EEE222 Circuit Theory II

OVERVIEW OF LAPLACE TRANSFORM

Lecture Notes 3

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Pierre Simon de Laplace (1749-1827)

- *Pierre Simon Laplace* was born in Normandy on March 23, 1749, and died at Paris on March 5, 1827
- French scientist, mathematician and astronomer; established mathematically the stability of the Solar system and its origin - without a divine intervention
- Professor of mathematics in the École militaire of Paris at the age of 19.
- Main publications:
  - *Mécanique céleste* (1771, 1787)
  - *Théorie analytique des probabilités* 1812 – first edition dedicated to Napoleon

History of the Transform

- Euler began looking at integrals as solutions to differential equations in the mid 1700's:

$$z = \int X(x) e^{ax}\, dx \qquad z(x; a) = \int_0^x e^{at} X(t)\, dt,$$

- Lagrange took this a step further while working on probability density functions and looked at forms of the following equation:

$$\int X(x) e^{-ax} a^x\, dx,$$

- Finally, in 1785, Laplace began using a transformation to solve equations of finite differences which eventually lead to the current transform

$$S = Ay_s + B\,\Delta y_s + C\,\Delta^2 y_s + \ldots, \qquad y_s = \int e^{-sx}\,\phi(x)\, dx,$$

Review of Laplace Transform

Time domain | Frequency domain

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\text{Problem } f(t) \arrow[r, "\mathcal{L}\{\ \}"] \arrow[d, "\text{difficult operations}"'] & F(s) \arrow[d, "\text{easy operations}"] \\
g(t) \text{ Solution} & G(s) \arrow[l, "\mathcal{L}^{-1}\{\ \}"']
\end{tikzcd}$$

Laplace Transform

$$\mathcal{L}\{f(t)\} = F(s) = \int_{0^-}^{\infty} f(t)e^{-st}dt, \quad s = \sigma + j\omega$$

Inverse Laplace Transform

$$\mathcal{L}^{-1}\{F(s)\} = f(t) = \frac{1}{2\pi j}\int_{\sigma_c - j\infty}^{\sigma_c + j\infty} F(s)e^{st}ds$$

Review of Laplace Transform

Time domain

Frequency domain

Hard Problem

$$y'' + y = te^{-t}$$
$$y(0) = y'(0) = 0$$

$\mathcal{L}\{\ \}$ →

$$Y(s^2+1) = \frac{1}{(s+1)^2}$$

difficult to solve ↓

algebraic operations ↓

Solution in *y*?

$$y(t) = -\tfrac{1}{2}\cos t + \tfrac{1}{2}te^{-t} + \tfrac{1}{2}e^{-t}$$

← $\mathcal{L}^{-1}\{\ \}$

$$Y = \frac{1}{(s^2+1)(s+1)^2}$$

Laplace Transform: Example

$$f(t) = a$$

$$\mathcal{L}\{a\} = \int_{0^-}^{\infty} a e^{-st}\,dt = -\frac{a}{s} e^{-st}\Big|_{0^-}^{\infty} = -\frac{a}{s}\left(e^{-\infty} - e^{0^-}\right) = \frac{a}{s}$$

$$f(t) = e^{-at}$$

$$\mathcal{L}\left\{e^{-at}\right\} = \int_{0^-}^{\infty} e^{-at} e^{-st}\,dt = \int_{0^-}^{\infty} e^{-(s+a)t}\,dt = -\frac{1}{s+a} e^{-(s+a)t}\Big|_{0^-}^{\infty} = \frac{1}{s+a}$$

Laplace Transform: Example

$$f(t) = \cos \omega t$$

$$\mathcal{L}\{\cos \omega t\} = \int_{0^-}^{\infty} \frac{1}{2}\left(e^{j\omega t} + e^{-j\omega t}\right)e^{-st}dt = \int_{0^-}^{\infty} \frac{1}{2} e^{-(s-j\omega)t}dt + \int_{0^-}^{\infty} \frac{1}{2} e^{-(s+j\omega)t}dt$$

$$= \frac{1}{2}\frac{1}{(s-j\omega)} + \frac{1}{2}\frac{1}{(s+j\omega)} = \frac{1}{2}\frac{(s+j\omega)}{s^2+\omega^2} + \frac{1}{2}\frac{(s-j\omega)}{s^2+\omega^2}$$

$$= \frac{s}{s^2+\omega^2}$$

$$f(t) = \sin \omega t$$

$$\mathcal{L}\{\sin \omega t\} = \int_{0^-}^{\infty} \frac{1}{2j}\left(e^{j\omega t} - e^{-j\omega t}\right)e^{-st}dt = \int_{0^-}^{\infty} \frac{1}{2j} e^{-(s-j\omega)t}dt - \int_{0^-}^{\infty} \frac{1}{2j} e^{-(s+j\omega)t}dt$$

$$= \frac{1}{2j}\frac{1}{(s-j\omega)} - \frac{1}{2j}\frac{1}{(s+j\omega)} = \frac{1}{2j}\frac{(s+j\omega)}{s^2+\omega^2} - \frac{1}{2j}\frac{(s-j\omega)}{s^2+\omega^2}$$

$$= \frac{\omega}{s^2+\omega^2}$$

Table of Laplace Transform

$f(t)$	$F(s)$
$\delta(t)$	1
$u(t)$	$\dfrac{1}{s}$
e^{-at}	$\dfrac{1}{s+a}$
t	$\dfrac{1}{s^2}$
t^n	$\dfrac{n!}{s^{n+1}}$
te^{-at}	$\dfrac{1}{(s+a)^2}$
$t^n e^{-at}$	$\dfrac{n!}{(s+a)^{n+1}}$

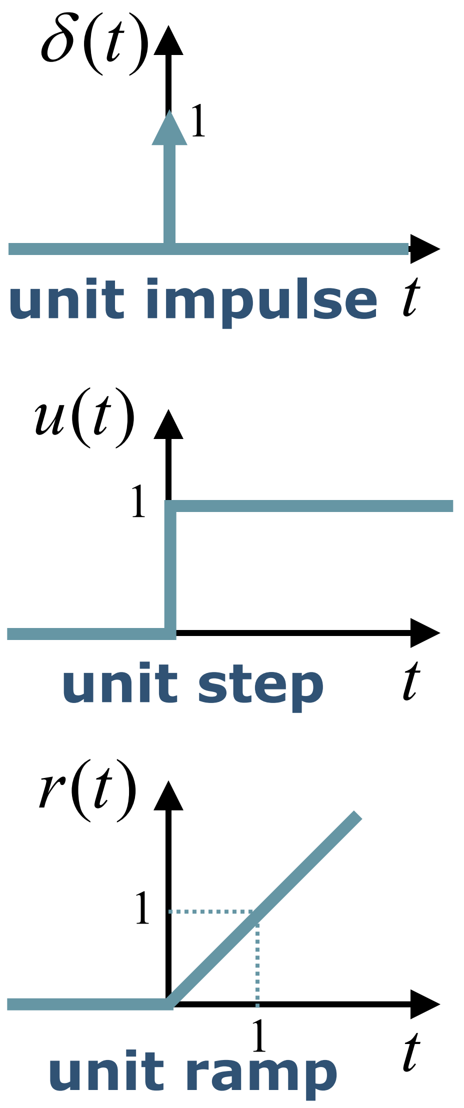


Table of Laplace Transform

$f(t)$	$F(s)$
$\sin \omega t$	$\dfrac{\omega}{s^2 + \omega^2}$
$\cos \omega t$	$\dfrac{s}{s^2 + \omega^2}$
$\sin(\omega t + \theta)$	$\dfrac{s \sin\theta + \omega \cos\theta}{s^2 + \omega^2}$
$\cos(\omega t + \theta)$	$\dfrac{s \cos\theta - \omega \sin\theta}{s^2 + \omega^2}$
$e^{-at} \sin \omega t$	$\dfrac{\omega}{(s + a)^2 + \omega^2}$
$e^{-at} \cos \omega t$	$\dfrac{s + a}{(s + a)^2 + \omega^2}$

Properties of Laplace Transform

1. Superposition

$$\mathcal{L}\left\{\alpha \cdot f_1(t) + \beta \cdot f_2(t)\right\} = \alpha \cdot F_1(s) + \beta \cdot F_2(s)$$

2. Time delay

$$\mathcal{L}\left\{f(t-\lambda) \cdot u(t-\lambda)\right\} = e^{-\lambda s} \cdot F(s)$$

3. Time scaling

$$\mathcal{L}\left\{f(at)\right\} = \frac{1}{a} \cdot F\left(\frac{s}{a}\right)$$

4. Shift in Frequency

$$\mathcal{L}\left\{e^{-at} \cdot f(t) \cdot u(t)\right\} = F(s+a)$$

5. Differentiation in Time

$$\mathcal{L}\left\{f'(t)\right\} = s \cdot F(s) - f(0^-)$$

$$\mathcal{L}\left\{f''(t)\right\} = s^2 \cdot F(s) - s \cdot f(0^-) - f'(0^-)$$

$$\mathcal{L}\left\{f^n(t)\right\} = s^n \cdot F(s) - s^{n-1} \cdot f(0^-) - \ldots - s \cdot f^{n-2}(0^-) - f^{n-1}(0^-)$$

Properties of Laplace Transform

6. Integration in Time

$$\mathcal{L}\left\{\int_0^t f(t)dt\right\} = \frac{1}{s}\cdot F(s)$$

7. Differentiation in Frequency

$$\mathcal{L}\left\{t\cdot f(t)\right\} = -\frac{dF(s)}{ds}$$

8. Convolution

$$F_1(s)\cdot F_2(s) = \mathcal{L}\left\{f_1(t) * f_2(t)\right\}$$

$$F_1(s) * F_2(s) = 2\pi j\cdot\mathcal{L}\left\{f_1(t)\cdot f_2(t)\right\}$$

$$f_1(t) * f_2(t) = \int_{0^-}^{\infty} f_1(\lambda)\cdot f_2(t-\lambda)d\lambda$$

Properties of Laplace Transform

- **Superposition**

$$f(t) = 2e^{-t} + t$$

$$\mathcal{L}\left\{2e^{-t} + t\right\} = 2\mathcal{L}\left\{e^{-t}\right\} + \mathcal{L}\left\{t\right\}$$

$$= 2\frac{1}{s+1} + \frac{1}{s^2}$$

$$= \frac{2s^2 + s + 1}{s^2\left(s+1\right)}$$

- **Superposition**

$$f(t) = 2\sin 3t + \cos 3t$$

$$\mathcal{L}\left\{2\sin 3t + \cos 3t\right\} = 2\mathcal{L}\left\{\sin 3t\right\} + \mathcal{L}\left\{\cos 3t\right\}$$

$$= 2\frac{3}{s^2 + 3^2} + \frac{s}{s^2 + 3^2}$$

$$= \frac{s+6}{s^2 + 3^2}$$

$$f(t) = t^3 + 2t^2 - 4t + 1 \quad \Rightarrow \mathcal{L}\left\{f(t)\right\} = ...$$

$f(t)$	$F(s)$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
e^{-at}	$\frac{1}{s+a}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
te^{-at}	$\frac{1}{(s+a)^2}$
$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
$\sin(\omega t + \theta)$	$\frac{s\sin\theta + \omega\cos\theta}{s^2+\omega^2}$
$\cos(\omega t + \theta)$	$\frac{s\cos\theta - \omega\sin\theta}{s^2+\omega^2}$
$e^{-at}\sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$

Properties of Laplace Transform

- **Shift in Frequency**

$$\mathcal{L}\left\{e^{-at} f(t)\cdot u(t)\right\} = F(s+a)$$

$$f(t) = e^{-t}t^3$$

$$\mathcal{L}\left\{t^3\right\} = \frac{3!}{s^{3+1}} = \frac{6}{s^4}$$

$$\mathcal{L}\left\{e^{-t}t^3\right\} = \frac{6}{(s+1)^4}$$

- **Differentiation in Frequency**

$$\mathcal{L}\left\{t\cdot f(t)\right\} = -\frac{dF(s)}{ds}$$

$$f(t) = t\cos 3t$$

$$\mathcal{L}\left\{\cos 3t\right\} = \frac{s}{s^2+3^2}$$

$$\mathcal{L}\left\{t\cos 3t\right\} = -\frac{d}{ds}\left(\frac{s}{s^2+3^2}\right)$$

$$= \frac{s^2-3^2}{(s^2+3^2)^2}$$

$f(t)$	$F(s)$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
e^{-at}	$\frac{1}{s+a}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
te^{-at}	$\frac{1}{(s+a)^2}$
$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
$\sin(\omega t+\theta)$	$\frac{s\sin\theta+\omega\cos\theta}{s^2+\omega^2}$
$\cos(\omega t+\theta)$	$\frac{s\cos\theta-\omega\sin\theta}{s^2+\omega^2}$
$e^{-at}\sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$

Laplace Transform

Example 1:

Obtain the Laplace Transform of $f(t) = \delta(t) + 2 \cdot u(t) - 3e^{-2t}, \ t \geq 0.$

$$F(s) = \mathcal{L}\{\delta(t)\} + \mathcal{L}\{2 \cdot u(t)\} - \mathcal{L}\{3e^{-2t}\}$$

$$= \mathcal{L}\{\delta(t)\} + 2 \cdot \mathcal{L}\{u(t)\} - 3 \cdot \mathcal{L}\{e^{-2t}\}$$

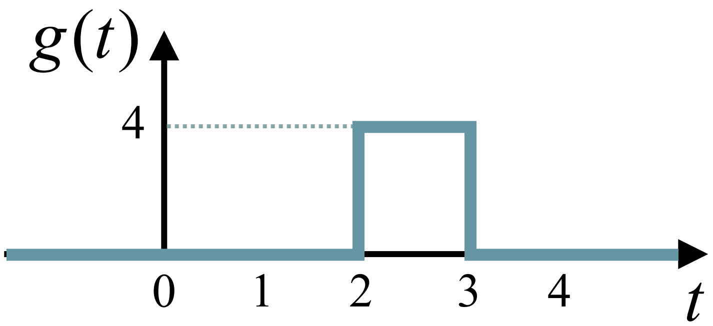
$$= 1 + 2 \cdot \frac{1}{s} - 3 \cdot \frac{1}{s+2}$$

$$\boxed{F(s) = \frac{s^2 + s + 4}{s(s+2)}}$$

Laplace Transform

Example 2:

Find the Laplace Transform of the function shown below.



$$g(t) = 4 \cdot u(t-2) - 4 \cdot u(t-3)$$

$$G(s) = \mathcal{L}\{g(t)\}$$

$$= 4 \cdot \frac{e^{-2s}}{s} - 4 \cdot \frac{e^{-3s}}{s}$$

$$\boxed{G(s) = \frac{4}{s}(e^{-2s} - e^{-3s})}$$

Inverse Laplace Transform

$$\mathcal{L}^{-1}\{F(s)\} = f(t) = \frac{1}{2\pi j}\int_{\sigma_c - j\infty}^{\sigma_c + j\infty} F(s)e^{st}\,ds$$

- Instead of using the formula above to calculate the Inverse Laplace Transform, we try to match entries in the Laplace Transform table.
- The easiest way to find *f*(*t*) from its Laplace Transform *F*(*s*), if *F*(*s*) is rational, is to expand *F*(*s*) as a sum of simpler terms that can be found in the tables.

The steps to find the Inverse Laplace Transform:

1. Decompose *F*(*s*) into simple terms using "Partial-Fraction Expansion Method".
2. Find the inverse of each term by using the table of Laplace Transform.

Inverse Laplace Transform

Example 3:

Find $y(t)$ for $Y(s)=\dfrac{(s+2)(s+4)}{s(s+1)(s+3)}$.

$$Y(s)=\frac{c_1}{s}+\frac{c_2}{s+1}+\frac{c_3}{s+3}$$

$$=\frac{c_1(s+1)(s+3)+c_2\cdot s(s+3)+c_3\cdot s(s+1)}{s(s+1)(s+3)}$$

$$Y(s)=\frac{(c_1+c_2+c_3)s^2+(4c_1+3c_2+c_3)s+3c_1}{s(s+1)(s+3)}\equiv\frac{1s^2+6s+8}{s(s+1)(s+3)}$$

- **Algebraic Method**

Inverse Laplace Transform

Comparing the coefficients

$$\left.\begin{aligned} c_1 + c_2 + c_3 &= 1 \\ 4c_1 + 3c_2 + c_3 &= 6 \\ 3c_1 &= 8 \end{aligned}\right\rangle \quad c_1 = \frac{8}{3},\ c_2 = -\frac{3}{2},\ c_3 = -\frac{1}{6}$$

$$Y(s) = \frac{8}{3}\left(\frac{1}{s}\right) - \frac{3}{2}\left(\frac{1}{s+1}\right) - \frac{1}{6}\left(\frac{1}{s+3}\right)$$

$$y(t) = \mathcal{L}^{-1}\left\{Y(s)\right\}$$

$$= \frac{8}{3}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - \frac{1}{6}\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}$$

$$\boxed{y(t) = \left(\frac{8}{3} - \frac{3}{2}e^{-t} - \frac{1}{6}e^{-3t}\right)u(t)}$$

- **Is there an easiest and faster method? "Cover-Up Method"**

Inverse Laplace Transform

Example 3 (Cover-up method):

Find $y(t)$ for $Y(s) = \dfrac{(s+2)(s+4)}{s(s+1)(s+3)}$.

$$Y(s) = \frac{c_1}{s} + \frac{c_2}{s+1} + \frac{c_3}{s+3}$$

$$c_1 = sY(s)\big|_{s=0} = \left.\frac{(s+2)(s+4)}{(s+1)(s+3)}\right|_{s=0} = \frac{8}{3}$$

$$c_2 = (s+1)Y(s)\big|_{s=-1} = \left.\frac{(s+2)(s+4)}{s(s+3)}\right|_{s=-1} = -\frac{3}{2}$$

$$c_3 = (s+3)Y(s)\big|_{s=-3} = \left.\frac{(s+2)(s+4)}{s(s+1)}\right|_{s=-3} = -\frac{1}{6}$$

$$Y(s) = \frac{8}{3}\left(\frac{1}{s}\right) - \frac{3}{2}\left(\frac{1}{s+1}\right) - \frac{1}{6}\left(\frac{1}{s+3}\right) \quad y(t) = \left(\frac{8}{3} - \frac{3}{2}e^{-t} - \frac{1}{6}e^{-3t}\right)u(t)$$

Inverse Laplace Transform

Different Examples

$$\mathcal{L}^{-1}\left\{\frac{s+2}{s^2-4}\right\} = \ldots. \qquad \ldots. = e^{2t}$$

$$\mathcal{L}^{-1}\left\{\frac{s+6}{s^2+4}\right\} = \ldots. \qquad \ldots. = \cos 2t + 3\sin 2t$$

$$\mathcal{L}^{-1}\left\{\frac{9s-8}{s^2-2s}\right\} = \ldots. \qquad \ldots. = 4\cdot u(t) + 5e^{2t}$$

$f(t)$	$F(s)$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
e^{-at}	$\frac{1}{s+a}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
te^{-at}	$\frac{1}{(s+a)^2}$
$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
$\sin(\omega t+\theta)$	$\frac{s\sin\theta+\omega\cos\theta}{s^2+\omega^2}$
$\cos(\omega t+\theta)$	$\frac{s\cos\theta-\omega\sin\theta}{s^2+\omega^2}$
$e^{-at}\sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$

Inverse Laplace Transform

EXAMPLE 4: Repeated Poles (Repeated Denominator Roots)

$$\mathcal{L}^{-1}\left\{\frac{s^2-15s+41}{(s+2)(s-3)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{A}{s+2}+\frac{B}{(s-3)^2}+\frac{C}{s-3}\right\}$$

$$A = \left.\frac{s^2-15s+41}{(s-3)^2}\right|_{s=-2} = 3$$

$$B = \left.\frac{s^2-15s+41}{(s+2)}\right|_{s=3} = 1$$

$$C = \frac{d}{ds}\left.\left(\frac{s^2-15s+41}{(s+2)}\right)\right|_{s=3}$$

$$= \left.\left(\frac{(2s-15)(s+2)-(1)(s^2-15s+41)}{(s+2)^2}\right)\right|_{s=3} = -2$$

- **Cover-Up Method + Differentiation**

$$\mathcal{L}^{-1}\left\{\frac{3}{s+2}+\frac{1}{(s-3)^2}+\frac{-2}{s-3}\right\} \boxed{= 3e^{-2t} + te^{3t} - 2e^{3t}}$$

Inverse Laplace Transform

EXAMPLE 5: Complex Poles (Complex Denominator Roots)

$$\mathcal{L}^{-1}\left\{\frac{4s^2-5s+6}{(s+1)(s^2+4)}\right\} = \mathcal{L}^{-1}\left\{\frac{A}{s+1}+\frac{Bs+C}{s^2+4}\right\}$$

$$A = \left.\frac{4s^2-5s+6}{(s^2+4)}\right|_{s=-1} = 3$$

- **Cover-Up Method + Algebraic Method**

$$\frac{4s^2-5s+6}{(s+1)(s^2+4)} = 3\frac{(s^2+4)}{(s+1)(s^2+4)}+\frac{(Bs+C)(s+1)}{(s+1)(s^2+4)}$$

$$= \frac{(3+B)s^2+(B+C)s+(12+C)}{(s+1)(s^2+4)} \quad \Rightarrow B=1, C=-6$$

$$\mathcal{L}^{-1}\left\{\frac{4s^2-5s+6}{(s+1)(s^2+4)}\right\} = \mathcal{L}^{-1}\left\{\frac{3}{s+1}+\frac{s-6}{s^2+2^2}\right\} = \mathcal{L}^{-1}\left\{\frac{3}{s+1}+\frac{s}{s^2+2^2}-3\frac{2}{s^2+2^2}\right\}$$

$$\boxed{= 3e^{-t}+\cos 2t-3\sin 2t}$$

Inverse Laplace Transform

EXAMPLE 6: Complex Poles (Complex Denominator Roots)

compute $\mathscr{L}^{-1}\{Y(s)\}$, where

$$Y(s) = \frac{s}{(s+1)(s^2+1)}$$

$$Y(s) = \frac{a}{s+1} + \frac{bs+c}{s^2+1} \qquad \text{(need } bs+c \text{ so that } \deg(\text{num}) = \deg(\text{den}) - 1\text{)}$$

$$a = (s+1)Y(s)\Big|_{s=-1} = -\frac{1}{2}$$

Find b: multiply by s^2+1 to isolate $bs+c$

$$(s^2+1)Y(s) = \frac{s}{s+1} = \frac{a(s^2+1)}{s+1} + bs + c$$

— now let $s = j$ to "kill" the first term on the RHS:

$$bj + c = (s^2+1)Y(s)\Big|_{s=j} = \frac{j}{1+j}$$

Match $\mathrm{Re}(\cdot)$ and $\mathrm{Im}(\cdot)$ parts:

$$c + bj = \frac{j}{1+j} = \frac{j(1-j)}{(1+j)(1-j)} = \frac{1}{2} + \frac{j}{2} \quad \Longrightarrow b = c = \tfrac{1}{2}$$

Inverse Laplace Transform

EXAMPLE 6 (cont'd): Complex Poles (Complex Denominator Roots)

We found that

$$Y(s) = -\frac{1}{2(s+1)} + \frac{s}{2(s^2+1)} + \frac{1}{2(s^2+1)}$$

Now we can use linearity and tables:

$$\begin{aligned}
y(t) &= \mathscr{L}^{-1}\left\{-\frac{1}{2(s+1)} + \frac{s}{2(s^2+1)} + \frac{1}{2(s^2+1)}\right\} \\
&= -\frac{1}{2}\mathscr{L}^{-1}\left\{\frac{1}{s+1}\right\} + \frac{1}{2}\mathscr{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + \frac{1}{2}\mathscr{L}^{-1}\left\{\frac{1}{s^2+1}\right\} \\
&= -\frac{1}{2}e^{-t} + \frac{1}{2}\cos t + \frac{1}{2}\sin t \quad \text{(from tables)} \\
&= -\frac{1}{2}e^{-t} + \frac{1}{\sqrt{2}}\cos(t - \pi/4) \quad (\cos(a-b) = \cos a \cos b + \sin a \sin b)
\end{aligned}$$

Solving ODEs Using Laplace Transform

EXAMPLE 7:

$$5y' + 4y = 2,\ y(0) = 1$$

- **Input (determine forced response)**
- **Initial condition (determine natural response)**

$$\Big\downarrow \mathscr{L}\{\ \}$$

$$5\left[sY(s) - y(0)\right] + 4Y(s) = \frac{2}{s}$$

$$Y(s)\left[5s + 4\right] = 5 + \frac{2}{s}$$

$$Y(s) = \frac{5s+2}{s(5s+4)} = \frac{A}{s} + \frac{B}{(5s+4)},$$

$$A = \left.\frac{5s+2}{(5s+4)}\right|_{s=0} = 0.5$$

$$B = \left.\frac{5s+2}{s}\right|_{s=-0.8} = 2.5$$

$$Y(s) = \frac{0.5}{s} + \frac{2.5}{(5s+4)}$$

$$\boxed{y(t) = 0.5 \cdot u(t) + 0.5e^{-0.8t}}$$

Solving ODEs Using Laplace Transform

EXAMPLE 8:

$$y'' - 3y' + 2y = 2e^{3t}, \; y(0) = 0, \; y'(0) = 7$$

- **Input (determine forced response)**
- **Initial condition (determine natural response)**

$$\Big\downarrow \mathcal{L}\{\ \}$$

$$\left[s^2Y(s) - sy(0) - y'(0)\right] - 3\left[sY(s) - y(0)\right] + 2Y(s) = \frac{2}{(s-3)}$$

$$\left[s^2Y(s) - 7\right] - 3\left[sY(s)\right] + 2Y(s) = \frac{2}{(s-3)}$$

$$Y(s)\left[s^2 - 3s + 2\right] = 7 + \frac{2}{(s-3)}$$

$$Y(s) = \frac{7s-19}{(s-1)(s-2)(s-3)} \quad = -\frac{6}{s-1} + \frac{5}{s-2} + \frac{1}{s-3}$$

$$\boxed{y(t) = -6e^{t} + 5e^{2t} + e^{3t}}$$

More Exercises in Solving ODEs

$$y' + 2y = 10e^{3t}, y(0) = 6$$

$$\text{....} \qquad \Rightarrow y(t) = 4e^{-2t} + 2e^{3t}$$

$$y'' - 3y' + 2y = 4t - 6, y(0) = 1, y'(0) = 3$$

$$\text{....} \qquad \Rightarrow y(t) = 2t + e^{t}$$

$$y'' + 4y = 3, y(0) = 1, y'(0) = 1$$

$$\text{....} \qquad \Rightarrow y(t) = \frac{3}{4}u(t) + \frac{1}{4}\cos 2t + \frac{1}{2}\sin 2t$$

Initial and Final Value Theorem

- An especially **useful** property of the Laplace Transform in control is Initial and Final Value Theorem.

- Initial Value Theorem (IVT) is useful to easily calculate the initial value of a time function $f(t)$, given its Laplace Transform $F(s)$.
- Final Value Theorem (FVT) allows us to compute the constant steady-state value of a time function $f(t)$ from its Laplace Transform $F(s)$.

$$\lim_{t \to 0^+} y(t) = \lim_{s \to \infty} s \cdot Y(s)$$

- **Initial Value Theorem (IVT)**

$$\lim_{t \to \infty} y(t) = \lim_{s \to 0} s \cdot Y(s)$$

- **Final Value Theorem (FVT)**

- The condition to use FVT: all poles of $Y(s)$ are in the left hand of the s-plane, except for one at $s = 0$.

Final Value Theorem is only applicable to stable system, *i.e.* a system with convergent step response

Initial and Final Value Theorem

Example 9:

Find the final value of the system corresponding to

$$Y(s) = \frac{3(s+2)}{s(s^2+2s+10)}$$

$$y(\infty) = \lim_{t\to\infty} y(t) = \lim_{s\to 0} s \cdot Y(s)$$

$$y(\infty) = \lim_{s\to 0} s \cdot \frac{3(s+2)}{s(s^2+2s+10)} = \frac{3\cdot 2}{10} = \boxed{0.6}$$

Initial and Final Value Theorem

Example 10:

Find the final value of the system corresponding to

$$Y(s) = \frac{3}{s(s-2)}$$

$$y(\infty) = \lim_{t\to\infty} y(t) = \lim_{s\to 0} s\cdot Y(s) = \lim_{s\to 0} s\cdot\frac{3}{s(s-2)} = \frac{3}{-2}$$

WRONG

$$Y(s) = \frac{3}{s(s-2)} = \frac{-3/2}{s} + \frac{3/2}{s-2}$$

$$y(t) = \mathcal{L}^{-1}\left\{Y(s)\right\} = -3/2\cdot 1(t) + 3/2\cdot e^{2t}\cdot u(t)$$

One pole ($s = 2$) is in the right half of the s-plane
→ Not convergent
→ FVT not applicable

Initial and Final Value Theorem

Example 11:

Find the final value of

$$Y(s) = \frac{2}{s^2 + 4}$$

$$y(\infty) = \lim_{t \to \infty} y(t) = \lim_{s \to 0} s \cdot Y(s) = \lim_{s \to 0} s \cdot \frac{2}{s^2 + 4} = 0$$

WRONG

$$Y(s) = \frac{2}{s^2 + 4} \Rightarrow y(t) = \sin 2t$$

A pair of poles on the imaginary axis ($s = \pm j2$)
→ Sinusoidal signal / periodic signal
→ Not convergent
→ FVT not applicable

Partial-Fraction Expansion with MATLAB

$$\frac{B(s)}{A(s)} = \frac{\text{num}}{\text{den}} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_n}{s^n + a_1 s^{n-1} + \cdots + a_n}$$

num = [b_0 b_1 ... b_n]
den = [1 a_1 ... a_n]

```
[r,p,k] = residue(num,den)
```

finds the residues (r), poles (p), and direct terms (k) of a partial-fraction expansion of the ratio of two polynomials $B(s)$ and $A(s)$.

$$\frac{B(s)}{A(s)} = \frac{r(1)}{s - p(1)} + \frac{r(2)}{s - p(2)} + \cdots + \frac{r(n)}{s - p(n)} + k(s)$$

Note that if $p(j) = p(j+1) = \cdots = p(j+m-1)$ $\left[\text{that is, } p_j = p_{j+1} = \cdots = p_{j+m-1}\right]$, the pole $p(j)$ is a pole of multiplicity m. In such a case, the expansion includes terms of the form

$$\frac{r(j)}{s - p(j)} + \frac{r(j+1)}{[s - p(j)]^2} + \cdots + \frac{r(j+m-1)}{[s - p(j)]^m}$$

Partial-Fraction Expansion with MATLAB

Example 12: Different Roots

$$\frac{B(s)}{A(s)} = \frac{2s^3 + 5s^2 + 3s + 6}{s^3 + 6s^2 + 11s + 6}$$

$$\frac{B(s)}{A(s)} = \frac{2s^3 + 5s^2 + 3s + 6}{s^3 + 6s^2 + 11s + 6}$$

$$= \frac{-6}{s+3} + \frac{-4}{s+2} + \frac{3}{s+1} + 2$$

```
num= [2  5  3  6]
 den= [1  6  11  6]
[r,p,k] = residue(num,den)

r =

     -6.0000
     -4.0000
      3.0000

p =

     -3.0000
     -2.0000
     -1.0000

k =

      2
```

Partial-Fraction Expansion with MATLAB

Example 13: Repeated Roots

$$\frac{B(s)}{A(s)} = \frac{s^2 + 2s + 3}{(s+1)^3} = \frac{s^2 + 2s + 3}{s^3 + 3s^2 + 3s + 1}$$

$$\frac{B(s)}{A(s)} = \frac{1}{s+1} + \frac{0}{(s+1)^2} + \frac{2}{(s+1)^3}$$

```
num = [1  2  3];
den = [1  3  3  1];
[r,p,k] = residue(num,den)

r =

    1.0000
    0.0000
    2.0000

p =

    -1.0000
    -1.0000
    -1.0000

k =

    []
```

Partial-Fraction Expansion with MATLAB

Example 14: Complex Roots

$$F(s) = \frac{3}{s(s^2+2s+5)}$$

$$F(s) = \frac{3/5}{s} - \frac{3}{20}\left(\frac{2+j1}{s+1+j2} + \frac{2-j1}{s+1-j2}\right)$$

```
 num=[3];
>> denum=[1 2 5 0];
>> [r,p,k]=residue(num,denum)

r =

  -0.3000 + 0.1500i
  -0.3000 - 0.1500i
   0.6000 + 0.0000i


p =

  -1.0000 + 2.0000i
  -1.0000 - 2.0000i
   0.0000 + 0.0000i


k =

     []
```

Laplace and Inverse Laplace with MATLAB

L = laplace(F) is the Laplace transform of the sym F with default independent variable t. The default return is a function of s.

```
>> syms t
>> L=laplace(2*exp(-t)+sin(2*t))
L =
2/(s + 1) + 2/(s^2 + 4)
```

Laplace and Inverse Laplace with MATLAB

F = ilaplace(L) is the inverse Laplace transform of the sym L with default independent variable s. The default return is a function of t

```
>> syms s
f=ilaplace(3/(s*(s^2+2 *s+5)));
pretty(f)
              /              sin(2 t) \
   exp(-t) | cos(2 t) + -------- | 3
3            \                2        /
- - ---------------------------------
5                  5
```

```
>> syms s
>> f=ilaplace((s^2+2*s+3)/(s+1)^3);
>> pretty(f)
              2
exp(-t) + t  exp(-t)
```

Exercise Problems

Find the Laplace transform of the following functions:

1. $$f(t) = 3\cos 6t$$
$$f(t) = \sin 2t + 2\cos 2t + e^{-t}\sin 2t$$
$$f(t) = t^2 + e^{-2t}\sin 3t$$

2. $$f(t) = 1 + 2t$$
$$f(t) = 3 + 7t + t^2 + \delta(t)$$
$$f(t) = e^{-t} + 2e^{-2t} + te^{-3t}$$
$$f(t) = (t+1)^2$$
$$f(t) = \sinh t$$

3. $$f(t) = \sin t \sin 3t$$
$$f(t) = \sin^2 t + 3\cos^2 t$$
$$f(t) = (\sin t)/t$$

Exercise Problems

Find the inverse Laplace transform of the following functions:

1. $$F(s) = \frac{2}{s(s+2)}$$

$$F(s) = \frac{10}{s(s+1)(s+10)}$$

$$F(s) = \frac{3s+2}{s^2+4s+20}$$

$$F(s) = \frac{3s^2+9s+12}{(s+2)(s^2+5s+11)}$$

2. $$F(s) = \frac{1}{s(s+2)^2}$$

$$F(s) = \frac{2s^2+s+1}{s^3-1}$$

$$F(s) = \frac{2(s^2+s+1)}{s(s+1)^2}$$

$$F(s) = \frac{s^3+2s+4}{s^4-16}$$

Exercise Problems

Solve the following diiferential equations using Laplace transform:

$$\ddot{y}(t) + \dot{y}(t) + 3y(t) = 0; y(0) = 1,\ \dot{y}(0) = 2$$

$$\ddot{y}(t) - 2\dot{y}(t) + 4y(t) = 0; y(0) = 1,\ \dot{y}(0) = 2$$

$$\ddot{y}(t) + \dot{y}(t) = \sin t; y(0) = 1,\ \dot{y}(0) = 2$$

$$\ddot{y}(t) + 3y(t) = \sin t; y(0) = 1,\ \dot{y}(0) = 2$$

$$\ddot{y}(t) + 2\dot{y}(t) = e^{t}; y(0) = 1,\ \dot{y}(0) = 2$$

$$\ddot{y}(t) + y(t) = t; y(0) = 1,\ \dot{y}(0) = -1$$