 & 2x_1 - 4 \\ -2x_1 & 8x_2 \end{bmatrix}$$

$DN1 : (4, 2) \rightarrow J(4, 2) = \begin{bmatrix} 4 & 4 \\ -8 & 16 \end{bmatrix}$ (5)

$$\begin{bmatrix} \Delta\dot{x}_1 \\ \Delta\dot{x}_2 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ -8 & 16 \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix}$$

$$\Delta(\lambda) = \det[\lambda I - A] = \begin{vmatrix} \lambda - 4 & -4 \\ 8 & \lambda - 16 \end{vmatrix} = \lambda^2 - 20\lambda + 96 = (\lambda - 12)(\lambda - 8) = 0$$

$\lambda_1 = 12$ ve $\lambda_2 = 8$ kararsız düzgün olmayan düğüm. (5)
(improper node)

$\underline{DN2 : (-2, -1)}$

$J(-2, -1) = \begin{bmatrix} -2 & -8 \\ 4 & -8 \end{bmatrix}$ (5)

$$\begin{bmatrix} \Delta\dot{x}_1 \\ \Delta\dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & -8 \\ 4 & -8 \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix}$$

$$\Delta(\lambda) = \det(\lambda I - A) = \begin{vmatrix} \lambda + 2 & 8 \\ -4 & \lambda + 8 \end{vmatrix} = \lambda^2 + 10\lambda + 48$$

$\lambda_{1,2} = -5 \pm j\sqrt{23}$ Kararlı spiral (5)

EXAMPLE 3: LINEARIZATION TUNNEL-DIODE CIRCUIT

Example 2.2 The state model of a tunnel-diode circuit is given by

$$\dot{x}_1 = \frac{1}{C}[-h(x_1) + x_2], \qquad \dot{x}_2 = \frac{1}{L}[-x_1 - Rx_2 + u]$$

Assume that the circuit parameters are[7] $u = 1.2\ V$, $R = 1.5\ k\Omega = 1.5 \times 10^3\ \Omega$, $C = 2\ pF = 2 \times 10^{-12}\ F$, and $L = 5\ \mu H = 5 \times 10^{-6}\ H$. Measuring time in nanoseconds and the currents x_2 and $h(x_1)$ in mA, the state model is given by

$$\begin{aligned} \dot{x}_1 &= 0.5[-h(x_1) + x_2] \stackrel{\text{def}}{=} f_1(x) \\ \dot{x}_2 &= 0.2(-x_1 - 1.5x_2 + 1.2) \stackrel{\text{def}}{=} f_2(x) \end{aligned}$$

Suppose $h(\cdot)$ is given by

$$h(x_1) = 17.76x_1 - 103.79x_1^2 + 229.62x_1^3 - 226.31x_1^4 + 83.72x_1^5$$

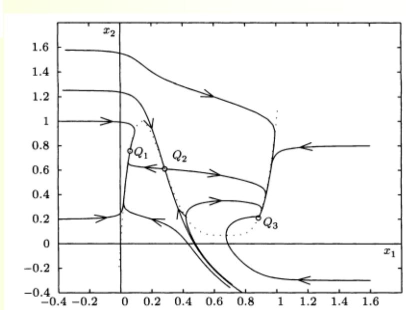
EXAMPLE 3: LINEARIZATION TUNNEL-DIODE CIRCUIT

$$\frac{\partial f}{\partial x} = \begin{bmatrix} -0.5h'(x_1) & 0.5 \\ -0.2 & -0.3 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} -3.598 & 0.5 \\ -0.2 & -0.3 \end{bmatrix}, \quad \text{Eigenvalues} : -3.57,\ -0.33$$

$$A_2 = \begin{bmatrix} 1.82 & 0.5 \\ -0.2 & -0.3 \end{bmatrix}, \quad \text{Eigenvalues} : 1.77,\ -0.25$$

$$A_3 = \begin{bmatrix} -1.427 & 0.5 \\ -0.2 & -0.3 \end{bmatrix}, \quad \text{Eigenvalues} : -1.33,\ -0.4$$

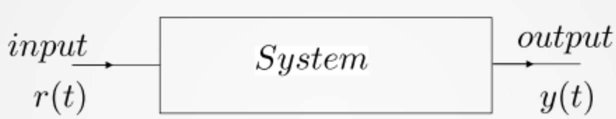


MODELING

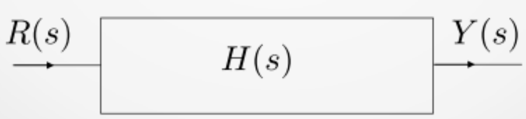
- This lecture we will concentrate on how to do system modeling based on two commonly used techniques
  - In frequency domain using Transfer Function (TF) representation
  - In time domain via using State Space representation
- Transition between the TF to SS and SS to TF will also be discussed

TRANSFER FUNCTION REPRESENTATION

Transfer functions is an Input/Output approach for system modelling



In Laplace Domain this becomes



where

$$H(s) = \frac{Y(s)}{R(s)}$$

Relating the output to the input is called the transfer function of the system

Transfer Function (TF)

For the differential equation of the form

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y^{(1)} + a_0$$
$$= b_{n-1} u^{(n-1)} + \dots + b_1 u^{(1)} + b_0 u$$

the transfer function $H(s)$ is

$$H(s) = \frac{Y(s)}{U(s)} = \frac{b_{n-1}s^{n-1} + \dots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0}$$

Note that transfer function is obtained by assuming that all the initial conditions are zero

Roots of the numerator of $H(s)$ are the zeros of $H(s)$

Roots of the denominator of $H(s)$ are the poles of $H(s)$

EXAMPLE:TRANSFER FUNCTION

$$T(s) = \frac{s+3}{s^3 + 11s^2 + 38s + 40}$$

Zeros: s=-3

Poles: s=-2,-4,-5

```
>> num=[1 3];
>> denum=[1 11 38 40];
>> roots(denum)

ans =

   -5.0000
   -4.0000
   -2.0000
```

```
>> [Z,P,K] = tf2zp(num,denum)
Z =
    -3
P =
   -5.0000
   -4.0000
   -2.0000
K =
     1
```

TF MODELS OF PHYSICAL SYSTEMS

Electrical Systems

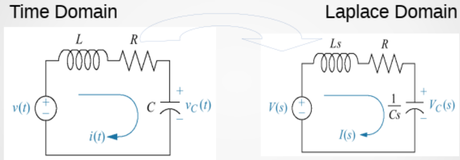
Capacitor	$v(t) = \frac{1}{C}\int_0^t i(\tau)\,d\tau$	$i(t) = C\frac{dv(t)}{dt}$	$v(t) = \frac{1}{C}q(t)$	$\frac{1}{Cs}$	Cs
Resistor	$v(t) = Ri(t)$	$i(t) = \frac{1}{R}v(t)$	$v(t) = R\frac{dq(t)}{dt}$	R	$\frac{1}{R} = G$
Inductor	$v(t) = L\frac{di(t)}{dt}$	$i(t) = \frac{1}{L}\int_0^t v(\tau)\,d\tau$	$v(t) = L\frac{d^2q(t)}{dt^2}$	Ls	$\frac{1}{Ls}$

Note: The following set of symbols and units is used throughout this book: $v(t)$ = V (volts), $i(t)$ = A (amps), $q(t)$ = Q (coulombs), C = F (farads), R = Ω (ohms), G = ℧ (mhos), L = H (henries).

ELECTRICAL SYSTEMS

Back to the basic example of RLC circuit

Time Domain　　Laplace Domain

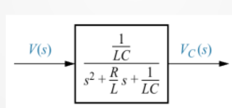


$$H(s) = \frac{V_C(s)}{V(s)} = \frac{\frac{1}{Cs}}{Ls + R + \frac{1}{Cs}}$$

After some mathematical manipulatons

$$H(s) = \frac{\frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

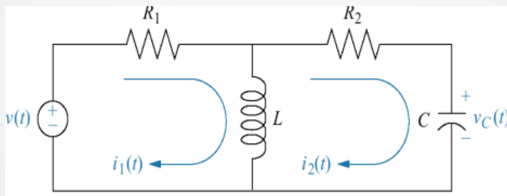
That is



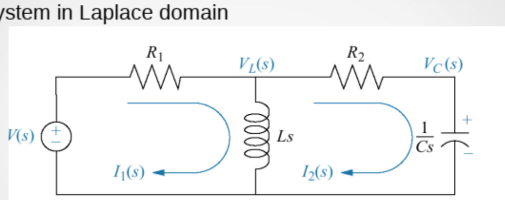
$$\frac{\frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

2-LOOP ELECTRICAL SYSTEM

Find the relation between the input voltage and the voltage across the capacitor



System in Laplace domain



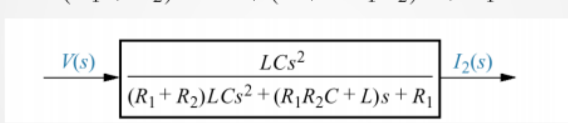
$$\begin{bmatrix} R_1 + Ls & -Ls \\ -Ls & R_2 + Ls + \frac{1}{Cs} \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} V(s) \\ 0 \end{bmatrix} \quad (1)$$

$$V_C(s) = \frac{1}{Cs} i_2 \quad (2)$$

2-LOOP ELECTRICAL SYSTEM

Solve for i_2 with respect to $V(s)$ from the mesh equation (1) and replace it in the output equation (2)

$$i_2 = \frac{CLs^2}{(R_1 + R_2)\, CLs^2 + (L + CR_1R_2)\, s + R_1} V(s)$$



$$\begin{aligned} V_C(s) &= \frac{1}{Cs} \frac{CLs^2}{(R_1 + R_2)\, CLs^2 + (L + CR_1R_2)\, s + R_1} V(s) \\ &= \frac{Ls}{(R_1 + R_2)\, CLs^2 + (L + CR_1R_2)\, s + R_1} V(s) \end{aligned}$$

Transfer function is then

$$H(s) = \frac{V_C(s)}{V(s)} = \frac{Ls}{(R_1 + R_2)\, CLs^2 + (L + CR_1R_2)\, s + R_1}$$

TRANSLATIONAL MOTION

- The cornerstone for obtaining a mathematical model, or the dynamic equations for any mechanical system is Newton's law

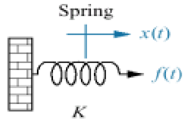
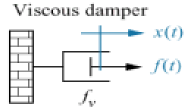
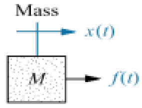
$$\mathbf{F} = m\mathbf{a},$$

F = the vector sum of all forces applied to each body in a system, newtons (N),

a = the vector acceleration of each body with respect to an **inertial reference frame** (that is, one that is neither accelerating nor rotating); often called **inertial acceleration**, m/sec^2,

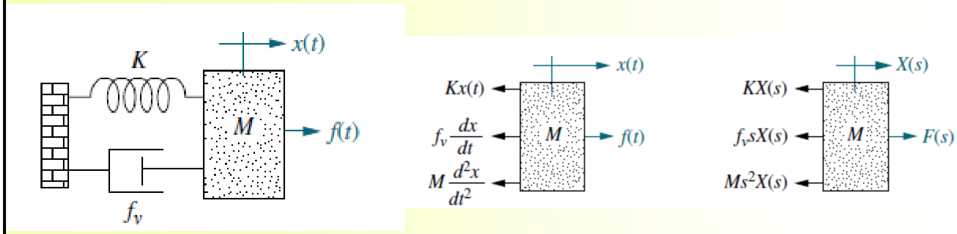
m = mass of the body, kg.

TRANSLATIONAL MOTION

Component	Force-velocity	Force-displacement	Impedance $Z_M(s) = F(s)/X(s)$
Spring 	$f(t) = K\int_0^t v(\tau)\,d\tau$	$f(t) = Kx(t)$	K
Viscous damper 	$f(t) = f_v v(t)$	$f(t) = f_v\frac{dx(t)}{dt}$	$f_v s$
Mass 	$f(t) = M\frac{dv(t)}{dt}$	$f(t) = M\frac{d^2x(t)}{dt^2}$	Ms^2

Note: The following set of symbols and units is used throughout this book: $f(t)$ = N (newtons), $x(t)$ = m (meters), $v(t)$ = m/s (meters/second), K = N/m (newtons/meter), f_v = N-s/m (newton-seconds/meter), M = kg (kilograms = newton-seconds^2/meter).

EXAMPLE: TRANSLATIONAL MECHANICAL SYSTEM



$$M\frac{d^2x(t)}{dt^2}+f_v\frac{dx(t)}{dt}+Kx(t)=f(t)$$

$$Ms^2X(s)+f_vsX(s)+KX(s)=F(s)$$

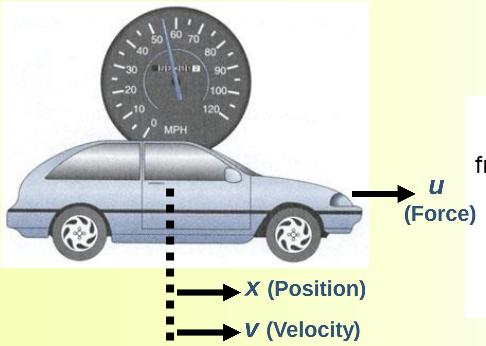
$$(Ms^2+f_vs+K)X(s)=F(s)$$

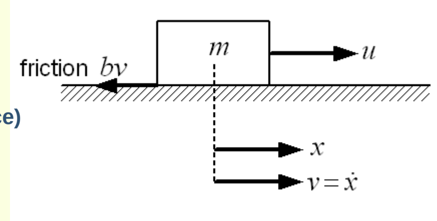
$$G(s)=\frac{X(s)}{F(s)}=\frac{1}{Ms^2+f_vs+K}$$

EXAMPLE: A SIMPLE SYSTEM-CRUISE CONTROL MODEL

Write the equations of motion for the speed and forward motion of the car shown below, assuming that the engine imparts a force *u*, and results the car velocity *v*, as shown.

Using the Laplace Transform, find the transfer function between the input *u* and the output *v*.





A Simple System: Cruise Control Model

Applying the Newton's Law for translational motion yields:

$$u - bv = ma$$

$$u - b\dot{x} = m\ddot{x}$$

$$u - bv = m\dot{v}$$

$$\dot{v} + \frac{b}{m}v = \frac{u}{m}$$

$$V(s + b/m) = U/m$$

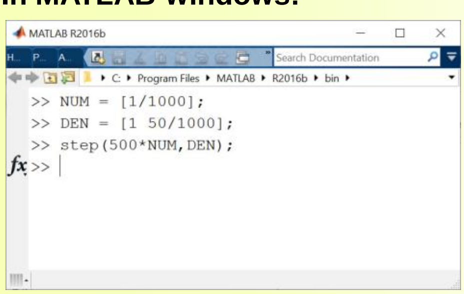
$$\boxed{\frac{V(s)}{U(s)} = \frac{1/m}{s + b/m}}$$

A Simple System: Cruise Control Model

With the parameters:

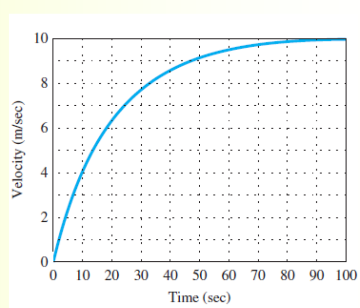
$$m = 1000 \text{ kg}$$
$$b = 50 \text{ Ns/m}$$
$$u = 500 \text{ N}$$

In MATLAB windows:



$$\boxed{\frac{V(s)}{U(s)} = \frac{1/m}{s + b/m}}$$

Response of the car velocity *v* to a step-shaped force *u*:



Rotational Motion

- Application of Newton's law to one-dimensional rotational systems requires

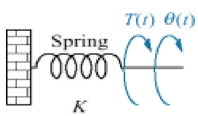
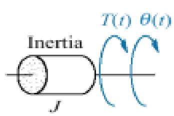
$$M = I\alpha$$

***M* =** the sum of all external moments about the center of mass of a body, N · m,

***I* =** the body's mass moment of inertia about its center of mass, kg·m^2,

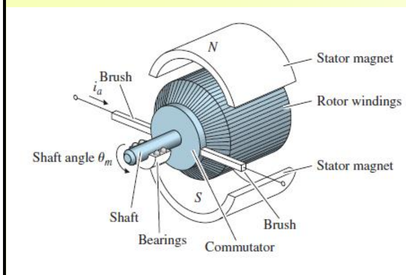
***α* =** the angular acceleration of the body, rad/sec^2

Rotational Motion

Component	Torque-angular velocity	Torque-angular displacement	Impedance $Z_M(s) = T(s)/\theta(s)$
Spring K 	$T(t) = K\int_0^t \omega(\tau)\,d\tau$	$T(t) = K\theta(t)$	K
Viscous damper D 	$T(t) = D\omega(t)$	$T(t) = D\frac{d\theta(t)}{dt}$	Ds
Inertia J 	$T(t) = J\frac{d\omega(t)}{dt}$	$T(t) = J\frac{d^2\theta(t)}{dt^2}$	Js^2

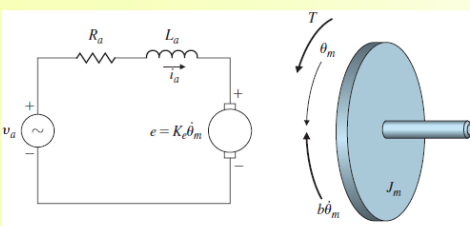
Note: The following set of symbols and units is used throughout this book: $T(t)$ = N-m (newton-meters), $\theta(t)$ = rad (radians), $\omega(t)$ = rad/s (radians/second), K = N-m/rad (newton-meters/radian), D = N-m-s/rad (newton-meters-seconds/radian), J = kg-m^2 (kilogram-meters^2 = newton-meters-seconds^2/radian).

Example: DC motor



- In addition to housing and bearings, the nonturning part (stator) has magnets, which establish a field across the rotor.
- The magnets may be electromagnets or, for small motors, permanent magnets.
- The brushes contact the rotating commutator, which causes the current always to be in the proper conductor windings so as to produce maximum torque. If the direction of the current is reversed, the direction of the torque is reversed.

Example: DC motor (cont'd)



Torque and back emf voltages:

$$T = K_t\, i_a,$$
$$e = K_e\, \dot{\theta}_m.$$

Electrical equation:

$$L_a \frac{di_a}{dt} + R_a i_a = v_a - K_e \dot{\theta}_m.$$

Newton's laws:

$$J_m \ddot{\theta}_m + b\dot{\theta}_m = K_t i_a.$$

Transfer function: $$\frac{\Theta_m(s)}{V_a(s)} = \frac{K_t}{s[(J_m s + b)(L_a s + R_a) + K_t K_e]}.$$

Show how to obtain transfer function on the board…

Example: DC motor (cont'd)

Simplified transfer function (neglecting the inductance):

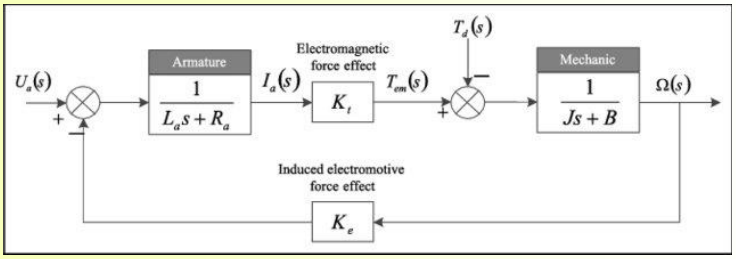
$$\frac{\Theta_m(s)}{V_a(s)} = \frac{\frac{K_t}{R_a}}{J_m s^2 + \left(b + \frac{K_t K_e}{R_a}\right) s} = \frac{K}{s(\tau s + 1)},$$

$$K = \frac{K_t}{bR_a + K_t K_e},$$
$$\tau = \frac{R_a J_m}{bR_a + K_t K_e}.$$

Transfer function between the motor input and the output speed (*ω*):

$$\frac{\Omega(s)}{V_a(s)} = s\frac{\Theta_m(s)}{V_a(s)} = \frac{K}{\tau s + 1}.$$

Example: DC motor (cont'd)



State-Space Equations

$$\text{state } x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n \qquad \text{input } u = \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix} \in \mathbb{R}^m$$

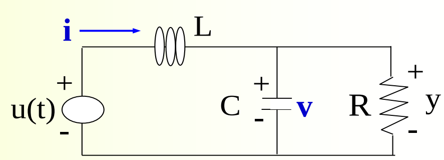
$$\text{output } y = \begin{pmatrix} y_1 \\ \vdots \\ y_p \end{pmatrix} \in \mathbb{R}^p \qquad y = Cx \qquad C - p \times n \text{ matrix}$$

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

Example: if we only care about (or can only measure) x_1, then

$$y = x_1 = \begin{pmatrix} 1 & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Example 1: Electrical Circuits



State variables?

- i and v

- **How to describe the evolution of the state variables?**

$$L\frac{di}{dt} = u - v \qquad C\frac{dv}{dt} = i - \frac{v}{R} \quad \Longrightarrow \quad \frac{di}{dt} = -\frac{1}{L}v + \frac{1}{L}u \qquad \frac{dv}{dt} = \frac{1}{C}i - \frac{v}{RC}$$

State Equation: Two first-order differential equations in terms of state variables and input

Output equation: $\mathbf{y = v} \quad = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} i \\ v \end{bmatrix} + 0u$

In matrix form:

$$\begin{bmatrix} \frac{di}{dt} \\ \frac{dv}{dt} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{RC} \end{bmatrix} \begin{bmatrix} i \\ v \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} u$$

Electrical Circuits: Steps Involved

- Steps to obtain state and output equations:

Step 1: Pick $\{i_L, v_C\}$ as state variables

Step 2:
$$L\frac{di_L}{dt} = v_L$$
$$C\frac{dv_C}{dt} = i_C$$

Express v_L and i_C in terms of state variables and input using KVL and KCL

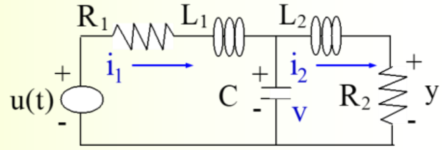
Step 3:
$$\frac{di_L}{dt} = \frac{1}{L}v_L(\text{state variables, input, nothing else})$$
$$\frac{dv_C}{dt} = \frac{1}{C}i_L(\text{state variables, input, nothing else})$$

Step 4: Put the above in matrix form

Step 5: Do the same thing for y in terms of state variables and input, and put in matrix form

Example 2

- State variables?
  - i_1, i_2, and v,



$$L_1\frac{di_1}{dt} = u - R_1 i_1 - v$$
$$L_2\frac{di_2}{dt} = v - R_2 i_2$$
$$C\frac{dv}{dt} = i_1 - i_2$$

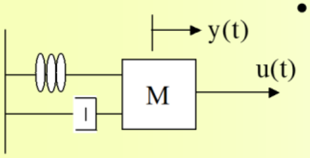
$\Longrightarrow$

$$\frac{di_1}{dt} = -\frac{R_1}{L_1}i_1 - \frac{1}{L_1}v + \frac{1}{L_1}u$$
$$\frac{di_2}{dt} = -\frac{R_2}{L_2}i_2 + \frac{1}{L_2}v$$
$$\frac{dv}{dt} = \frac{1}{C}i_1 - \frac{v}{C}i_2$$

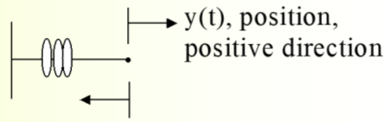
$$\begin{bmatrix} \frac{di_1}{dt} \\ \frac{di_2}{dt} \\ \frac{dv}{dt} \end{bmatrix} = \begin{bmatrix} -\frac{R_1}{L_1} & 0 & -\frac{1}{L_1} \\ 0 & -\frac{R_2}{L_2} & \frac{1}{L_2} \\ \frac{1}{C} & -\frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ v \end{bmatrix} + \begin{bmatrix} \frac{1}{L_1} \\ 0 \\ 0 \end{bmatrix} u$$

$$y = R_2 i_2 = \begin{bmatrix} 0 & R_2 & 0 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ v \end{bmatrix}$$

Mechanical Systems

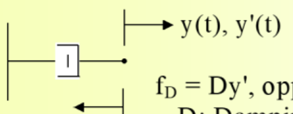


- Elements: Spring, dashpot, and mass
  - Spring:



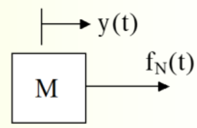
$f_S = Ky$, opposite direction
~ Hooke's law, K: Stiffness

  - Dashpot:



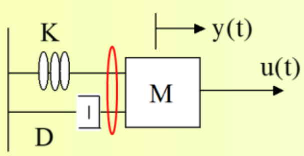
$f_D = Dy'$, opposite direction
~ D: Damping coefficient

  - Mass: M, Newton's law of motion

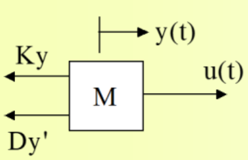


$M\ddot{y} = f_N$ ~ Net force

Mechanical Systems (cont'd)



- How to describe the system?
- Free body diagram: 1 junction (where 2 or more elements connects)



$$M\ddot{y} = u - Ky - D\dot{y} \qquad \ddot{y} + \frac{D}{M}\dot{y} + \frac{K}{M}y = \frac{1}{M}u$$

~ Input/Output description

- Number of state variables? Which ones?
  - 2 state variables: $x_1 \equiv y,\ x_2 \equiv x_1'$

$$\dot{x}_1 = x_2 \qquad \dot{x}_2 = \ddot{y} = \frac{u - Ky - D\dot{y}}{M} = \frac{u - Kx_1 - Dx_2}{M}$$

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{K}{M} & -\frac{D}{M} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix} u \qquad y = x_1 = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 0u$$

Mechanical Systems: Steps Involved

- Steps to obtain state and output equations:

  Step 1: Determine ALL junctions (where 2 or more elements are connected), and label the motion of each one

  Step 2: Draw a free body diagram for each junction to obtain the net force of that junction

  Step 3: Apply Newton's law of motion to each diagram

  Step 4: Select appropriate variables as state variables, and write the state and output equations in matrix form
- For rotational systems: $\tau = J\alpha$
  - τ: Torque = Tangential force·distance
  - J: Moment of inertia = $\int r^2 dm$
  - α: Angular acceleration
  - There are also angular spring/damper

Obtaining Transfer Function from SSE

Realisations

The realisation problem is the converse to obtaining $\mathbf{G}(s)$ from $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$. That is, it is the problem of obtaining the system state equations from its transfer matrix.

$$\begin{tikzcd}
\mathbf{G}(s) \arrow[rr, bend right, "\text{Realisation}"'] & & \{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\} \arrow[ll, bend right, "\text{Transfer Matrix Computation}"']
\end{tikzcd}$$

A transfer matrix $\mathbf{G}(s)$ is said to be realisable if there exists a finite-dimensional state equation, or simply a quadruple $\{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\}$ such that

$$\mathbf{G}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}.$$

The quadruple $\{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\}$ is then called a realisation of $\mathbf{G}(s)$.

Obtaining Transfer Function from SSE

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$
$$y = \mathbf{C}\mathbf{x} + Du$$

$$\Rightarrow \quad \frac{Y(s)}{U(s)} = G(s)$$

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}U(s)$$
$$Y(s) = \mathbf{C}\mathbf{X}(s) + DU(s)$$

$$s\mathbf{X}(s) - \mathbf{A}\mathbf{X}(s) = \mathbf{B}U(s)$$

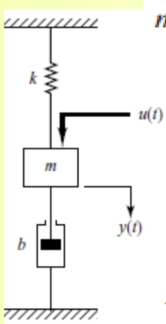
$$(s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{B}U(s)$$

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}U(s)$$

$$Y(s) = \left[\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D\right]U(s)$$

$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D$$

Example 2: Obtaining Transfer Function from SSE



$$m\ddot{y} + b\dot{y} + ky = u$$

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{k}{m}x_1 - \frac{b}{m}x_2 + \frac{1}{m}u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u \qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}, \qquad \mathbf{C} = [1 \quad 0], \qquad D = 0$$

$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D$$
$$= [1 \quad 0]\left\{\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix}\right\}^{-1} \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} + 0$$
$$= [1 \quad 0]\begin{bmatrix} s & -1 \\ \frac{k}{m} & s + \frac{b}{m} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}$$

$$G(s) = [1 \quad 0]\frac{1}{s^2 + \frac{b}{m}s + \frac{k}{m}} \begin{bmatrix} s + \frac{b}{m} & 1 \\ -\frac{k}{m} & s \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}$$
$$= \frac{1}{ms^2 + bs + k}$$

Example 3:Obtaining Transfer Function from SSE

Obtain the transfer function of the system defined by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$G(s) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} s+1 & -1 & 0 \\ 0 & s+1 & -1 \\ 0 & 0 & s+2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{s+1} & \frac{1}{(s+1)^2} & \frac{1}{(s+1)^2(s+2)} \\ 0 & \frac{1}{s+1} & \frac{1}{(s+1)(s+2)} \\ 0 & 0 & \frac{1}{s+2} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{(s+1)^2(s+2)} = \frac{1}{s^3 + 4s^2 + 5s + 2}$$

Creating Continuous-Time Models

MATLAB is quite useful to transform the system model from transfer function to state space, and vice versa

- **Transfer function (TF) models**
- **Zero-pole-gain (ZPK) models**
- **State-space (SS) models**

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}$$

$$\frac{Y(s)}{U(s)} = \frac{\text{numerator polynomial in } s}{\text{denominator polynomial in } s} = \frac{\text{num}}{\text{den}}$$

Transfer Function (TF) Model

- **SYS = tf(NUM,DEN)** creates a continuous-time transfer function SYS with numerator NUM and denominator DEN. SYS is an object of type tf when NUM,DEN are numeric arrays

```
num = [ 1 0 ]; % Numerator: s
 den = [ 1 2 10 ]; % Denominator: s^2 + 2 s + 10
H = tf(num,den)
H =
      s
--------------
s^2 + 2 s + 10
Continuous-time transfer function.
```

Transfer Function (TF) Model

- Alternatively, you can specify this model as a rational expression of the Laplace variable *s*:

```
s = tf('s');  % Create Laplace variable
H = s / (s^2 + 2*s + 10)
H =
          s
     --------------
    s^2 + 2 s + 10
```

Zero-pole-gain (ZPK) Model

- **SYS = zpk(Z,P,K)** creates a continuous-time zero pole-gain (zpk) model SYS with zeros Z, poles P, and gains K.

$$H(s) = k\frac{(s-z_1)\ldots(s-z_n)}{(s-p_1)\ldots(s-p_m)} \qquad H(s) = \frac{-2s}{(s-2)(s^2-2s+2)}$$

```
z = 0; % Zeros
 p = [ 2 1+i 1-i ]; % Poles
 k = -2; % Gain
H = zpk(z,p,k)
H =
         -2 s
--------------------
(s-2) (s^2 - 2s + 2)
```

```
s = zpk('s');
H = -2*s / (s - 2) / (s^2 - 2*s + 2)
H =
       -2 s
--------------------
(s-2) (s^2 - 2s + 2)
```

Creating State-Space Model

- **SYS = ss(A,B,C,D)** creates an object SYS representing the continuous-time state-space model

$$\frac{dx}{dt} = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

```
A = [ 0 1 ; -5 -2 ];
B = [ 0 ; 3 ];
 C = [ 1 0 ];
 D = 0;
H = ss(A,B,C,D)
```

From State-Space Model to Transfer Function Model

- **ss2tf State-space to transfer function conversion.**
  [NUM,DEN] = ss2tf(A,B,C,D,iu) calculates the transfer function:

```
          NUM(s)           -1
  H(s) = -------- = C(sI-A) B + D
          DEN(s)
```

of the system:

```
   .
   x = Ax + Bu
   y = Cx + Du
```

from the iu'th input. Vector DEN contains the coefficients of the denominator in descending powers of s. The numerator coefficients are returned in matrix NUM with as many rows as there are outputs y

From State-Space Model to Transfer Function Model: Example

```
A = [0  1  0;  0  0  1;  -5  -25  -5];
B = [0; 25; -120];
C = [1  0   0];
D = [0];
[num,den] = ss2tf(A,B,C,D)

num =

   0  0.0000  25.0000  5.0000

den

   1.0000  5.0000  25.0000  5.0000

% ***** The same result can be obtained by entering the following command:

[num,den] = ss2tf(A,B,C,D,1)

num =

   0  0.0000  25.0000  5.0000

den =

   1.0000  5.0000  25.0000  5.0000
```

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & -25 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 25 \\ -120 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\frac{Y(s)}{U(s)} = \frac{25s + 5}{s^3 + 5s^2 + 25s + 5}$$

From Transfer Function Model to State-Space Model

- tf2ss Transfer function to state-space conversion.

[A,B,C,D] = tf2ss(NUM,DEN) calculates the state-space representation:

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

of the system:

$$H(s) = \frac{NUM(s)}{DEN(s)}$$

from a single input. Vector DEN must contain the coefficients of the denominator in descending powers of s. Matrix NUM must contain the numerator coefficients with as many rows as there are outputs y. The A,B,C,D matrices are returned in controller canonical form.

From Transfer Function Model to State-Space Model-Example

```
num = [1    0];
den = [1   14   56   160];
[A,B,C,D] = tf2ss(num,den)

A =

   -14   -56  -160
     1     0     0
     0     1     0

B =

     1
     0
     0

C =

     0     1     0

D =

     0
```

$$\frac{Y(s)}{U(s)} = \frac{s}{(s+10)(s^2+4s+16)}$$

$$= \frac{s}{s^3+14s^2+56s+160}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -14 & -56 & -160 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = [0 \quad 1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [0]u$$

Stability Definitions

$$u \longrightarrow \boxed{h} \longrightarrow y$$

One reasonable definition is as follows:

A linear time-invariant system is *Bounded-Input, Bounded-Output* (BIBO) *stable* provided either one of the following three equivalent conditions is satisfied:

1. If every bounded input $u(t)$ results in a bounded output $y(t)$, regardless of initial conditions.
2. If the impulse response $h(t)$ is absolutely integrable:

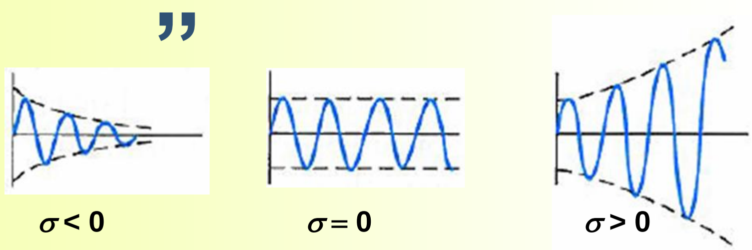
$$\int_{-\infty}^{\infty} |h(t)| \, \mathrm{d}t < \infty.$$

3. If all poles of the transfer function $H(s)$ are *strictly stable* (lie in open LHP).

Stability

- Consider the linear time-invariant system (LTI system). For those systems, the following condition for stability applies:

> “A linear time-invariant system is said to be stable if all the roots of the transfer function denominator polynomial have negative real parts (*i.e.*, they are all in the left half of *s*-plane) and is unstable otherwise.”



$\sigma < 0$ $\sigma = 0$ $\sigma > 0$

- A system is stable if its impulse response decays to zero, and unstable if diverge.

Stability of Linear Time-Invariant Systems

- Consider the linear time-invariant system whose transfer function denominator polynomial (or characteristic equation) is given by

$$s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_{n-1} s + a_n = 0$$

- Assume that the roots $\{p_i\}$ of the characteristic equation are real or complex, but are distinct; so that the transfer function can be given as:

$$T(s) = \frac{Y(s)}{R(s)} = \frac{b_0 s^m + b_1 s^{m-1} + b_2 s^{m-2} + \ldots + b_{m-1} s + b_m}{s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_{n-1} s + a_n}$$

$$= \frac{K \prod_{i=1}^{m} (s - z_i)}{\prod_{i=1}^{n} (s - p_i)}, \quad m \le n$$

Stability of Linear Time-Invariant Systems

- The solution of the system response, found using partial fraction expansion, may be written as:

$$y(t) = \sum_{i=1}^{n} K_i e^{p_i t}$$

- The system is stable if and only if (necessary and sufficient condition) every term in the equation above goes to zero as $t \to \infty$.

$$e^{p_i t} \to 0 \quad \text{for all } p_i$$

- This situation will happen if all the poles of the system are strictly in the LHP.

$$\mathrm{Re}\{p_i\} < 0$$

Stability of Linear Time-Invariant Systems

- If any LHP poles are repeated, the response will change because a polynomial in t must be included in place of K_i. However, the conclusion is the same: as $t \to \infty$, $y(t) \to 0$.

$$\lim_{t\to\infty} \frac{1}{n!} t^n e^{-at} \equiv 0 \qquad \text{for any } n \geq 0$$

$f(t)$	$F(s)$
$t^n e^{-at}$	$\dfrac{n!}{(s+a)^{n+1}}$

- Thus, the stability of a system can be determined by computing the location of the roots of the characteristic equation and determining whether they are all in the LHP. This is called internal stability.
  - If a system has any poles in the RHP, it is unstable.
  - If a system has non-repeated $j\omega$-axis poles, then it is said to be neutrally stable.
  - If the system has repeated $j\omega$-axis poles, then it is unstable, as it results in $t^n e^{\pm j\omega t}$ in the solution equation

Exponential Series, Power Series

- Using Taylor series approximation, we can see that exponential series $e^{\lambda t}$ increases faster than power series of t^k.

$$e^{\lambda t} = 1 + \lambda t + \frac{(\lambda t)^2}{2!} + \frac{(\lambda t)^3}{3!} + \frac{(\lambda t)^4}{4!} + \ldots$$

$$e^{\lambda t} >>> t^n \quad \text{for any value of } n$$

- Therefore, e^{-at} decreases faster than the increase of t^n.

$$\lim_{t\to\infty} \frac{1}{n!} t^n e^{-at} \equiv 0 \qquad \text{for any } n \geq 1$$

- Also, since $e^{\pm j\omega t} = \cos\omega t \pm j\sin\omega t$ then $\lim_{t\to\infty} t^n e^{\pm j\omega t} \neq 0$
- Repeated $j\omega$-axis poles will make the system unstable.

Routh's Stability Criterion

- The roots of the characteristic equation determine whether the system is stable or unstable.
- Consider the characteristic equation

$$a(s) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_{n-1} s + a_n$$

- Routh's stability criterion allows us to make certain statements about the stability of the system without actually solving for the roots of the polynomial.
- Routh's stability criterion is also useful for determining the ranges of coefficients of polynomials for stability, especially when the coefficients are in symbolic (non-numerical) form.

Routh-Hurwitz Criterion: A Bit of History

J.C. Maxwell, "On governors," Proc. Royal Society, no. 100, 1868

... [Stability of the governor] is mathematically equivalent to the condition that all the possible roots, and all the possible parts of the impossible roots, of a certain equation shall be negative. ... I have not been able completely to determine these conditions for equations of a higher degree than the third; but I hope that the subject will obtain the attention of mathematicians.

In 1877, Maxwell was one of the judges for the Adams Prize, a biennial competition for best essay on a scientific topic. The topic that year was stability of motion. The prize went to Edward John Routh, who solved the problem posed by Maxwell in 1868.

In 1893, Adolf Hurwitz solved the same problem, using a different method, independently of Routh.

Routh-Hurwitz Criterion: A Bit of History

Edward John Routh, 1831–1907

Adolf Hurwitz, 1859–1919

Routh's Stability Criterion

- A *necessary condition for stability* of the system is that all of the roots of its characteristic equation have negative real parts, which in turn requires that all the coefficients $\{a_i\}$ be positive.

"A necessary (but not sufficient) condition for stability is that *all* the coefficients of the characteristic polynomial be positive."

$\equiv$ (identical to)

"If a system is stable, then *all* the coefficients of the characteristic polynomial are positive."

Routh's Stability Criterion

- Once the elementary necessary conditions have been satisfied, a more powerful test is needed.
- Routh in 1874 proposed a test that requires the computation of a triangular array that is a function of the coefficients of the characteristic equation.

> "A system is stable if and only if *all* the elements in the first column of the Routh array are positive."

$\equiv$ (identical to)

> "If a system is stable then *all* the elements in the first column of the Routh array are positive, and vice versa."

Routh's Stability Criterion

- Consider the characteristic equation

$$a(s) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_{n-1} s + a_n$$

- First, arrange the coefficients of the characteristic polynomial in two rows, beginning with the first and second coefficients and followed by the even-numbered and odd-numbered coefficients:

$$\begin{array}{lllll} s^n: & 1 & a_2 & a_4 & \ldots \\ s^{n-1}: & a_1 & a_3 & a_5 & \ldots \end{array}$$

Routh's Stability Criterion

- Then add subsequent rows to complete the Routh array:

$$
\begin{array}{lr|cccc}
\textbf{Row } n & s^n: & 1 & a_2 & a_4 & \dots \\
\textbf{Row } n-1 & s^{n-1}: & a_1 & a_3 & a_5 & \dots \\
\textbf{Row } n-2 & s^{n-2}: & \boxed{b_1} & \boxed{b_2} & \boxed{b_3} & \dots \\
\textbf{Row } n-3 & s^{n-3}: & \boxed{c_1} & \boxed{c_2} & \boxed{c_3} & \dots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \\
\textbf{Row } 2 & s^2: & \boxed{*} & \boxed{*} & & \\
\textbf{Row } 1 & s^1: & \boxed{*} & & & \\
\textbf{Row } 0 & s^0: & \boxed{*} & & &
\end{array}
$$

$$b_1 = \frac{a_1 a_2 - a_3 1}{a_1},\ c_1 = \frac{b_1 a_3 - b_2 a_1}{b_1},$$

$$b_2 = \frac{a_1 a_4 - a_5 1}{a_1},\ c_2 = \frac{b_1 a_5 - b_3 a_1}{b_1},$$

$$b_3 = \frac{a_1 a_6 - a_7 1}{a_1},.c_3 = \frac{b_1 a_7 - b_4 a_1}{b_1},\dots$$

First column of Routh's array

- If the elements of the first column are all positive, then all the roots are in the LHP.
- If the elements are not all positive, then the number of roots in the RHP equals the number of <u>sign changes</u> in the column.

Example 1

- **<u>Example :</u>** **D(s)=s^3+20s^2+9s+100**

Passes Hurwitz test !

$$
\begin{array}{c|cc}
s^3 & 1 & 9 \\
s^2 & 20 & 100 \\
\hline
s^1 & \boxed{4} & \\
s^0 & \boxed{100} &
\end{array}
$$

$$b_1 = \frac{(20)(9)-(1)(100)}{20} = 4$$

$$c_1 = \frac{(4)(100)-(20)(0)}{4} = 100$$

Example 2

- **Example :** $D(s)=s^3+s^2+2s+24$

Passes Hurwitz test !

$s_1 = -3.0000$
$s_2 = 1.0000 + 2.6458i$
$s_3 = 1.0000 - 2.6458i$

$$
\begin{array}{c|cc}
s^3 & 1 & 2 \\
s^2 & 1 & 24 \\
\hline
s^1 & -22 & 0 \\
s^0 & 24 &
\end{array}
$$

Knight's move

$$b_1 = \frac{(1)(2)-(1)(24)}{1} = -22$$

$$c_1 = \frac{(-22)(24)-(1)(0)}{-22} = 24$$

Sign changes in the 1st column : Unstable system.

2 sign changes : two roots with positive real parts.

Example 3: Routh's Test

All the coefficients of the characteristic equation

$$a(s) = s^6 + 4s^5 + 3s^4 + 2s^3 + 1s^2 + 4s + 4$$

are positive. This means that ...

→ the system maybe stable or maybe not.
We have to determine whether all of the roots are in the LHP

$$
\begin{array}{lcccc}
s^6: & 1 & 3 & 1 & 4 \\
s^5: & 4 & 2 & 4 & 0 \\
s^4: & ? & ? & ? & \\
s^3: & ? & ? & & \\
s^2: & ? & ? & & \\
s^1: & ? & & & \\
s^0: & ? & & &
\end{array}
$$

Example 3: Routh's Test

$$\begin{array}{llllll} s^6: & 1 & 3 & 1 & 4 \\ s^5: & 4 & 2 & 4 & 0 \\ s^4: & \boxed{\mathbf{2.5}} & \boxed{\mathbf{0}} & \boxed{\mathbf{4}} \\ s^3: & ? & ? \\ s^2: & ? & ? \\ s^1: & ? \\ s^0: & ? \end{array}$$

$$b_1 = \frac{4 \cdot 3 - 2}{4} = 2.5$$

$$b_2 = \frac{4 \cdot 1 - 4}{4} = 0$$

$$b_3 = \frac{4 \cdot 4 - 0}{4} = 4$$

Example 3: Routh's Test

$$\begin{array}{llllll} s^6: & 1 & 3 & 1 & 4 \\ s^5: & 4 & 2 & 4 & 0 \\ s^4: & 2.5 & 0 & 4 \\ s^3: & \boxed{\mathbf{2}} & \boxed{\mathbf{-2.4}} \\ s^2: & \boxed{\mathbf{3}} & \boxed{\mathbf{4}} \\ s^1: & ? \\ s^0: & ? \end{array}$$

$$c_1 = \frac{2.5 \cdot 2 - 0 \cdot 4}{2.5} = 2$$

$$c_2 = \frac{2.5 \cdot 4 - 4 \cdot 4}{2.5} = -2.4$$

$$d_1 = \frac{2 \cdot 0 - (-2.4)2.5}{2} = 3$$

$$d_2 = \frac{2 \cdot 4 - 0 \cdot 2.5}{2} = 4$$

Example 3: Routh's Test

$$
\begin{array}{l|c|ccc}
s^6: & 1 & 3 & 1 & 4 \\
s^5: & 4 & 2 & 4 & 0 \\
s^4: & 2.5 & 0 & 4 & \\
s^3: & 2 & -2.4 & & \\
s^2: & 3 & 4 & & \\
s^1: & \boxed{-5.067} & & & \\
s^0: & \boxed{4} & & &
\end{array}
$$

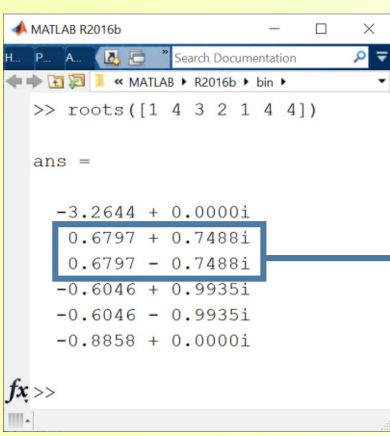
$$e_1 = \frac{3\cdot(-2.4)-4.2}{3} = -5.067$$

$$f_1 = \frac{-5.067\cdot 4 - 0.3}{-5.067} = 4$$

- **The elements of the first column are <u>not</u> all positive**
  ➔ The characteristic equation has at least one RHP root
  ➔ The system is unstable
- **There are two sign changes (+ to – and – to +)**
  ➔ There are two poles in the RHP

Example 3: Routh's Test

- Roots of polynomials can also be found by using MATLAB:



- Roots in the RHP, *i.e.*, roots with positive real parts
- There are two roots of characteristic equation in the RHP
- There are two unstable poles

Example 4: Low-Order Polynomials

$n = 2$ $\quad p(s) = s^2 + a_1 s + a_2$

$$\begin{array}{lll} s^2 & : 1 & a_2 \\ s^1 & : a_1 & 0 \\ s^0 : & b_1 & \end{array} \qquad b_1 = -\frac{1}{a_1}\det\begin{pmatrix} 1 & a_2 \\ a_1 & 0 \end{pmatrix} = a_2$$

— p is stable iff $a_1, a_2 > 0$ (necessary *and* sufficient).

$n = 3$ $\quad p(s) = s^3 + a_1 s^2 + a_2 s + a_3$

$$\begin{array}{lll} s^3 & : 1 & a_2 \\ s^2 & : a_1 & a_3 \\ s^1 : & b_1 & 0 \\ s^0 : & c_1 & \end{array} \qquad \begin{array}{l} b_1 = -\frac{1}{a_1}\det\begin{pmatrix} 1 & a_2 \\ a_1 & a_3 \end{pmatrix} = \frac{a_1 a_2 - a_3}{a_1} \\ c_1 = -\frac{1}{b_2}\det\begin{pmatrix} a_1 & a_3 \\ b_1 & 0 \end{pmatrix} = a_3 \end{array}$$

— p is stable iff $a_1, a_2, a_3 > 0$ (necc. cond.) and $a_1 a_2 > a_3$

Example 5: Routh's Test

Given the characteristic equation:

$$a(s) = s^6 + 4s^5 - 4s^4 + 6s^3 + 8s^2 + 1s + 1$$

Is the system described by this characteristic equation stable?

" If a system is stable, then *all* the coefficients of the characteristic polynomial are positive. "

$\equiv$ (identical to)

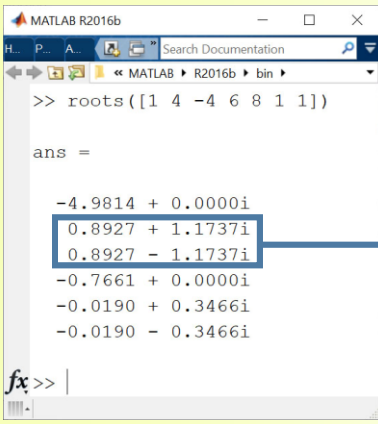
" If not *all* the coefficients of the characteristic polynomial are positive, then a system is not stable. "

$$\therefore\ p \rightarrow q \ \Leftrightarrow\ \sim q \rightarrow \sim p$$

Example 5: Routh's Test

$$a(s) = s^6 + 4s^5 - 4s^4 + 6s^3 + 8s^2 + 1s + 1$$

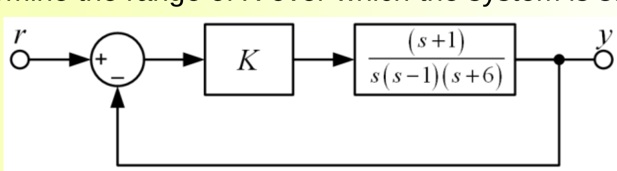
- There is a negative coefficient
  ➔ The system is <u>not</u> stable
- Necessary condition for stability is not even fulfilled
  ➔ No need to continue to Routh's Test



- Roots in the RHP, *i.e.*, roots with positive real parts

Example 6: Stability Versus Parameter Range

Consider the system shown below. The stability properties of the system are a function of the proportional feedback gain *K*.
Determine the range of *K* over which the system is stable.



$$\frac{Y(s)}{R(s)} = \frac{K\dfrac{(s+1)}{s(s-1)(s+6)}}{1+K\dfrac{(s+1)}{s(s-1)(s+6)}} = \frac{K(s+1)}{\underbrace{s(s-1)(s+6)+K(s+1)}} = \frac{b(s)}{a(s)}$$

- The characteristic equation
- Which is the denominator of the transfer function

Example 6: Stability Versus Parameter Range

$$a(s) = s^3 + 5s^2 + (K-6)s + K$$

$$\begin{array}{lll} s^3: & 1 & (K-6) \\ s^2: & 5 & K \\ s^1: & b_1 & b_2 \\ s^0: & c_1 & c_2 \end{array}$$

$$b_1 = \frac{5\cdot(K-6)-K}{5} = \frac{4K-30}{5}$$

$$b_2 = 0$$

$$c_1 = \frac{b_1\cdot K - b_2\cdot 5}{b_1} = K$$

$$c_2 = 0$$

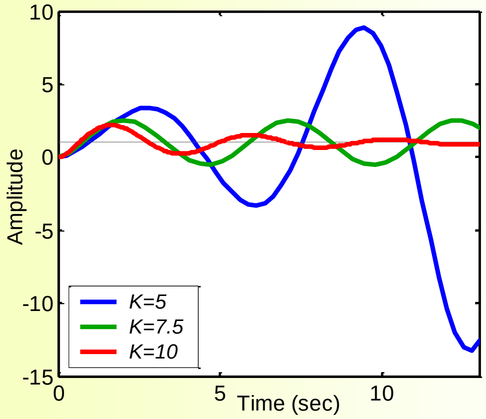
The system is stable if and only if b_1 and c_1 are positive.

$$\left.\begin{array}{l} 4K-30>0 \Rightarrow K>7.5 \\ K>0 \end{array}\right\} \therefore \underline{\underline{K>7.5}}$$

Example 6: Stability Versus Parameter Range

Generating the step responses of the transfer function in MATLAB, for 3 different values of *K*:

$$\frac{Y(s)}{R(s)} = \frac{Ks+K}{s^3+5s^2+(K-6)s+K}$$



Special Cases

- **Special Cases :**

  There are some cases in which problems appear in completing the Routh's array.

  They are encountered in the case of systems that are not stable, and means are devised to allow the completion of the Routh's array.

Special Case 1

- **Special Case 1 :**

  When a first column term in a row becomes zero with all other terms being nonzero, the calculation of the rest of the terms becomes impossible due to division by zero.

  In such a case the system is unstable and the procedure is continued just to determine the number of roots with positive real parts.

Example 1: Special Case 1

- **Example :** $D(s)=s^5+2s^4+2s^3+4s^2+11s+10$

s^5	1	2	11
s^4	2	4	10
s^3	0	6	0
s^2	???		
s^1			
s^0			

$$b_1 = \frac{(2)(2)-(1)(4)}{2} = 0$$

$$b_2 = \frac{(2)(11)-(1)(10)}{2} = 6$$

$$c_1 = \frac{(0)(4)-(2)(6)}{0}$$

Example 1: Special Case 1

- **Example :** $D(s)=s^5+2s^4+2s^3+4s^2+11s+10$

s^5	1	2	11
s^4	2	4	10
s^3	0	6	0
s^2	???		
s^1			
s^0			

In such a case, replace zero term by a very small and positive number ε.

Example 1: Special Case 1

- **Example :** $D(s)=s^5+2s^4+2s^3+4s^2+11s+10$

s^5	1	2	11
s^4	2	4	10
s^3	$0 \rightarrow \varepsilon$	6	0
s^2	$-\frac{12}{\varepsilon}$	10	0
s^1	6	0	
s^0	10		

$$c_1 = \frac{(4)(\varepsilon)-(2)(6)}{\varepsilon} = 4 - \frac{12}{\varepsilon}$$

$$\varepsilon \rightarrow 0 \quad \Rightarrow \quad c_1 \cong -\frac{12}{\varepsilon}$$

$$d_1 = \frac{\left(-\frac{12}{\varepsilon}\right)(6)-(10)(\varepsilon)}{-\frac{12}{\varepsilon}} = 6 + \frac{10}{12}\varepsilon^2$$

$$\varepsilon \rightarrow 0 \quad \Rightarrow \quad d_1 \cong 6$$

Example 1: Special Case 1

- **Special Case 1 :** $D(s)=s^5+2s^4+2s^3+4s^2+11s+10$

s^5	1	2	11
s^4	2	4	10
s^3	$0 \rightarrow \varepsilon$	6	0
s^2	$-\frac{12}{\varepsilon}$	10	
s^1	6		
s^0	10		

2 sign changes : 2 roots with positive real part.

$s_1 = 0.8950 + 1.4561i$
$s_2 = 0.8950 - 1.4561i$
$s_3 = -1.2407 + 1.0375i$
$s_4 = -1.2407 - 1.0375i$
$s_5 = -1.3087$

Example 2: Special Case 1

- If the sign of the coefficient above the zero (ε) is opposite that below it, it indicates that there is one sign change.

$$a(s) = s^5 + 3s^4 + 2s^3 + 6s^2 + 6s + 9$$

$$\begin{array}{llll} s^5 : + & 1 & 2 & 6 \\ s^4 : + & 3 & 6 & 9 \\ s^3 : + & 0 \equiv \varepsilon & 3 & \\ s^2 : - & \frac{6\varepsilon - 9}{\varepsilon} & 9 & \\ s^1 : + & \dfrac{3 \cdot \left(\frac{6\varepsilon - 9}{\varepsilon}\right) - 9\varepsilon}{\frac{6\varepsilon - 9}{\varepsilon}} & & \\ s^0 : + & 9 & & \end{array}$$

1st

The opposite signs above and below ε
➔ There is one sign change
➔ The 1st root in the RHP

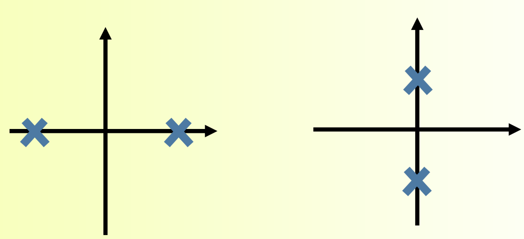
2nd

Another sign change between s^2 and s^1
➔ The 2nd root in the RHP

$$\begin{aligned} s_1 &= -2.9043 \\ s_{2,3} &= 0.6567 \pm j1.2881 \\ s_{4,5} &= -0.7046 \pm j0.9929 \end{aligned}$$

Special Case 2

- If all the coefficients in any derived row are zero, it indicates that there are roots of equal magnitude lying radially opposite in the s-plane, that is, two real roots with equal magnitudes and opposite signs and/or two conjugate imaginary roots.



- In such a case, the evaluation of the rest of the array can be continued by forming an auxiliary polynomial from the last nonzero row, and then using the coefficients of the derivative of this auxiliary polynomial to replace the zero row.

Example 1: Special Case 2

- **Example :** $D(s)=s^3+2s^2+s+2$ **Passes Hurwitz test !**

s^3	1	1
s^2	2	2
s^1	0	0
	4	0
s^0	2	

$Q(s)=(2)s^2+(2)s^0$

Replace row of zeroes with dQ/ds

$\frac{dQ(s)}{ds}=(4)s+(0)s^0$

No sign changes in the 1st column :

No roots with positive real parts.

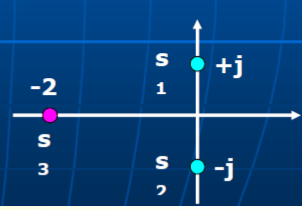
Example 1: Special Case 2

- **Example :** $D(s)=s^3+2s^2+s+2$ **Passes Hurwitz test !**

s^3	1	1
s^2	2	2
s^1	0	0
	4	0
s^0	2	

$Q(s)=2s^2+2=0$ $\quad s_{1,2}=\pm j$

$\frac{s^3+2s^2+s+2}{2s^2+2}=\frac{s}{2}+1$ $\quad s_3=-2$



Example 2: Special Case 2

$$a(s) = s^5 + 5s^4 + 11s^3 + 23s^2 + 28s + 12$$

	$s^5:$	1	11	28
	$s^4:$	5	23	12
	$s^3:$	6.4	25.6	0
	$s^2:$	3	12	$\leftarrow a_1(s) = 3s^2 + 12$
	$s^1:$	0	0	
New	$s^1:$	6	0	$\leftarrow \dfrac{da_1(s)}{ds} = 6s$
	$s^0:$	12		

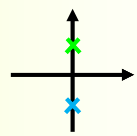
Auxiliary polynomial from the last nonzero row

Zero row

Derivative of the auxiliary polynomial

Zero row is replaced

- One zero row
  ➔ Radially opposite roots
- No sign change
  ➔ No root in the RHP
  ➔ Means, mirrored by real axis



$$s_1 = -3$$
$$s_{2,3} = \pm j2$$
$$s_4 = -1$$
$$s_5 = -1$$

Example 3: Special Case 2

$$a(s) = s^5 + 2s^4 + 24s^3 + 48s^2 - 25s - 50$$

Unstable

	$s^5:$	1	24	-25
	$s^4:$	2	48	-50 $\leftarrow a_1(s) = 2s^4 + 48s^2 - 50$
	$s^3:$	0	0	0
New	$s^3:$	8	96	0 $\leftarrow \dfrac{da_1(s)}{ds} = 8s^3 + 96s$
	$s^2:$	24	-50	
	$s^1:$	112.7	0	
	$s^0:$	-50		

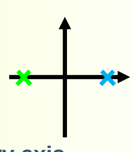
Auxiliary polynomial from the last nonzero row

Zero row

Derivative of the auxiliary polynomial

Zero row is replaced

- One zero row
  ➔ Radially opposite roots
- One sign change
  ➔ One root in the RHP
  ➔ Means, mirrored by imaginary axis



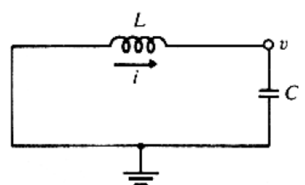
$$s_{1,2} = \pm j5$$
$$s_3 = 1$$
$$s_4 = -1$$
$$s_5 = -2$$

LYAPUNOV METHODS

- Aleksandr Mikhailovich Lyapunov (1857-1918) developed an approach to stability analysis
- The highlight of the method is that "<u>only the form of differential equations need be known</u>"
- There is no need for solving the equations
- It is widely used in stability analysis
- Energy concept is a way of viewing Lyapunov methods (direct method)

ENERGY CONCEPT

- Consider



- Let the capacitor voltage (v) and inductor current (i) be state variables $\mathbf{x}$
- The total energy at any given time is

$$E = \frac{L}{2}i^2 + \frac{C}{2}v^2$$

- Thus

$$E > 0 \quad \text{if} \quad \mathbf{x} \neq 0, \qquad E = 0 \quad \text{only if} \quad \mathbf{x} = 0$$

ENERGY CONCEPT

- If $\dot{E}$ is always negative, the E will decrease and approach zero $\left(\mathbf{x}(t) \to 0\right)$
- The above argument is asymptotic stability
- If $\dot{E}$ is never positive, $\dot{E} \leq 0$ E can never increase, and it need not approach zero.
- This is i.s.L. stability
- We express the energy in the form of
- and

$$E(x) = \frac{1}{2}a_1 x_1^2 + \frac{1}{2}a_2 x_2^2$$

$$\dot{E} = a_1 x_1 \dot{x}_1 + a_2 x_2 \dot{x}_2$$

(No knowledge of the solutions is required)

THE DIRECT METHOD

- Let a single value function $\mathrm{V}(\mathbf{x})$ be continuous and has continuous partial derivatives
- $\mathrm{V}(\mathbf{x})$ is said to be positive definite (P.D.) if
  - $\mathrm{V}(\mathbf{0}) = 0$
  - $\mathrm{V}(\mathbf{x}) > 0$ for all nonzero $\mathbf{x}$
- If we relax the 2nd condition to $\mathrm{V}(\mathbf{x}) \geq 0$ then $\mathrm{V}(\mathbf{x})$ is positive semidefinite
- Reversing the inequalities leads to corresponding negative definite and negative semidefinite

THE DIRECT METHOD

- Consider the following autonomous system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

- The origin is assumed to be an equilibrium point that $\mathbf{f}(\mathbf{0}) = \mathbf{0}$
- Theorem : if a positive definite function $V(\mathbf{x})$ can be determined s.t. $\dot{V}(\mathbf{x}) \le 0$, then the origin is stable i.s.L.
- Theorem: if a positive definite function $V(\mathbf{x})$ can be determined s.t. $\dot{V}(\mathbf{x}) < 0$, then the origin is asymptotically stable

****Inability to find, V(x), does not mean that the system is unstable****

THE DIRECT METHOD

- Theorem: The origin is a globally asymptotically stable equilibrium point if a Lyapunov function, $V(\mathbf{x})$ can be found s.t. (1) $V(\mathbf{x}) > 0$ for all $\mathbf{x} \neq 0$ and $V(\mathbf{0}) = 0$, (2) $\dot{V}(\mathbf{x}) < 0$ for all $\mathbf{x} \neq 0$, and (3) $V(\mathbf{x}) \to \infty$ as $\|\mathbf{x}\| \to \infty$
- The Lyapunov theorems give no indication of how a Lyapunov function might be found
- The Lyapunov function for any given system is not unique
- If a system is stable, it is ensured that an appropriate Lyapunov function does exist

EIGENVALUES

- Consider

$$\mathbf{Av} = \lambda \mathbf{v} \quad \rightarrow \quad (\mathbf{A} - \lambda \mathbf{I})\mathbf{v} = 0$$

  - where $\mathbf{A}$ is an nxn matrix, $\mathbf{v}$ is an nx1 vector
- A nontrivial solution

$$|\mathbf{A} - \lambda \mathbf{I}| = 0$$

  - Expansion of the determinant the characteristic equation
  - The n solutions of λ are eigenvalues of **A**
  - Eigenvalues may be real or complex
  - If **A** is real, complex eigenvalues must occur in conjugate pairs
  - A matrix and its transpose have the same eigenvalues

EIGENVECTORS

- Consider $\mathbf{Av}_i = \lambda_i \mathbf{v}_i$
  - For any eigenvalue λ_i, the eigenvector $\mathbf{v}_i$ has the form

$$\mathbf{v}_i = \begin{bmatrix} v_{1i} & v_{2i} & \cdots & v_{ni} \end{bmatrix}^T$$

  - $\mathbf{v}_i$ is known as the right eigenvector
- Similarly,
  - The n-row vector $\mathbf{w}_i$ satisfies $\mathbf{w}_i \mathbf{A} = \lambda_i \mathbf{w}_i$
  - $\mathbf{w}_i$ is known as the left eigenvector
- For different λ_i and λ_j $\rightarrow$ $\mathbf{w}_j \mathbf{v}_i = 0$
- The same eigenvalue $\mathbf{w}_i \mathbf{v}_i = 1$

MODAL MATRICES

- Introduce $\mathbf{V} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix}$

$$\mathbf{W} = \begin{bmatrix} \mathbf{w}_1^T & \mathbf{w}_2^T & \cdots & \mathbf{w}_n^T \end{bmatrix}^T$$

$$\Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

- This can be expanded as $\mathbf{AV} = \mathbf{V}\Lambda$
- Recall that $\mathbf{WV} = \mathbf{I}$ and $\mathbf{W} = \mathbf{V}^{-1}$
- It follows that $\boxed{\mathbf{V}^{-1}\mathbf{AV} = \Lambda}$

FREE MOTION OF DYNAMIC SYSTEM

- Consider a zero input system $\Delta\dot{\mathbf{x}} = \mathbf{A}\Delta\mathbf{x}$
- Define a new state vector $\mathbf{z}$ s.t.

$$\Delta\mathbf{x} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix}\mathbf{z} = \mathbf{Vz}$$

- This means that

$$\mathbf{V}\dot{\mathbf{z}} = \mathbf{AVz}$$

- The new state equation is

$$\dot{\mathbf{z}} = \begin{bmatrix} \mathbf{V}^{-1}\mathbf{AV} \end{bmatrix}\mathbf{z} = \Lambda\mathbf{z}$$

- This gives us

$$\dot{z}_i = \lambda_i z_i \qquad i = 1, 2, \ldots, n$$

- and the solution is

$$z_i(t) = z_i(0)e^{\lambda_i t}$$

FREE MOTION OF DYNAMIC SYSTEM

- The original system state vector is

$$\Delta \mathbf{x} = \mathbf{V}\mathbf{z}(t) = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \\ \vdots \\ z_n(t) \end{bmatrix}$$

- This implies that

$$\Delta \mathbf{x} = \sum_{i=1}^{n} v_i z_i(0) e^{\lambda_i t}$$

- The time dependent characteristic is upto an eigenvalue
- Stability can be determined by eigenvalues

EIGENVALUE AND STABILITY

- Real eigenvalue → non-oscillatory mode
  - Negative real eigenvalue = decaying mode
  - Positive real eigenvalue = aperiodic instability
- Complex eigenvalues → Each pair corresponds to an oscillatory mode

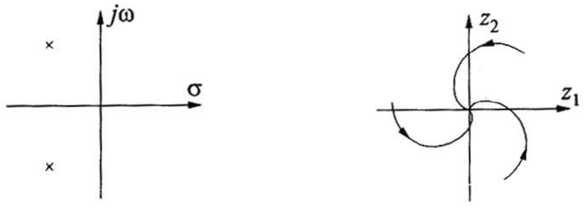
$$\lambda = \sigma \pm j\omega$$

  - Negative real part = damped oscillation
  - Positive real part = oscillation of increasing amplitude
- Frequency of oscillation $f = \omega / 2\pi$
- Damping ratio (rate of decay of amplitude)

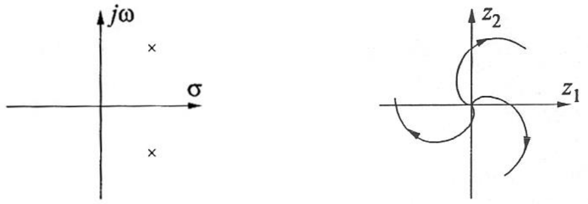
$$\varsigma = -\sigma / \sqrt{\sigma^2 + \omega^2}$$

EIGENVALUE AND STABILITY

- Stable spiral node



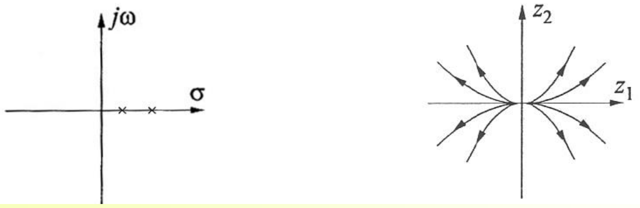
- Unstable spiral node



EIGENVALUE AND STABILITY

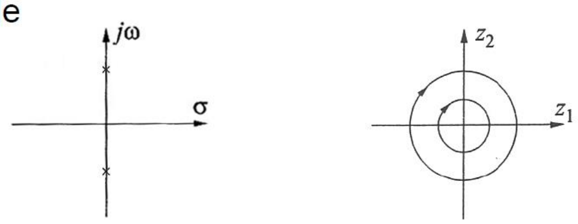
- Stable node

- Unstable node

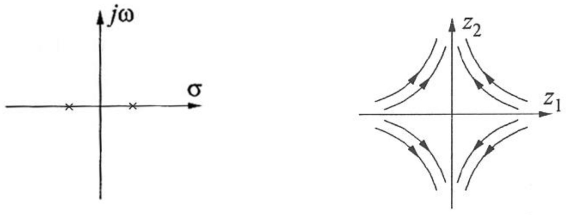


EIGENVALUE AND STABILITY

- Cycle

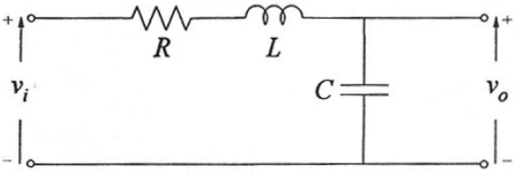


- Saddle node



EXAMPLE

- Examine the modal characteristics of the following second order system



- The governed differential equation for input vs. output

$$LC\frac{d^2v_o}{dt^2}+RC\frac{dv_o}{dt}+v_o=v_i$$

- Standard form

$$\frac{d^2v_o}{dt^2}+(2\varsigma\omega_n)\frac{dv_o}{dt}+\omega_n^2v_o=\omega_n^2v_i \qquad \begin{cases}\omega_n=1/\sqrt{LC}\\ \varsigma=(R/2)\sqrt{L/C}\end{cases}$$

EXAMPLE

- State-space form
  - Let $x_1 = v_o,\ x_2 = \dot{x}_1 = \dot{v}_o,\ u = v_i,\ y = v_o = x_1$
  - The system becomes

$$\frac{dx_1}{dt} = x_2$$

$$\frac{dx_2}{dt} = -\omega_n^2 x_1 - (2\varsigma\omega_n)x_2 + \omega_n^2 u$$

  - In matrix form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\varsigma\omega_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \omega_n^2 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

EXAMPLE

- The eigenvalue

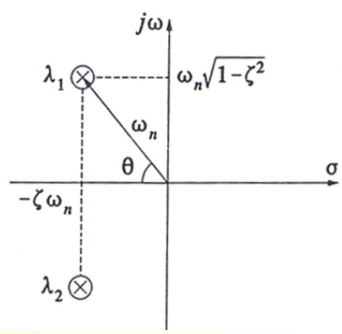
$$\begin{vmatrix} -\lambda & 1 \\ -\omega_n^2 & -2\varsigma\omega_n - \lambda \end{vmatrix} = 0 \quad \Rightarrow \quad \lambda^2 + 2\varsigma\omega_n\lambda + \omega_n^2 = 0$$

- This gives us

$$\lambda_1, \lambda_2 = -\varsigma\omega_n \pm j\omega_n\sqrt{\varsigma^2 - 1}$$

- Damping angle

$$\theta = \tan^{-1}\left(\frac{\sqrt{1-\varsigma^2}}{\varsigma}\right)$$



EXAMPLE

- The right eigenvector

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{v}=0 \quad \Rightarrow \quad \begin{bmatrix} -\lambda_i & 1 \\ -\omega_n^2 & -2\varsigma\omega_n-\lambda_i \end{bmatrix}\begin{bmatrix} v_{1i} \\ v_{2i} \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- The first eigenvector

$$\mathbf{v}_1=\begin{bmatrix} v_{11} \\ v_{21} \end{bmatrix}=\begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix}=\begin{bmatrix} 1 \\ -\varsigma\omega_n+\omega_n\sqrt{\varsigma^2-1} \end{bmatrix}$$

- Similarly, the 2nd eigenvector

$$\mathbf{v}_1=\begin{bmatrix} v_{11} \\ v_{21} \end{bmatrix}=\begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix}=\begin{bmatrix} 1 \\ -\varsigma\omega_n-\omega_n\sqrt{\varsigma^2-1} \end{bmatrix}$$